A CONSTRUCTION FOR THE NORMALIZER OF A RING WITH LOCAL UNIT WITH APPLICATIONS TO THE THEORY OF L-ALGEBRAS[1]

BY

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Introduction. The purpose of this paper is to extend some of the author's results [4; 5] about F-rings to a wider class of lattice ordered rings called generalized F-rings. A *generalized F-ring* or GFR is a partially ordered real algebra R which forms a σ-complete vector lattice with respect to addition, scalar multiplication, and order, and for which the following statements are valid:

(G1) For $a, b \in R$ with $a \geqq 0$, $b \geqq 0$, $ab = 0 \Leftrightarrow a \wedge b = 0$.

(G2) For each $a \in R$ there exists an ideal I of R such that $a \in I$ and I possesses a unit element.

An arbitrary ring satisfying G2 is called a *ring with local unit.* Such rings have been previously studied by Morrison [12].

It is possible that many of the results of [4; 5] can be proved for GFR's by a direct attempt to reproduce the arguments of [4; 5] in the wider setting. Instead, however, it is shown that every GFR can be embedded in an F-ring and then this result is used to generalize the results of [4; 5]. We prove the embedding theorem for the more general class of rings with local unit. In particular, in §1, a construction is given for the normalizer of a ring with local unit which involves an inverse limit process. As a corollary we show that the normalizer of a strongly regular ring is also strongly regular. The *normalizer* N of a faithful ring R is the maximal subring of the ring C of endomorphisms of R (considered as a left R-module) relative to the condition that N contain R as an ideal. This concept has been studied extensively by Johnson. See for example [11]. The normalizer of a GFR is again a GFR, and since it contains an identity, it is an F-ring.

In §2, a certain class of f-rings is shown to be embeddable in our F-rings. Birkhoff and Pierce [3] define an f-ring to be a lattice-ordered ring in which $a \wedge b = 0$ and $c \geqq 0$ imply $ca \wedge b = ac \wedge b = 0$.

§3 deals with the relationship between a GFR and its conditionally σ-complete Boolean ring of idempotents. It should be remarked at this point that no distinction is made here between a Boolean ring and a relatively complemented distributive lattice with zero, since a homomorphism which preserves one of these structures also preserves the other. A number of results of

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[5] are generalized to GFR's. In particular, if R is a regular GFR and $B(R)$ its Boolean ring of idempotents, then $R \rightarrow B(R)$ is a biunique (up to isomorphism) mapping from the class of regular GFR's onto the class of conditionally σ-complete Boolean rings.

§4 deals with the relationship between the maximal modular [10] ideals of a GFR R and the maximal ideals of its normalizer $\hat{R}$. The main results are (i) if R is identified with its isomorph in $\hat{R}$, then $\hat{M} \rightarrow \hat{M} \cap R$ is a biunique mapping of the maximal ideals of $\hat{R}$ which contain a maximal modular ideal of R but do not contain R onto the maximal modular ideals of R, and (ii) a maximal modular ideal of R is real if and only if it is σ-closed. The definitions of "real" and "σ-closed" are given in §4.

The main result of §5 is that every GFR with the property that the intersection of its real maximal modular ideals is the zero-ideal is isomorphic to a GFR of real valued functions. In §6 some examples of normalizers of well known conditionally σ-complete Boolean rings are discussed.

1. **The normalizer of a ring with local unit.** Let R be a ring with local unit, that is, a ring which satisfies (G2). Morrison [12] has proved that in a general ring A, the set of central idempotents forms a Boolean ring under the operations of "$\oplus$" and "$\cdot$", where $a \oplus b = (a-b)^2$, and "$\cdot$" is the multiplication of A. For a ring R with local unit, the class of central idempotents is designated by $B(R)$. Morrison has shown that the mapping ϕ: $I \rightarrow I \cap B(R)$ is an isomorphism of the lattice of ideals with local unit of R onto the ideals of $B(R)$ considered as a Boolean ring, and that $\phi^{-1}(J) = RJ$, the ideal of R generated by the subset J of R.

The Boolean ring $B(R)$ is a relatively complemented distributive lattice with respect to the order relation: $a \leqq b$ if and only if $ab = a$. It is clear that any lattice homomorphism of the relatively complemented distributive lattice $\langle B(R), \vee, \wedge \rangle$ which preserves relative complements is a ring homomorphism of $\langle B(R), \oplus, \cdot \rangle$ and conversely. In the course of this paper, we shall consider many times homomorphisms from one ring R_1 with local unit into another R_2; such homomorphisms when restricted to $B(R_1)$ preserve lattice operations as well as relative complements, and therefore are homomorphisms of the ring structure of $\langle B(R_1), \oplus, \cdot \rangle$.

For any $x \in R$, there is an element $e \in B(R)$ such that $ex = xe = x$. Such an element is called a *local unit for* x.

For $a \leqq b$ in $B(R)$, let the mapping π_{ab} be defined on Rb as follows: $\pi_{ab}x = ax$. π_{ab} is a homomorphism of the ideal Rb onto the ideal Ra. It is clear that $\{(Ra, \pi_{ab}) \mid a, b \in B(R)\}$ is an inverse limit system [8] and hence an inverse limit $\hat{R} = \lim_{\leftarrow}(Ra, \pi_{ab})$ exists. The elements of $\hat{R}$ are functions f from $B(R)$ into R such that for $a \leqq b$, $f(b)a = f(a)$. In the sequel, the ring constructed by this method from a ring R with local unit will be referred to as $\hat{R}$. In virtue of the inverse limit construction one defines for $f, g \in \hat{R}$, $(f+g)(u) = f(u) + g(u)$ and $(f \cdot g)(u) = f(u) \cdot g(u)$ for all $u \in B(R)$.

If R is a ring with local unit, then the mapping i of R into $\hat{R}$ is defined as follows: $i(x)=g_x$ where $g_x(u)=xu$ for each $u\in B(R)$.

PROPOSITION 1.1. *The mapping i is an injection of R into $\hat{R}$.*

Proof. The mapping i is clearly a homomorphism of R into $\hat{R}$. To show it is an injection let $i(x)=i(y)$. Then $xu=yu$ for all $u\in B(R)$ and in particular for $u_0=e_x+e_y-e_xe_y$ where e_x and e_y are local units for x and y respectively. Thus $x=xu_0=yu_0=y$.

The mapping i is referred to as the *canonical injection* of R into $\hat{R}$. The symbol iR is used to designate the image of R under i when it is necessary to distinguish this object from R.

If A is a subset of an arbitrary ring Q then A^l stands for the class of elements $x\in Q$ such that $xa=0$ for all $a\in A$. Let 1 be the element of $\hat{R}$ with the form $1(u)=u$ for each $u\in B(R)$.

PROPOSITION 1.2. *The ring $\hat{R}$ has* 1 *for its unit element; the ring iR is an ideal of $\hat{R}$; $(iR)^l=0$ in $\hat{R}$; and, if R contains a unit element, then $\hat{R}\cong R$.*

Proof. Consider the function $1(u)=u$. If $f\in\hat{R}$, then $(f1)(u)=(1f)(u)=1(u)f(u)=uf(u)$, and if $a\leqq b$, then by the definition of inverse limit $f(b)a=f(a)=f(a)a$. Therefore $1f=f1=f$ and hence 1 is the unit element of $\hat{R}$.

It is clear that iR is a subgroup of $\hat{R}$. To show it is an ideal, let $f\in iR$ and $g\in\hat{R}$. The function f is of the form $f(u)=xu$ for some $x\in R$. Now $(gf)(u)=g(u)xu$. If $a\geqq e$ where e is a local unit for x, then since $x=xe=xa=xae$, it follows that $g(a)f(a)=g(a)xa=g(a)xea=g(a)exa=[g(e)x]a$. If $a\ngeqq e$, then $a\vee e=a+e-ae\geqq e$ and hence $g(a\vee e)f(a\vee e)=g(a\vee e)x(a\vee e)=[g(e)x](a\vee e)$. Therefore

$$\begin{aligned} g(a)f(a) &= [g(a\vee e)f(a\vee e)]a = [g(e)x](a\vee e)a \\ &= [g(e)x]a. \end{aligned}$$

Hence for all $a\in B(R)$, $g(a)f(a)=[g(e)x]a$ and thus $gf\in iR$. In a similar fashion it is possible to show $fg\in iR$ as well.

To show $(iR)^l=0$, let $g\in(iR)^l$. Then for each $a\in B(R)$, $g(a)(xa)=0$ for all $x\in R$. In particular, $g(a)ua=0$ for $u\in B(R)$ and each $a\in B(R)$. Thus $g(u)u=g(u)=0$ for each $u\in B(R)$. Therefore $g=0$, and $(iR)^l=0$.

If R contains a unit element q, then $B(R)$ contains q as a maximal element, and for all $a\in B(R)$, $f(a)=f(q)a$. Therefore every $f\in\hat{R}$ is of the form $f(u)=xu$ and hence the canonical injection is an isomorphism of R onto $\hat{R}$.

An arbitrary ring Q is *faithful* if $Q^l=0$ in Q. Clearly all rings with local unit are faithful.

Johnson [11] has shown that the normalizer of a faithful ring Q is (up to an isomorphism) the universal faithful ring $\overline{Q}$ containing Q as an ideal such that $Q^l=0$ in $\overline{Q}$. Thus $\hat{R}$ can be embedded in the normalizer of R.

THEOREM 1.1. *If R is a ring with local unit, then $\hat{R}$ is isomorphic to the normalizer of R.*

Proof. Let $\overline{R}$ designate the normalizer of R. If $x\in\overline{R}$, then $xa\in R$ for all $a\in B(R)$. The function ψ_x from $B(R)$ to R where $\psi_x(u)=xu$, is a member of $\hat{R}$. Indeed, if $a\leqq b$, then $\psi_x(b)a=xba=xa=\psi_x(a)$. In addition $\psi_{x+y}=\psi_x+\psi_y$, and

$$\psi_{xy}(a) = xya = (xya)a = x(ya)a = xaya = \psi_x(a)\psi_y(a).$$

Therefore $x\rightarrow\psi_x$ is a homomorphism of $\overline{R}$ into $\hat{R}$. Suppose $\psi_x=\psi_y$. Then $xa=ya$ for all $a\in B(R)$, and hence $(x-y)z=0$ for all $z\in R$. Since $R^l=0$ in $\overline{R}$, $x=y$ and $x\rightarrow\psi_x$ is an injection.

The mapping ψ is clearly an extension of the mapping i which embeds R in $\hat{R}$. From [11, p. 527], there is an injection ϕ of $\hat{R}$ into $\overline{R}$ which is an extension of the natural injection of R into the ring of endomorphisms of R taken as an additive group. Thus the mapping $\phi\psi$ embeds $\overline{R}$ in $\overline{R}$, and the isomorph of R in $\overline{R}$ is strongly invariant under this mapping. For each $x\in\overline{R}$, $\phi\psi(x)-x\in\overline{R}$, and if R is identified with its isomorph in $\overline{R}$, then (because R is an ideal of $\overline{R}$)

$$\begin{aligned}(\phi\psi(x) - x)r &= \phi\psi(x)r - xr\\ &= \phi\psi(xr) - xr = \phi\psi(xr) - \phi\psi(xr) = 0\end{aligned}$$

for all $r\in R$. Since $R^l=0$ in $\overline{R}$, it follows that $\phi\psi(x)=x$ for each $x\in\overline{R}$. Thus $\phi\psi$ is the identity mapping on $\overline{R}$, and so ψ is an isomorphism of $\hat{R}$ onto $\overline{R}$ because the domain of ϕ is the entire ring $\hat{R}$.

It follows from the method of construction of $\hat{R}$ that if R is commutative or without nilpotent elements, then so is $\hat{R}$. However, it is still a matter of conjecture as to whether $\hat{R}$ is regular or biregular if R is. We can, however, establish that if R is strongly regular, then $\hat{R}$ is as well. See [1] for definitions of the various forms of regularity.

LEMMA 1.1. *Let A be a ring with unit element* 1. *A is strongly regular if and only if it has property*

(α) *For each $x\in A$ there is a central idempotent $a_x\in A$ such that $xa_x=0$ and $x+a_x$ has an inverse.*

Proof. If one remembers that a ring A is strongly regular if for each $x\in A$, there is an $x^0\in A$ such that $x^2x^0=x$ and that [1] the element $x^0x=xx^0$ is a central idempotent, then one can easily prove this lemma by rephrasing the proof of [4, Theorem 1] with $a_x=1-xx^0=1-x^0x$.

COROLLARY 1.1. *There is at most one element $a_x\in A$ which satisfies condition* (α) *for $x\in A$.*

Proof. Suppose there are two such elements a_x and b. There is a $y\in A$ such that $y(x+a_x)=1$, and hence $bya_x=b$. However, $a_x=y(x+a_x)a_x=ya_x$, so

$ba_x = b$. Similarly $ba_x = a_x$, and $a_x = b$.

PROPOSITION 1.3. *If R is strongly regular, then $\hat{R}$ is as well.*

Proof. Let f belong to $\hat{R}$. Then $f(u) \in Ru$ for each $u \in B(R)$. The ideal Ru is a strongly regular ring with unit element u, and hence by Lemma 1.1, for each u there is a unique element a_u which satisfies condition (α) relative to f in Ru. Let a_f stand for the function on $B(R)$ with $a_f(u) = a_u$. Now for each u there is a unique element $y_u \in Ru$ such that $y_u(f(u)+a_u) = (f(u)+a_u)y_u = u$. Let y stand for the function on $B(R)$ with $y(u) = y_u$. The proposition is valid if it can be shown that a_f and y belong to $\hat{R}$. To show this, let $u \leqq v$ and $u, v \in B(R)$. Then $f(v)a_v u = f(u)(a_v u) = 0$, and $y_v(f(v) + a_v)u = vu = u = y_v u(f(u)+(a_v u))$. Therefore by Corollary 1.1, $a_v u = a_v$, and hence $y_v u = y_u$. Thus a_f and y belong to $\hat{R}$.

If R is a partially ordered ring with local unit, then with respect to the order relation: $f \leqq g$ in $\hat{R}$ if $f(u) \leqq g(u)$ for all $u \in B(R)$, the ring $\hat{R}$ is a partially ordered ring. This partial order on $\hat{R}$ is used throughout the sequel.

LEMMA 1.2. *If R is a partially ordered ring with local unit such that $a^2 \geqq 0$ for all $a \in R$, then for $a, b \in B(R)$, $a \leqq b$ in the partially ordered ring R if and only if $ab = a$.*

Proof. Suppose $a \leqq b$; then $b - a \geqq 0$, and $ab \geqq a$. Since $a(a-b)^2 = a + ab - 2ab \geqq 0$, it follows that $a \geqq ab$, and hence $a = ab$.

Conversely, if $b, a \in B(R)$, then $b - ab \in B(R)$, and from $ab = a$, it follows that $b - a \in B(R)$. Thus $b \geqq a$.

PROPOSITION 1.4. *If R is a partially ordered ring with local unit such that $a^2 \geqq 0$ in R, then $\hat{a}^2 \geqq 0$ in $\hat{R}$, and the isomorphism i: $R \rightarrow iR \subseteq \hat{R}$ preserves order.*

Proof. The first statement of the conclusion is clear. Since $a \in B(R)$ implies $a \geqq 0$, it follows that $x \geqq 0$ in R implies $xa \geqq 0$, and thus that $x \geqq 0$ implies $i(x) \geqq 0$. Hence the canonical isomorphism preserves order.

A lattice-ordered ring R with local unit is called a *function ring* if for $0 \leqq x \in R$ and $0 \leqq y \in R$ the statement, $xy = 0$ if and only if $x \wedge y = 0$, is valid. If in addition R is an l-algebra, then it is called a *function algebra.*

LEMMA 1.3. *If R is a function ring, then for $a, b, c \in R$*
(i) $a \geqq 0 \Rightarrow a(b \vee c) = ab \vee ac$ *and* $a(b \wedge c) = ab \wedge ac$,
(ii) $a^2 \geqq 0$, *and*
(iii) $|a|\,|b| = |ab|$.

Proof. See [3, p. 57].

PROPOSITION 1.5. *If R is a function ring, then so is $\hat{R}$. The unit element of $\hat{R}$ is a weak order unit.*

Proof. This Proposition follows by direct verification.

PROPOSITION 1.6. *If R is a function ring, then for each $f \in \hat{R}$ such that $f \geqq 0$ the following equation is valid*:

$$f = \sup_{u \in B(R)} \{i(f(u))\}.$$

Proof. For u, $v \in B(R)$, $f(u \vee v) \geqq f(u)$ and $f(u \vee v)v \geqq f(u)v$ imply that $f \geqq i(f(u))$ for all $u \in B(R)$. If $g \geqq i(f(u))$ for all $u \in B(R)$ then $g(v) \geqq [i(f(v))](v) = f(v)$ for all $v \in B(R)$ and hence $g \geqq f$. Thus f is the required supremum.

Let m be a cardinal number. A lattice L is conditionally m-complete if every subset A of L of cardinality $\leqq m$ and bounded above (below) by an element of L possesses a supremum (infimum). A lattice L is conditionally complete if it is conditionally m-complete for each m.

LEMMA 1.4. *If R is an m-complete function ring, then for $g \in B(R)$ and $\{f_\gamma | \gamma \in \Gamma\} \subseteq R^+$ with the cardinality of $\Gamma \leqq m$, we have*

$$\left(\bigwedge_{\gamma \in \Gamma} f_\gamma\right) g = \bigwedge_{\gamma \in \Gamma} (f_\gamma g).$$

Proof. Clearly $(\bigwedge_{\gamma \in \Gamma} f_\gamma) g \leqq \bigwedge_{\gamma \in \Gamma} (f_\gamma g)$. Then because $i(g) \leqq 1$, the unit element of $\hat{R}$, it follows that

$$0 \leqq \bigwedge_{\gamma \in \Gamma} (f_\gamma g) - \left(\bigwedge_{\gamma \in \Gamma} f_\gamma\right) g = \bigwedge_{\gamma \in \Gamma} \left\{f_\gamma g - \left(\bigwedge_{\sigma \in \Gamma} f_\sigma\right) g\right\}$$

$$= \bigwedge_{\gamma \in \Gamma} \left[\left\{f_\gamma - \left(\bigwedge_{\sigma \in \Gamma} f_\sigma\right)\right\} g\right] \leqq \bigwedge_{\gamma \in \Gamma} \left\{f_\gamma - \left(\bigwedge_{\sigma \in \Gamma} f_\sigma\right)\right\} = 0.$$

Hence the equality is valid.

COROLLARY 1.2. *If R is a conditionally m-complete function ring, then so is $\hat{R}$. The canonical injection of R into $\hat{R}$ preserves m-operations.*

Proof. If $\{f_\gamma | \gamma \in \Gamma\} \subseteq R^+$ where the cardinality of $\Gamma \leqq m$, then for each $u \in B(R)$, $f(u) = \bigwedge_{\gamma \in \Gamma} f_\gamma(u)$ belongs to R. Using Lemma 1.4, the reader can verify that the function f belongs to $\hat{R}$ and is the infimum of the set $\{f_\gamma | \gamma \in \Gamma\}$. Thus $\hat{R}$ is conditionally m-complete. The remainder of the corollary follows analogously.

REMARK 1.1. A conditionally $\aleph_0$-complete function algebra is a generalized F-ring. This follows [3] because a conditionally σ-complete lattice ring is archimedean, and hence is commutative as well as an algebra over the real field. In addition, if R is a GFR, it follows from Corollary 1.2 that $\hat{R}$ is an F-ring.

THEOREM 1.2. *The normalizer of a GFR is an F-ring, and the normalizer of a regular GFR is a regular F-ring.*

Proof. This follows from Propositions 1.1, 1.3, and 1.5, Corollary 1.2, Theorem 1.1, and Remark 1.1.

2. **An embedding theorem for a class of function rings.** Let R be an archimedean l-algebra over the real field with local unit such that $\inf\{a_\gamma \mid \gamma\in\Gamma\} = 0$ and $c\geqq 0$ imply $\inf\{ca_\gamma \mid \gamma\in\Gamma\} = \inf\{a_\gamma c \mid \gamma\in\Gamma\} = 0$. In this paper such an algebra is called an a-l-algebra. The main result of §2 is that every a-l-algebra can be embedded in a regular F-ring. This is a generalization of a result in [5].

LEMMA 2.1. *If A is a lattice-ordered ring with local unit, then among the following four statements* (i)$\Leftrightarrow$(iii) *and* (ii)$\Leftrightarrow$(iv).

(i) $\forall_a\forall_b\forall_c[a\wedge b=0$ and $c\geqq 0\Rightarrow ac\wedge b=0]$,

(ii) $\forall_a\forall_b\forall_c[a\wedge b=0$ and $c\geqq 0\Rightarrow ca\wedge b=0]$,

(iii) $\forall_a\forall_b\forall_c[a\wedge b=0$ and $c\geqq 0\Rightarrow ac\wedge bc=0]$,

(iv) $\forall_a\forall_b\forall_c[a\wedge b=0$ and $c\geqq 0\Rightarrow ca\wedge cb=0]$.

Thus an a-l-algebra is an f-ring of Birkhoff and Pierce [3].

Proof. (i)$\Rightarrow$(iii): From (i) we deduce that for $a\wedge b=0$ and $c\geqq 0$, $ac\wedge b=0$. However, $ac\wedge b=b\wedge ac=0$ and $c\geqq 0$, so $bc\wedge ac=0$.

(iii)$\Rightarrow$(i): Let e be a local unit for b. Suppose $a\wedge b=0$ and $c\geqq 0$. Then $0\leqq ac\wedge b$. In addition $ac\leqq a(c\vee e)$ and $b\leqq b(c\vee e)$. Hence $0\leqq ac\wedge b\leqq a(c\vee e)\wedge b(c\vee e)$, but since $c\vee e\geqq 0$, it follows that $ac\wedge b=0$.

By an analogous procedure it is possible to prove (ii)$\Leftrightarrow$(iv).

LEMMA 2.2. *An archimedean f-ring R with local unit contains no nilpotent elements.*

Proof. First suppose $a\geqq 0$ and $a^n=0$. If e is a local unit for a, then $a\in Re$ where Re is an archimedean f-ring with positive unit element e. By [3, Corollary 3, p. 63], $a=0$.

Now if R contains a nonzero nilpotent element, it contains a positive nilpotent element. Indeed, let $a\neq 0$ be a nilpotent element of order n. Then $a^{n-1}=b$ is nilpotent of order 2, and hence $b^2=|b|^2=0$. Thus $|b|$ is a nonnegative nilpotent and the lemma follows.

COROLLARY 2.1. *An a-l-algebra A is a function ring.*

Proof. By Lemma 2.1, an a-l-algebra is an f-ring. From Lemma 2.2 and [3, p. 57, (17) and p. 63, Corollary 2], it follows that A is a function ring.

REMARK 2.1. Lemma 2.1 and [3, Theorem 13] yield the result that every a-l-algebra is commutative.

THEOREM 2.1. *If R is an a-l-algebra, then R can be embedded in a regular F-ring R^*.*

Proof. By Proposition 1.5, $\hat{R}$ is a function ring with positive unit element. The proof consists of showing that $\hat{R}$ satisfies the conditions of [5, Theorem

9], and hence, by the remarks preceding that theorem in [5], $\hat{R}$ can be embedded in a regular F-ring. Thus we must show (i) that $\hat{R}$ is archimedean and (ii) that in $\hat{R}$, $\bigwedge_\gamma a_\gamma = 0$ and $c \geqq 0$ imply $\bigwedge_\gamma ca_\gamma = 0$.

To prove (i), note that if $a \geqq 0$ in $\hat{R}$ and if $0 \leqq b \in \hat{R}$ satisfies the relation $b \leqq a/n$ for all $n \geqq 1$, then $b(u) \leqq a(u)/n$ for all $n \geqq 1$ and each $u \in B(R)$. Therefore $b(u) = 0$ for all $u \in B(R)$, whence $b = 0$ and $\hat{R}$ is archimedean.

Property (ii) can be verified in a similar fashion.

The embedding in Theorem 2.1 is in some sense "dense" as is indicated by the following corollary.

COROLLARY 2.2. *For every $a^* \in R^*$ with $a^* \geqq 0$, there is a subset A of the isomorph of R in R^* with the property*:

$$a^* = \sup A.$$

Proof. This follows from the remarks of Nakano [13, Chapter 5] and Proposition 1.6. Indeed, if i stands for the canonical injection of R into $\hat{R}$ and j stands for Nakano's injections of $\hat{R}$ into its cut extension R^*, then Nakano shows $a^* = \bigvee_{\gamma \in \Gamma} j(\hat{a}_\gamma)$ for some system $\{\hat{a}_\gamma \in \hat{R} \mid \gamma \in \Gamma\}$, and Proposition 1.6 implies that

$$\hat{a}_\gamma = \bigvee_{u \in B(R)} i(\hat{a}_\gamma(u)).$$

From [13, Theorem 30.1 and 2.5], it follows that if

$$A = \{ji(\hat{a}_\gamma(u)) \mid \gamma \in \Gamma, u \in B(R)\}, \quad \text{then} \quad a^* = \sup A.$$

3. **GFR's and their Boolean rings of idempotents.** In this section let R stand for a GFR and R^* stand for the regular F-ring containing $\hat{R}$ such that $B(\hat{R}) = B(R^*)$. The existence of R^* is proved in [5]. The main result of this section is that the correspondence $R \rightarrow B(R)$ maps the class of GFR's into the class of conditionally σ-complete Boolean rings in a manner which is biunique up to isomorphism: that is, if two GFR's are isomorphic, then the corresponding Boolean rings are isomorphic and conversely.

To begin with, a few preliminary results are proved. For a ring L with local unit, the set $B(L)$ can be considered as a relatively complemented distributive lattice or as a Boolean ring. If $B(L)$ is considered as a Boolean ring, then $[B(L)]\hat{}$ exists and is a Boolean ring.

LEMMA 3.1. *If L is a ring with local unit, then $B(\hat{L})$ is isomorphic to $[B(L)]\hat{}$.*

The term *l-ideal* is used here as in [3] to mean a ring ideal with the added property: If a belongs to the l-ideal and $|b| \leqq |a|$, then b also belongs to the l-ideal.

PROPOSITION 3.1. *If R is a regular F-ring and B is an ideal of the Boolean algebra $B(R)$, then there is a unique regular GFR, $R(B)$, such that $B(R(B)) = B$. In addition, $R(B)$ is an l-ideal of R.*

Proof. First it should be noted that B must be a conditionally σ-complete Boolean ring because $a_n \in B$ and $a_n < a \in B$ imply $\bigvee a_n \in B(R)$ and $\bigvee a_n < a$, and hence $\bigvee a_n \in B$.

Consider the class $R_1 = \{f \in R \mid \bar{e}_f \in B\}$ where $\bar{e}_f = \bigvee_{n=1}^{\infty} n|f| \wedge 1$. The ring R_1 is an l-ideal of R. Indeed, if $f, g \in R_1$, then it can be shown by employing the inequality $|f-g| \leqq 2|f| \vee 2|g|$ that $\bar{e}_{f-g} \leqq \bar{e}_f \vee \bar{e}_g$, and hence $f-g \in R_1$. Since $|g| \leqq |f|$ implies that $\bar{e}_g \leqq \bar{e}_f$, it follows that $f \in R_1$ implies $g \in R_1$. By [5, Theorem 7], $\bar{e}_{fg} \leqq \bar{e}_f$ for any $f, g \in R$; hence R_1 can be shown to be an l-ideal.

Since R_1 is an l-ideal of R, it is easy to see that R_1 is a sub-GFR of R. To show R_1 is regular, note that if $ff^0f = f$, then $(f\bar{e}_f)f^0f = f(\bar{e}_f f^0)f = f$, and $\bar{e}_f f^0 \in R_1$ if $f \in R_1$.

To ensure R_1 is a candidate for the role of $R(B)$, we must show $B(R_1) = B$. Certainly $B \subseteq B(R)$. Suppose $a = a^2 \in R_1$; then $\bar{e}_a = a \in B$ by [4, Theorem 2]. Hence $B(R_1) \subseteq B$ and $B = B(R_1)$.

It remains only to show that R_1 is unique with respect to the property: $B(R_1) = B$. Suppose there is another regular GFR R_2 such that $B(R_2) = B$. If $f \in R_2$, then there is a local unit e for f in R_2. If $\bar{e}_f = \bigvee_{n=1}^{\infty} n|f| \wedge 1$, then from [4, Theorem 2], $\bar{e}_f = ff^0$ where $f^0 \in R$ is an element with the property $ff^0f = f$. It is clear that $e\bar{e}_f = \bar{e}_f$. Now the element $e\bar{e}_f = \bigvee_{n=1}^{\infty} n|ef| \wedge e$ belongs to $B(R_2)$, and hence $\bar{e}_f \in B(R_2) = B$. Therefore $R_2 \subseteq R_1$.

Suppose $f \in R_1$, $f \geqq 0$, and $f \leqq \lambda \cdot 1$ for some real $\lambda \geqq 0$ where 1 is the unit element of R. From the spectral theorem [2, p. 251], it follows that f is the supremum of a sequence of finite linear combinations of elements of B. Therefore $f \in R_2$. Thus the bounded part of R_1 belongs to R_2. Suppose $f \geqq 0$ is not bounded (that is, there is no $\lambda > 0$ such that $f \leqq \lambda \cdot 1$) and $f \in R_1$; then $\bar{e}_f \in B \subseteq R_2$. The ideals $\bar{e}_f R_1$ of R_1 and $\bar{e}_f R_2$ of R_2 are both regular F-rings with unit element $\bar{e}_f$, and $\bar{e}_f R_2 \subseteq \bar{e}_f R_1$. It is also clear that the bounded part of $\bar{e}_f R_1$ coincides with that of $\bar{e}_f R_2$. The regularity of $\bar{e}_f R_1$ and [4, Theorems 1 and 2] imply the existence of $y \in \bar{e}_f R_1$ such that $y(f + \bar{e}_f) = \bar{e}_f$, $0 \leqq y \leqq \bar{e}_f$, and $\bigvee_{n=1}^{\infty} ny \wedge \bar{e}_f = \bar{e}_f$. Thus the regularity of $\bar{e}_f R_2$ implies that the inverse of y in $\bar{e}_f R_1$ belongs to $\bar{e}_f R_2$, but this inverse is just $f + \bar{e}_f$. Hence $f \in R_2$, and $R_1 = R_2$.

COROLLARY 3.1. *For every GFR R, there is a regular GFR, R_1, and an injection k of R into R_1 which preserves the ring and σ-lattice operations such that $B(R_1) = B(kR)$.*

Proof. From Remark 1.1 and from [5] it follows that the GFR $\hat{R}$ is an F-ring and that there is a regular F-ring R^* and an injection i^* which maps $\hat{R}$ into R^*. In addition, $B(i^*\hat{R}) = B(R^*)$ and i^* preserves the ring and σ-lattice operations. If i stands for the canonical injection of R into $\hat{R}$, then i^*iR is a sub GFR of R^* isomorphic to R. By the nature of i, it is clear that $i^*iB(R)$ is an ideal of $i^*B(\hat{R}) = B(R^*)$. From Proposition 3.1, $R(i^*iB(R))$ is the required regular GFR, and i^*i is the required injection.

In order to prove that two regular GFR's with isomorphic rings of idempotents are isomorphic, the following construction due to Olmsted [14] is useful. Let B be a σ-complete Boolean algebra and let $\Omega(B)$ stand for the class of functions f from the real line to B which satisfy the following conditions:

1. $f(\xi)\downarrow$ as $\xi\uparrow$.
2a. $\bigvee_\xi f(\xi)=1$.
2b. $\bigwedge_\xi f(\xi)=0$.
3. $\bigvee_{\beta<\alpha} f(\alpha)=f(\beta)$ for every β.

Olmsted has shown that ring and lattice operations can be defined in $\Omega(B)$ which make it an F-ring and the author has shown [5] that $\Omega(B)$ is a regular F-ring.

Let j represent the injection of $\hat{B}$ into $\Omega(\hat{B})$ defined as follows: $j(a)=f_a(\lambda)$ where

$$\begin{aligned} f_a(\lambda) &= 1 \qquad \text{for } \lambda < 0,\\ &= a \qquad \text{for } 0 \leqq \lambda < 1,\\ &= 0 \qquad \text{for } 1 \leqq \lambda. \end{aligned}$$

PROPOSITION 3.2. *If B is a conditionally σ-complete Boolean ring, then $Q=\{f\in\Omega(\hat{B})\mid f^+(0)$ and $f(0)$ belong to $iB\}$ is an l-ideal of $\Omega(\hat{B})$, and $B(Q)=jiB$.*

Proof. If $f,\ g\in Q$, then $(f+g)^+\leqq f^++g^+$, and hence from [14, p. 166], it follows that $(f+g)^+(0)\leqq (f^++g^+)(0)=\bigvee_\beta f^+(\beta)\wedge g^+(-\beta)\leqq f^+(0)\vee g^+(0)$ where β ranges over a dense subset of the real numbers. Therefore $(f+g)^+(0)\leqq f^+(0)\vee g^+(0)\in iB$, and hence $(f+g)^+(0)\in iB$. In an analogous fashion, it follows that $(f+g)^-(0)\in iB$. By similar arguments the reader, using the definitions of [14], can verify that Q is an l-ideal of $\Omega(\hat{B})$ and hence that Q is a regular GFR.

To show $B(Q)=jiB$, first suppose $f\in jiB$. Then $f^-(0)=0$ and $f^+(0)=f(0)\in iB$. Thus $f\in Q$, but since f is idempotent, $f\in B(Q)$.

Conversely, suppose $f\in B(Q)$. Then $f\in B(\Omega(\hat{B}))$, and hence from [14] it follows that $f=j(a)$ for $a\in\hat{B}$. By the definition of Q, $f(0)=a\in iB$ and hence $f\in jiB$. Therefore $B(Q)=jiB$.

If Q is defined as in Proposition 3.2, then the following theorem can be proved.

THEOREM 3.1. *If R is a GFR such that $B=B(R)$, then R can be embedded in Q.*

Proof. Let i be the canonical injection of R into $\hat{R}$. The restriction of i to $B=B(R)$ is the canonical injection of B into $\hat{B}=B(\hat{R})$. If $x\in iR$, then both $\bar{e}_{x^+}=\bigvee_{n=1}^\infty n|x^+|\wedge 1$ and $\bar{e}_{x^-}=\bigvee_{n=1}^\infty n|x^-|\wedge 1$ belong to iB. Similarly the mapping $k\colon x\to e_x(\lambda)$ where

$$e_x(\lambda) = \bigvee_{n=1}^{\infty} n(x-\lambda)^+ \wedge 1$$

is an injection (see [14]) of $\hat{R}$ into $\Omega(\hat{B})$ as well as an extension of the mapping j (discussed in Proposition 3.2) to $\hat{R}$. Indeed, if $a \in B(\hat{R}) = \hat{B}$, then $e_a(\lambda) = 0$ if $\lambda \geqq 1$, $e_a(\lambda) = 0$ if $\lambda < 0$, and $e_a(\lambda) = a$ if $0 \leqq \lambda < 1$, that is, $k(a) = j(a)$. The mapping k preserves the ring and σ-lattice operations by [14, Theorem 2.2; and 5, Theorem 3].

Since $x \in iR$ implies that $\bar{e}_{x^+}$ and $\bar{e}_{x^-}$ both belong to iB and since $\bar{e}_{x^+} = e_{x^+}(0)$ and $\bar{e}_{x^-} = e_{x^-}(0)$, it follows that $k(x) \in Q$. Therefore ki is an injection of R into Q.

COROLLARY 3.2. *If R is a regular GFR, then $R \cong Q$.*

Proof. Since kiB is an ideal of $k\hat{B}$, it follows from Proposition 3.1 that $kiR = Q$ and hence $R \cong Q$.

THEOREM 3.2. *Every GFR R is contained in a regular GFR R^* such that $B(R) = B(R^*)$, and R^* is determined uniquely up to an isomorphism. The mapping $R \to B(R)$ from the class of regular GFR's into the class of conditionally σ-complete Boolean rings is "onto" in the sense that every conditionally σ-complete Boolean ring can be embedded in a regular GFR and is biunique up to isomorphism in the sense that two regular GFR's map into isomorphic conditionally σ-complete Boolean rings if and only if they are isomorphic.*

Proof. The proof of this theorem follows directly from Theorem 3.1, Proposition 3.1, and Corollaries 3.1 and 3.2 .

4. **The maximal modular ideals of rings with local unit.** In this section a number of results are proved which relate the maximal modular ideals of a ring R with local unit to the maximal ideals of $\hat{R}$. Again let i stand for the canonical injection of R into $\hat{R}$.

To begin we prove a result concerning general modular ideals of R. An element $j \in R$ is a left identity modulo an ideal I of R if $jy - y \in I$ for all $y \in R$.

PROPOSITION 4.1. *For every modular left ideal I of R there is a central idempotent identity e modulo I.*

Proof. Let j be a left identity of I. If ϕ stands for the natural homomorphism associated with I and if $e = e^2$ is a local unit for j, then $je - j \in I$ and $je - e \in I$. For $y \in R$, $jey - ey \in I$ and $jey - jy \in I$. Therefore $\phi(jey) = \phi(ey) = \phi(jy)$, but since $\phi(jy) = \phi(y)$, it follows that $\phi(ey - y) = 0$. Hence $ey - y \in I$ for all $y \in R$.

PROPOSITION 4.2. *Every ideal I of iR is an ideal of $\hat{R}$.*

Proof. If $x \in \hat{R}$ and $y \in I$, then xy and yx belong to iR. In addition let e be a local unit for y; then xe and ex belong to iR, and hence $xey = xy$ and $yex = yx$ belong to I. Therefore, I is an ideal of $\hat{R}$.

It should be noted at this point that in Propositions 4.3 and 4.4 as well as Corollary 4.1 and Lemma 5.1, $\hat{R}$ is assumed to be biregular. This condition is certainly satisfied if R is strongly regular by Proposition 1.3.

PROPOSITION 4.3. *If $\hat{R}$ is a biregular ring, then every maximal modular ideal M of iR is contained in a unique maximal ideal $\hat{M}$ of $\hat{R}$ which coes not contain iR.*

Proof. By Proposition 4.2, M is an ideal of $\hat{R}$, and hence by [1, Corollary 3, p. 459] it follows that there is at least one maximal ideal $\hat{M}$ which contains M but not iR.

To show the uniqueness of $\hat{M}$ suppose there are two different maximal ideals M_1, M_2 of $\hat{R}$ which contain M but which do not contain iR. There is an element $y_k \in iR$ such that $y_n \notin M_k$ for $k=1, 2$. Let e_k be the local unit for y_k; then $e_k \notin M_k$. Thus the idempotent $e=e_1 \vee e_2 = e_1+e_2-e_1e_2$ belongs to iR but does not belong to either M_1 or M_2.

The ideal $M_k \cap iR$ contains M and is modular for $k=1, 2$; hence $M_k \cap iR = M$. The element e can be written in the form $e=e_1'+e_2'$ where e_n' belongs to $M_k \cap B(\hat{R})$. Indeed, since for $k=1, 2$ the ideal $M_k \cap B(\hat{R})$ is a maximal ideal of the Boolean algebra $B(\hat{R})$, e can be written $e=\hat{e}_1 \vee \hat{e}_2$ where $\hat{e}_k \in M_k \cap B(\hat{R})$. Then $e=e_1'+e_2'$ with $e_1'=\hat{e}_1$ and $e_2'=\hat{e}_2-\hat{e}_1\hat{e}_2$. It follows that $e_k' \in iR$ for $k=1, 2$; hence $e \in M$ which is contrary to the hypotheses that $M_1 \neq M_2$. Therefore the uniqueness of $\hat{M}$ is established.

COROLLARY 4.1. *If M is a maximal modular ideal of iR and $\hat{M}$ is the corresponding maximal ideal of $\hat{R}$ which contains M but not iR, then $M=\hat{M} \cap iR$.*

PROPOSITION 4.4. *If $\hat{R}$ is biregular, if M is a maximal modular ideal of R, if $\hat{M}$ is the maximal ideal of $\hat{R}$ for which $iM=\hat{M} \cap iR$, and if $T=\{f \in \hat{R} \mid f(u) \in M$ for each $u \in B(R)\}$, then $T=\hat{M}$.*

Proof. Clearly T is an ideal of $\hat{R}$, $iM \subseteq T$, and iR is not a subset of T. In addition, if T is a maximal ideal of $\hat{R}$, then $T=\hat{M}$.

To show T is maximal, suppose $f \in \hat{R}$ and $f \notin T$. Since $\hat{R}$ is biregular, there is a central idempotent e which generates the principal ideal generated by f, and hence $e \notin T$. Therefore there is an element $u \in B(R)$ such that $e(u) \notin M$. Let $g=i(e(u))$. Then $g(v)=e(u)v$ for all $v \in B(R)$ and $g \notin iM$. Since $g \in iR$, $g \notin \hat{M}$ and hence $1-g \in \hat{M}$ where 1 is the unit element of $\hat{R}$. Thus g is an identity modulo iM, whence $e(u)$ is an identity modulo M. Therefore for each $v \in B(R)$, $v-e(u)v \in M$ and $1-g \in Q$. Since both e and $1-g$ belong to the ideal I generated by T and $\{f\}$ and since $ge=g$, it follows that $1 \in I$. Therefore $I=\hat{R}$, and T is indeed maximal.

An ideal I of a GFR R is said to be *σ-closed* provided $\{a_n \in R \mid a_n \geqq 0, n \geqq 1\} \subseteq I$ and $\bigvee_{n=1}^{\infty} a_n$ exists in R imply that $\bigvee_{n=1}^{\infty} a_n \in I$.

COROLLARY 4.2. *If R is a GFR, then M is a σ-closed maximal modular ideal of R if and only if $\hat{M}$ is a σ-closed maximal ideal of $\hat{R}$.*

Proof. Suppose M is σ-closed, $f_n \in \hat{M}$ and $f_n \geqq 0$ for each $n \geqq 1$, and $\bigvee_{n=1}^{\infty} f_n \in \hat{R}$. Then $\bigvee_{n=1}^{\infty} f_n(u)$ exists in R for each $u \in B(R)$ and $[\bigvee_{n=1}^{\infty} f_n](u) = \bigvee_{n=1}^{\infty} f_n(u)$. Thus by Proposition 4.4, if M is σ-closed, then so is $\hat{M}$.

Conversely, from Corollary 4.1, it follows that if $\hat{M}$ is σ-closed, then $iM = \hat{M} \cap iR$ is also σ-closed. Hence so is M because i preserves σ-lattice operations.

LEMMA 4.1. *Let R be a ring with local unit. The ring $A + iR$ generated by A and iR is equal to $\hat{R}$ if A is a maximal ideal of $\hat{R}$ which does not contain iR.*

Proof. The ring generated by A and iR is a subset of the ideal generated by A and iR by definition. However, A and iR are ideals, so every element of the ring generated by A and iR is a sum of elements in $A \cup iR$. Hence this ring is the ideal I generated by A and iR. The ideal $I = \hat{R}$ by the maximality of A.

A maximal modular ideal M of a GFR R is *real* if R/M is isomorphic to the GFR of real numbers.

The following theorem is a direct generalization of a result of [6].

THEOREM 4.1. *If R is a regular GFR and M is a maximal modular ideal, then M is real if and only if it is σ-closed.*

Proof. From Lemma 4.1, Corollary 4.1, and the second homomorphism theorem for rings, it follows that

$$\hat{R}/\hat{M} \cong iR/iM \cong R/M$$

where $\hat{M}$ is the maximal ideal of $\hat{R}$ with the property $\hat{M} \cap iR = iM$. Thus $\hat{M}$ is real if and only if M is real. By [6, Corollary, p. 83] and Corollary 4.2 above, M is real if and only if it is σ-closed.

5. **Generalized F-rings of functions.** In this section we discuss GFR's of functions and their normalizers. In particular, we show that every regular GFR with the property that the intersection of all its real maximal modular ideals is the zero-ideal is isomorphic to the GFR of all real functions defined on a certain space Ω and measurable with respect to a certain σ-clan of subsets of Ω. A *σ-clan* $\mathfrak{A}$ of subsets of a space Ω is a collection with the following properties:

1. $\phi \in \mathfrak{A}$.
2. $A, B \in \mathfrak{A} \Rightarrow A - B \in \mathfrak{A}$ and $A \cup B \in \mathfrak{A}$.
3. $A_n \in \mathfrak{A}$ for $n \geqq 1 \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathfrak{A}$.

It is easy to show that a σ-clan is a conditionally σ-complete Boolean ring with respect to the natural order relation of sets. If $\mathfrak{A}$ is a σ-clan of subsets of Ω, then a function f from Ω into the real field is said to be $(\Omega, \mathfrak{A})$-*measurable* if the set $E_f(\lambda) = \{\omega \in \Omega \mid f(\omega) \leqq \lambda \text{ and } f(\omega) \neq 0\}$ is a member of $\mathfrak{A}$ for $-\infty \leqq \lambda \leqq \infty$. Let $\mathfrak{M}(\Omega, \mathfrak{A})$ stand for the class of $(\Omega, \mathfrak{A})$-measurable functions.

Note that the concept of σ-clan is an essential generalization of the concept of σ-ring. A necessary and sufficient condition on $\mathfrak{M}(\Omega, \mathfrak{A})$ is given in order that $\mathfrak{A}$ be a σ-ring. In addition we show that if $\mathfrak{A}$ is a σ-clan, then in essence $\hat{\mathfrak{A}}$ is a σ-algebra of sets.

Let R be a ring with local unit and let i be the canonical injection of R into $\hat{R}$. In §4 it is shown that for each maximal modular ideal M of R there is a unique maximal ideal $\hat{M}$ of $\hat{R}$ such that $\hat{M} \cap iR = iM$.

LEMMA 5.1. *Let $\hat{R}$ be biregular. If $\{M_\alpha | \alpha \in A\}$ is a class of maximal modular ideals of R, then*

$$\bigcap_{\alpha\in A} \hat{M}_\alpha = \{0\} \text{ if and only if } \bigcap_{\alpha\in A} M_\alpha = \{0\}.$$

Proof. Since $(\bigcap_{\alpha\in A} \hat{M}_\alpha) \cap iR = \bigcap_{\alpha\in A} (\hat{M}_\alpha \cap iR)$, the "if" part follows because $\hat{M}_\alpha \cap iR = \hat{M}_\alpha$.

Conversely, suppose $\bigcap_{\alpha\in A} M_\alpha = \{0\}$. Then $\bigcap_{\alpha\in A} \hat{M}_\alpha$ meets R in $\{0\}$. Suppose $f \neq 0$ belongs to $\bigcap_{\alpha\in A} \hat{M}_\alpha$. Then there is an idempotent $e \neq 0$ in the center of $\hat{R}$ such that $e \in \bigcap_{\alpha\in A} \hat{M}_\alpha$ because $\hat{R}$ is biregular. By the definition of $\hat{R}$, there is a central idempotent $e_0 \in iR$ such that $e_0 e = e_0$ and $e_0 \neq 0$. Therefore $e_0 \in (\bigcap_{\alpha\in A} \hat{M}_\alpha) \cap iR$. Thus the assumption that $\bigcap_{\alpha\in A} \hat{M}_\alpha \neq \{0\}$ leads to a contradiction.

A GFR with the property that the intersection of its σ-closed maximal modular ideals is the zero-ideal is called a *generalized M-ring* or GMR.

THEOREM 5.1. *A generalized M-ring R is isomorphic to a generalized M-ring $R(\Omega, \mathfrak{A})$ of $(\Omega, \mathfrak{A})$-measurable function where $\mathfrak{A}$ is a σ-clan of subsets of a space Ω.*

Proof. If R is a GMR, then $\hat{R}$ is an M-ring (see [4, p. 674]) by Lemma 5.1. By [4, Theorem 7], $\hat{R}$ is isomorphic to an F-ring $A(\Omega, \mathfrak{B})$ of $(\Omega, \mathfrak{B})$-measurable functions where Ω is the set of all σ-closed maximal ideals of $\hat{R}$ and $e \rightarrow \mathfrak{U}(e) = \{\hat{M} \in \Omega | e(\hat{M}) = 1\}$ is an isomorphism of the idempotent algebra $B(\hat{R})$ of $\hat{R}$ onto $\mathfrak{B}$. The symbol $e(\hat{M})$ stands for the image of e under the natural homomorphism associated with $\hat{M} \in \Omega$.

In $\hat{R}$, an element $f \in iR$ if and only if $\bar{e}_f \in iR$. Thus if ϕ stands for the isomorphism of $\hat{R}$ onto $A(\Omega, \mathfrak{B})$, then $\phi(f) \in \phi iR$ if and only if $\phi(\bar{e}_f) = \chi_{\{\omega\in\Omega | f(\omega)\neq 0\}}$ belongs to ϕiR. Let $\mathfrak{A}$ be the collection of all support sets of idempotents in ϕiR. Since R is a GFR, $\mathfrak{A}$ is a σ-clan. In addition, each function $\phi(f) \in \phi iR$ is $(\Omega, \mathfrak{A})$-measurable because $\mathfrak{A}$ is an ideal of $\mathfrak{B}$ and $\{\omega \in \Omega | f(\omega) \leqq \lambda, f(\omega) \neq 0\} \leqq \{\omega \in \Omega | f(\omega) \neq 0\}$. ϕiR is then the GMR, $R(\Omega, \mathfrak{A})$, mentioned in the statement of the theorem.

REMARK 5.1. Since in the above proof $R(\Omega, \mathfrak{A})$ is an ideal of $A(\Omega, \mathfrak{B})$, it follows that $R(\Omega, \mathfrak{A})$ is an order-convex subset of $A(\Omega, \mathfrak{B})$: that is, $f \leqq g \leqq h$ where $f, h \in R(\Omega, \mathfrak{A})$ and $g \in A(\Omega, \mathfrak{B})$ imply $g \in R(\Omega, \mathfrak{A})$.

COROLLARY 5.1. *If R is regular, then $R(\Omega, \mathfrak{A})$ is the generalized M-ring of all $\mathfrak{A}$-measurable functions on Ω.*

Proof. From Proposition 1.3 and [4, Corollary, p. 682], it follows that $A(\Omega, \mathfrak{B})$ is the M-ring of all $(\Omega, \mathfrak{B})$-measurable functions. $R(\Omega, \mathfrak{A})$ is a regular ideal of $A(\Omega, \mathfrak{B})$, and hence $B(R(\Omega, \mathfrak{A}))$ is an ideal of $B(A(\Omega, \mathfrak{B}))$. Therefore since $A \rightarrow \chi_A$ is an isomorphism between $\mathfrak{A}$ and $B(R(\Omega, \mathfrak{A}))$ and since $\mathfrak{M}(\Omega, \mathfrak{A})$ (the set of all (Ω) $\mathfrak{A})$-measurable functions) is a regular GFR, it follows from Proposition 3.1 that $R(\Omega, \mathfrak{A}) = \mathfrak{M}(\Omega, \mathfrak{A})$.

A σ-clan $\mathfrak{A}$ of subsets of Ω is said to be *full with respect to* Ω if every $\omega \in \Omega$ is contained in at least one member of $\mathfrak{A}$. If $\mathfrak{A}$ is not full with respect to Ω then let Ω^* be the set union of all elements of $\mathfrak{A}$. It is clear that $\mathfrak{A}$ is a σ-clan of subsets of Ω^*; and if f is an $(\Omega, \mathfrak{A})$-measurable function, then f vanishes outside Ω^*, and the restriction f^* of f to Ω^* is an $(\Omega^*, \mathfrak{A})$-measurable function. Thus in Theorem 5.1 and Corollary 5.1, it may be assumed that $\mathfrak{A}$ is full with respect to Ω.

In the remainder of the section we study full σ-clans and classes of functions measurable with respect to these σ-clans. The following propositions could be stated in terms of general σ-clans, but it is clear from the remarks of the previous paragraph that no greater generality is achieved by so doing.

PROPOSITION 5.1. *Let $\mathfrak{A}$ be a full σ-clan of subsets of Ω. Then there exists a σ-algebra $\mathfrak{B}$ of subsets of Ω with the property that for every $f \in \hat{\mathfrak{A}}$ there is a set $A \in \mathfrak{B}$ such that $f(U) = A \cap U$ for each $U \in \mathfrak{A}$, and the mapping $\phi: f \rightarrow A$ is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ such that if i is the canonical injection of $\mathfrak{A}$ into $\hat{\mathfrak{A}}$, then the composition ϕi is the identity map on $\mathfrak{A}$.*

Proof. Consider for each $f \in \hat{\mathfrak{A}}$, the set $E_f = \bigcup_{U \in \mathfrak{A}} f(U)$. Then $f(U) = E_f \cap U$. Indeed,

$$E_f \cap U = \bigcup_{V \in \mathfrak{A}} f(V) \cap U.$$

If $V \supseteq U$, then $f(V) \cap U = f(U)$, and if $V \not\supseteq U$, then $f(V) \cap U \subseteq f(V \cup U) \cap U = f(U)$. Therefore $E_f \cap U = f(U)$.

It is clear that the mapping $f \rightarrow E_f$ preserves order and is onto the class $\mathfrak{B} = \{E_f | f \in \hat{\mathfrak{A}}\}$. It is biunique because $E_f = E_g$ implies that for each $U \in \mathfrak{A}$, $f(U) = E_f \cap U = E_g \cap U = g(U)$.

Since $\hat{\mathfrak{A}}$ is a σ-complete Boolean algebra, it follows that $\mathfrak{B}$ is also. It remains only to show that $\mathfrak{B}$ is σ-complete with respect to the set theoretic operations. If $E_{f_n} \in \mathfrak{B}$ for each $n \geq 1$, then $E_{\vee f_n} = \bigcup_{U \in \mathfrak{A}} (\bigvee_{n=1}^{\infty} f_n)(U)$. Since $1 - \bigwedge_{n=1}^{\infty} (1 - f_n) = \bigvee_{n=1}^{\infty} f_n$ in $\hat{\mathfrak{A}}$, it follows that $(\bigvee_{n=1}^{\infty} f_n)(U) = U - \bigcap_{n=1}^{\infty} [U - f_n(U)]$. Therefore

$$E_{\vee_{n=1}^{\infty} f_n} = \bigcup_{n=1}^{\infty} E_{f_n},$$

and $\mathfrak{B}$ is a σ-algebra. It is clear that the composition ϕi is the identity on $\mathfrak{A}$.

Let $\mathfrak{A}$, $\mathfrak{B}$, and Ω be defined as in Proposition 5.1 and let $\mathfrak{M}(\Omega, \mathfrak{A})$ stand for the class of all $(\Omega, \mathfrak{A})$-measurable functions. From Corollary 5.1, it follows that $\mathfrak{M}(\Omega, \mathfrak{A})$ is a regular GMR.

PROPOSITION 5.2. *$\mathfrak{A}$ is a σ-ring if and only if $\mathfrak{M}(\Omega, \mathfrak{A})$ has the following property*: (*) *If $f_n \in \mathfrak{M}(\Omega, \mathfrak{A})$ and $f_m \wedge f_n = 0$ for all m and $n \geqq 1$, $m \neq n$, then $\bigvee_{n=1}^{\infty} f_n \in \mathfrak{M}(\Omega, \mathfrak{A})$.*

Proof. If $\mathfrak{M} = \mathfrak{M}(\Omega, \mathfrak{A})$ satisfies the property (*), then since $A \leftrightarrow X_A$ is an isomorphism between $\mathfrak{A}$ and $B(\mathfrak{M})$, it follows that $\mathfrak{A}$ is a σ-ring because disjoint countable unions (and hence arbitrary countable unions) exist in $B(\mathfrak{M})$.

Conversely, from Proposition 3.2 it follows that $f \in \mathfrak{M}(\Omega, \mathfrak{B})$ belongs to $\mathfrak{M}$ if and only if $\{\omega \in \Omega \mid f(\omega) \neq 0\} \in \mathfrak{A}$. If $\mathfrak{A}$ is a σ-ring and $f_n \in \mathfrak{M}$, $f_n \geqq 0$, and $f_m \wedge f_n = 0$ for $m \neq n$, then since $\bar{e}_{\bigvee_{n=1}^{\infty} f_n} = \bigvee_{n=1}^{\infty} \bar{e}_{f_n}$ and since by [7] the regular M-ring $\mathfrak{M}(\Omega, \mathfrak{B})$ has property (*), it follows that $\bigvee_{n=1}^{\infty} f_n$ exists and belongs to $\mathfrak{M}(\Omega, \mathfrak{A})$.

6. **Examples.** In §1 we established that for a ring R with local unit, the ring $\hat{R}$ constructed by the inverse limit process is indeed the normalizer of R. We are therefore able to construct normalizers for rings with local unit.

EXAMPLE 1. Let L be the Boolean ring of Lebesgue measurable subsets of the real line of finite diameter. It follows from Proposition 5.1 that there is a σ-algebra L^* of subsets of the real line which contains L as an ideal and $L^* \cong \hat{L}$. If i is the canonical injection of L into $\hat{L}$ and ϕ is the isomorphism of $\hat{L}$ onto L^* which was defined in Proposition 5.1, then the restriction of ϕi to L is the identity mapping. L is an ideal of the Boolean ring of all subsets of the real line if and only if there exist no nonmeasurable sets. Therefore, if we assume the axiom of choice, then L^* must be a proper σ-subalgebra of the algebra of all subsets of the real line. By Proposition 5.1, the elements of L^* are of the form $E_f = \bigcup_{U \in L} f(U)$ for $f \in \hat{L}$. *For each $f \in \hat{L}$, E_f is Lebesgue measurable.* Indeed, L contains a cofinal increasing subsequence, namely $\{[-N, N]\}_{N=1}^{\infty}$. Therefore for each U there is an N such that $f(U) \subseteq f([-N, N])$, and hence

$$E_f = \bigcup_{N=1}^{\infty} f([-N, N]).$$

Thus E_f is Lebesgue measurable. However, L is an ideal of $\mathfrak{L}$, the σ-algebra of all Lebesgue measurable subsets of the real line; hence $L^* = \mathfrak{L}$.

EXAMPLE 2. In contrast with Example 1, consider $\mathfrak{A}$, an ideal of $\mathfrak{L}$ composed entirely of sets of measure zero, which as a σ-clan is full with respect to the real line. If $A \in \mathfrak{A}$, then any subset of A belongs to $\mathfrak{A}$, and hence $\mathfrak{A}$ is an ideal of $2^{(-\infty,\infty)}$, the σ-algebra of all subsets of the real line. From Proposition 5.1 it follows that $2^{(-\infty,\infty)}$ is the normalizer of $\mathfrak{A}$.

In particular, if $\mathfrak{A}$ is the σ-clan of null sets of finite diameter, then $\mathfrak{A}$ is an ideal of L, while $\hat{L}$ can be embedded as a proper subring of $\hat{\mathfrak{A}}$.

EXAMPLE 3. Let $\mathfrak{B}$ be the σ-algebra of Borel subsets of the real line, and let $\mathfrak{B}_0$ be the σ-clan of Borel sets of Lebesgue measure zero. If $\mathfrak{B}$ does not coincide with $\mathfrak{L}$, then $\mathfrak{B}_0^*$, the isomorph of the normalizer of $\mathfrak{B}_0$ guaranteed by Proposition 5.1, is properly contained in $2^{(-\infty,\infty)}$. In addition $\mathfrak{B}_0^*$ contains $\mathfrak{B}$ as a σ-subalgebra which is possibly not proper.

It is also possible to show that $\mathfrak{B}_0^* \cap \mathfrak{L} = \mathfrak{B}$. Indeed, if $A \in \mathfrak{B}_0^* \cap \mathfrak{L}$ and $A \notin \mathfrak{B}$, then there is a $B \in \mathfrak{B}$ and a Lebesgue null set $M \notin \mathfrak{B}_0$ such that $A = B \cup M$ and $M \cap B = \phi$. Thus $M \in \mathfrak{B}_0^*$. Since M is Lebesgue measurable, there is an element $M_0 \in \mathfrak{B}_0$ such that $M \subseteq M_0$, and hence $M \in \mathfrak{B}_0$ because $\mathfrak{B}_0$ is an ideal of $\mathfrak{B}_0^*$. This contradiction yields the result that $\mathfrak{B}_0^* \cap \mathfrak{L} = \mathfrak{B}$.

Finally, $\mathfrak{B}_0^* = \mathfrak{B}$ *if and only if every set* A, *for which* $M \in \mathfrak{B}_0$ *implies* $A \cap M \in \mathfrak{B}_0$, *is Lebesgue measurable.* Assume the latter condition holds. Then since $A \in \mathfrak{B}_0^*$ implies that $A \cap U \in \mathfrak{B}_0$ for each $U \in \mathfrak{B}_0$, it follows that $A \in \mathfrak{L}$ which by the previous paragraph implies that $A \in \mathfrak{B}$. Thus $\mathfrak{B}_0^* = \mathfrak{B}$. Conversely, if $\mathfrak{B} = \mathfrak{B}_0^*$ and $A \cap U \in \mathfrak{B}_0$ for each $U \in \mathfrak{B}_0$, then $A \in \mathfrak{B}_0^* = \mathfrak{B} \subseteq \mathfrak{L}$.

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