ABELIAN TORSION GROUPS HAVING A MINIMAL SYSTEM OF GENERATORS(1)

BY

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NOTATION AND TERMINOLOGY. Let T denote an arbitrary Abelian torsion group. Let G denote an arbitrary primary p-group to be considered fixed in each separate proposition. Let the symbol "iff" mean "if and only if," $<$ mean properly contained in, $\subset$ mean contained in, $N\backslash M$ mean the set of elements in N and not in M, $\cong$ mean isomorphic to, $(N_a)_{a\in A}$ denote a family of sets, elements or groups—as the case may be—N_a indexed by members of an index set A; and if for each $a\in A$ N_a is a group let $\oplus_{a\in A} N_a$ denote the direct sum of the N_a's—$\oplus$ denotes a direct sum. If $x\in G$ let $h(x)$ denote the ordinary height of x—see [2].

If S is a subset or subgroup of T, let $|S|$ denote the power—cardinal number—of S. Let (S) mean the same thing as S, and $\{S\}$ mean the subgroup generated by the elements of S. If $x_1, x_2, x_3, \cdots \in T$, let $(x_1, x_2, x_3, \cdots)$ denote the set whose only elements are $x_1, x_2, x_3, \cdots$, and let $\{x_1, x_2, x_3, \cdots\} = \{(x_1, x_2, x_3, \cdots)\}$.

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I. On primary groups.

REMARK. Since most of the following propositions are trivially true for finite groups, we shall often assume while proving them that G is infinite.

LEMMA 1. *Let G have a (m.s.g.) S, and let $x\in S$. Then $h(x)=0$.*

Proof. Assume for some $x\in S$ $\exists y\in G$ such that $py=x$. Then $\exists$ $x_1, x_2, \cdots, x_m\in S$ and integers $n_1, \cdots, n_m$ such that $y=n_1x_1+\cdots+n_mx_m$. By the minimality of S we may assume that $x_1=x$ and $n_1\neq 0$. Then $x=py=n_1px+n_2px_2+\cdots+n_mpx_m$. Thus $(S\backslash(x))\cup(px)$ is a (m.s.g.) of G. Continuing this with $p^{i+1}x$ in place of p^ix we arrive finally at $(S\backslash(x))\cup(p^nx)$ is a (m.s.g.) of G where $p^nx=0$, which is impossible.

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LEMMA 2. *Let G have a (m.s.g.) S. Suppose that $x = n_1x_1 + \cdots + n_mx_m$, where $x_1, \cdots, x_m \in S$ and for some integer i, $1 \leqq i \leqq m$, $(n_i, p) = 1$. Then $(S\backslash(x_i)) \cup (x)$ is a (m.s.g.) of G.*

Proof. We may take $i = 1$. That $(S\backslash(x_1)) \cup (x)$ generates G is trivial. To show it is minimal it suffices to show that no x_i for $1 < i \leqq m$ is a linear combination of elements of $(S\backslash(x_1, x_i)) \cup (x)$. So assume one such x_i, say x_m, is a linear combination of the elements of $(S\backslash(x_1, x_m)) \cup (x)$. Then $\exists$ elements $y_m, \cdots, y_k$ of S different from the x_i's and integers $l_1, \cdots, l_k$, such that $x_m = l_1x + l_2x_2 + \cdots + l_{m-1}x_{m-1} + l_my_m + \cdots + l_ky_k$. Thus $x_m = l_1n_1x_1 + (l_1n_2 + l_2)x_2 + (l_1n_3 + n_3)x_3 + \cdots + (l_1n_{m-1} + l_{m-1})x_{m-1} + l_1n_mx_m + l_my_m + \cdots + l_ky_k$. Since S is minimal, the coefficients of all the x's and y's on the right hand side of the last equation which are not equal to x_m must be multiples of p. In particular, l_1n_1 is a multiple of p. But n_1 was assumed to be relatively prime to p. Hence l_1 is a multiple of p. Hence also the coefficient l_1n_m of x_m is a multiple of p. Thus $h(x_m) > 0$, contrary to Lemma 1.

LEMMA 3. *Let G have a (m.s.g.) S. Then $x \in G$ can be extended to a (m.s.g.) of G iff $h(x) = 0$.*[2]

REMARK. Before stating our next theorem, we remark that all basic subgroups of a primary group are isomorphic; and for nonprimary groups while I of the following theorem is implied by VI, I does not in general imply VI, as is the case in a group which is the direct sum of an infinite number of cyclic groups having different primes for orders.

THEOREM 1. *Let G be a primary p-group. Then the following statements are equivalent:*

(I) *G has a minimal system of generators.*

(II) *Some system of representatives of some basis of G/pG generates G.*

(III) *G is finite or G/pG has the same power as G.*

(IV) *G has the same power as a basic subgroup of itself.*

(V) *G has a direct summand which is a direct sum of cyclic groups and has the same power as G.*

(VI) *G has a minimum system of generators.*

(VII) *G is finite or the automorphism group of G/D has power equal to that of the set of all subsets of G, where D is the divisible part of G.*

(VIII) *G is finite or the automorphism group of G/D has power exceeding that of G, D being as in* (VII).

Proof of (I) **implies** (II). Assume G has a (m.s.g.) S. Then if σ is the natural homomorphism of G onto G/pG, certainly the set $(\sigma(x))_{x\in S}$ generates G/pG, and it remains to show that this set is independent. So assume that for some $x \in S$, $x + pG = (n_1x_1 + pG) + \cdots + (n_mx_m + pG)$, where each $x_i \in S$, and

[2] *Added in proof.* Lemma 3 has an analogue for an arbitrary Abelian group G, in which case Lemma 1 is best proved by observing that $G/\{S\backslash(x)\}$ is cyclic.

$x_i \neq x$. Then $\exists g_1, \cdots, g_m \in G$ such that $x = n_1x_1 + \cdots + n_mx_m + p(g_1 + \cdots + g_m)$. Then by Lemma 2 the set $(S\backslash(x)) \cup (p(g_1 + \cdots + g_m))$ is a (m.s.g.) of G. This contradicts Lemma 1 since $h(p(g_1 + \cdots + g_m)) \geqq 1$.

Next we prove (II) **implies** (III) **implies** (IV). If G is finite there is nothing to prove. So assume G is infinite; then since some system of representatives of some basis of G/pG generates G, we must have $|G| = |G/pG|$. But by a theorem of Kulikov which holds for an arbitrary primary group G and any basic subgroup B of it we have $G/pG \cong B/pB$. Thus $|G| = |G/pG| = |B/pB|$. Since $|G| \geqq |B| \geqq |B/pB|$, we obtain $|G| = |B|$. This completes the proof of (II) implies (III) implies (IV).

Proof of (IV) **implies** (V). Write $G = L \oplus D$ where D is divisible and L is reduced, and let B be basic in L. Then B is basic in G, so that $|G| = |B| \leqq |L| \leqq |G|$. Thus we may assume that G itself is reduced, since this would be the same as showing L has a direct summand as desired.

We may assume that G is infinite.

Case 1. G is countable.

Any countable reduced p-group is known to have a direct summand which is a direct sum of a countable number of cyclic groups. See [1, p. 143].

Now let $B = \oplus_{i=1}^{\infty} B_i$, where B_i is a direct sum of cyclic groups of order p^i, and B is a basic subgroup of G.

Case 2. $|G| > \aleph_0$, and $\exists$ an integer n such that $|\oplus_{i=1}^{n} B_i| = |G|$.

It is then known that—see [1, p. 98, Baer's Theorem]—$G = (\oplus_{i=1}^{n} B_i) \oplus \{\oplus_{i=n+1}^{\infty} B_i, p^nG\}$, and in this case we may take our direct summand to be $\oplus_{i=1}^{n} B_i$.

Case 3. $|G| > \aleph_0$, G has no elements of infinite height, and for no integer n is $|\oplus_{i=1}^{n} B_i| = |G|$.

Then there is no greatest $|B_i|$, and in this case $|B| = \sum_{i=1}^{\infty} |B_i|$, and $|B|$ is a "limit" cardinal. Let i_1 be the first integer such that $|B_{i_1}|$ is infinite, and assume i_n has been defined. Let i_{n+1} be the first integer larger than i_n such that $|B_{i_{n+1}}| > |B_{i_n}|$. Then $\sum_{n=1}^{\infty} |B_{i_n}| = |B| = |G|$.

Now if we denote by $\overline{B}$ the closure of B, i.e., the torsion subgroup of the strong direct sum of the B_i's, then—since G has no elements of infinite height—it is well known that G may be considered as a pure subgroup of $\overline{B}$—see [1, p. 112]. Then G/B is divisible and of power not exceeding that of B. Let S be a system of representatives of the nonzero cosets of G/B. Then each element of S is a sequence $(v_1, v_2, v_3, \cdots)$ with $v_i \in B_i$, and where an infinite number of the v_i's are different from zero. Moreover, any element differing from $(v_1, v_2, v_3, \cdots)$ in only a finite number of places belongs to G and represents the same coset of G/B as $(v_1, v_2, v_3, \cdots)$.

Now S may be divided into a sequence of disjoint subsets $S_1, S_2, S_3, \cdots$ such that $|S_j| \leqq |B_{i_j}|$, for $j = 1, 2, 3, \cdots$.

By our last remark, we may assume in addition that each element of S_j has its first i_j components equal to zero. Now since each $|B_{i_j}|$ is infinite, every B_{i_j} is a direct sum of cyclic groups $\oplus_{a\in A_{i_j}}\{v_a\}$ for some index set A_{i_j} where $|A_{i_j}|=|B_{i_j}|$.

Now if $x\in S$ has some nonzero component from B_{i_j}, then $x\in S_l$ where $l<j$; moreover this component is of the form $k_1v_{a_1}+k_2v_{a_2}+\cdots+k_mv_{a_m}$ for some integers $k_1,\cdots,k_m$ and $a_t\in A_{i_j}$, for $t=1,\cdots,m$. Thus the set C_{i_j} of all $a\in A_{i_j}$ such that v_a occurs in an expression of a component of some element of S is of cardinal $\leqq\aleph_0(\sum_{l<j}|B_{i_l}|)=\aleph_0|B_{i_{j-1}}|=|B_{i_{j-1}}|<|B_{i_j}|$.

Now $B_{i_j}=(\oplus_{a\in C_{i_j}}\{v_a\})\oplus(\oplus_{a\in(A_{i_j}\setminus C_{i_j})}\{v_a\})$ where $|A_{i_j}\setminus C_{i_j}|=|B_{i_j}|$. Also G is generated by S and $\oplus_{i=1}^{\infty}B_i$. Moreover it is clear from the above that $G=\oplus_{j=1}^{\infty}(\oplus_{a\in(A_{i_j}\setminus C_{i_j})}\{v_a\})\oplus\{S,\ \oplus_{j=1}^{\infty}(\oplus_{a\in C_{i_j}}\{v_a\}),$ all B_i with $i\neq i_j$ for any $j\}$.

Thus $\oplus_{j=1}^{\infty}(\oplus_{a\in(A_{i_j}\setminus C_{i_j})}\{v_a\})$ is a direct summand as required.

Case 4. $|G|>\aleph_0$, G has elements of infinite height, and for no integer n is $|\oplus_{i=1}^{n}B_i|=|G|$.

Let $\overline{G}$ be the subgroup of G consisting of all the elements of G of infinite height. Then by a theorem of Kulikov, [1, p. 103], the image $\overline{B}$ of B under the natural homomorphism of G onto $G/\overline{G}$ is a basic subgroup of $G/\overline{G}$ and $\overline{B}\cong B$. Now $\overline{B}$ and $G/\overline{G}$ satisfy the conditions of case 3—see [1, p. 112]. Hence $G/\overline{G}$ may be written as $\overline{H}\oplus\overline{K}$ where $\overline{H}$ is a direct sum of cyclic groups, $|\overline{H}|\geqq|G/\overline{G}|$, and as in the proof of case 3, $\overline{H}$ is contained in $\overline{B}$. Then let H be the subgroup of B generated by representatives from B of the elements of $\overline{H}$. Then we have $(H+\overline{G})/\overline{G}$ is a direct summand of $G/\overline{G}$, $H\cap\overline{G}=0$, and hence—see [2, p. 18]—H is a direct summand of G. Then we have $|H|=|\overline{H}|\geqq|G/\overline{G}|\geqq|\overline{B}|=|B|=|G|$. Hence $|H|=|G|$. Moreover, H being a subgroup of B which is a direct sum of cyclic groups, H is also a direct sum of cyclic groups. This completes the proof of (IV) implies (V).

Proof of (V) **implies** (I). Let $G=H\oplus K$, where $|H|\geqq|K|$ and H is a direct sum of cyclic groups. Then we must show that G has a (m.s.g.). So let $B=\oplus_{a\in A}\{v_a\}$ be a basic subgroup of K. Then it is readily checked that pB is a basic subgroup of pK and $|pK/pB|\leqq|K|\leqq|H|$. Let $H=\oplus_{b\in Q}\{w_b\}$, and $pK/pB=\oplus_{z\in Z}D_z$, where each D_z is an indecomposable divisible group. Then Q may be divided into $|Z|+1$ sets B_z, B_1 such that for each $z\in Z$, $|B_z|=\aleph_0$. Let D_z be generated by $d_1^z, d_2^z, d_3^z,\cdots$ where $pd_1^z=0$, and $pd_{i+1}^z=d_i^z$ Let $e_i^z\in pK$ represent d_i^z. Now if each set $(w_b)_{b\in B_z}$ is arranged in a sequence w_1^z, $w_2^z, w_3^z,\cdots$ where the order of w_i^z is $p^{n_i^z}$, then for each $z\in Z$ form the set

$$E_z=(w_1^z+e_{n_1^z}^z,\ w_2^z+e_{n_1^z+n_2^z}^z,\cdots,w_m^z+e_{n_1^z+\cdots+n_m^z}^z,\cdots).$$

Then we assert that the set

$$T=\left(\bigcup_{a\in A}(v_a)\right)\cup\left(\bigcup_{b\in B_1}(w_b)\right)\cup\left(\bigcup_{z\in Z}E_z\right)$$

is a (m.s.g.) of G. First we observe that the images of the elements of

$$\left(\bigcup_{b\in B_1}(w_b)\right)\cup\left(\bigcup_{z\in Z}E_z\right)$$

under the natural homomorphism of $H+pK$ onto $(H+pK)/pB$ generate $(H+pK)/pB$, and that $\bigcup_{a\in A}(v_a)$ generates B which contains pB. Thus $\{T\}\supset\{H+pK, B\}$. Now $\{pK, B\}=K$—see [1, p. 109]. Thus $\{T\}=G$. To show T is minimal, we first observe that $\{T\backslash(w_b)\}$ does not contain w_b for any $b\in B_1$, and that $\{T\backslash(w_m^z+e^z_{n_1^z+\cdots+n_m^z})\}$ does not contain w_m^z for any $z\in Z$ and any integer m. Moreover we will show that for no $a_0\in A$ does $\{T\backslash(v_{a_0})\}$ contain v_{a_0}.

The last assertion follows from: $\{T\backslash(v_{a_0})\}\subset\{H, pK, (v_a)_{a_0\neq a\in A}\}$ which does not contain v_{a_0}, since the images of the v_a's are independent in $G/(H+pK)$, as mentioned above. This completes the proof of (V) implies (I).

Proof of (I) **is equivalent to** (VI). We will prove the stronger statement, namely, S is a (m.s.g.) of G iff S is a (M.s.g.) of G. It follows at once from the definitions that if S is a (M.s.g.) of G, then S is a (m.s.g.) of G.

Next assume that S is a (m.s.g.) of G, and that for some $n\geqq 1$ the n elements $x_1, \cdots, x_n$ of S may be replaced by the $n-1$ elements $y_1, \cdots, y_{n-1}$ of G in such a way that the resulting set Q still generates G. Hence for each $x_i\,\exists x_1^i, \cdots, x_{m_i}^i\in S\backslash(x_1, \cdots, x_n)$ such that $x_i\in\{x_1^i, \cdots, x_{m_i}^i, y_1, \cdots, y_{n-1}\}$.

Consider the subset

$$T=(x_1,\cdots,x_n)\cup\left(\bigcup_{i=1}^{n}(x_1^i,\cdots,x_{m_i}^i)\right)$$

of S. T is finite and consists, say, of k elements. Hence $Q=(T\backslash(x_1, \cdots, x_n))\cup(y_1, \cdots, y_{n-1})$ has $k-1$ elements. If σ is the natural homomorphism of G onto G/pG, this implies that $\sigma(T)$ which by the proof of (I) implies (II) forms part of a basis of G/pG consisting of k elements is dependent in G/pG on the $k-1$ elements $\sigma(Q)$ which is impossible. This contradiction proves our assertion.

Finally, we prove that (VII) **is equivalent to** (V).

Assume that G is infinite, and that it has a direct summand as specified in (V). Then it is easy to verify that the power of the automorphism group of G is equal to $2^{|G|}$. Next assume (V) is not satisfied. If $|G|>|G/D|$ the conclusion is obvious. If not, we may assume G is reduced, and by the equivalence of (IV) and (V), the power of G would exceed that of any of its basic subgroups. Then an unpublished theorem of E. Walker, delivered to the author by oral communication, see [4, p. 867], asserts that the power of the automorphism group of G is $|G|$. This also proves that (V) is equivalent to (VIII).

With this the proof of Theorem 1 is complete.

REMARK. From the proof of the equivalence of (I) and (IV) of Theorem 1, Lemma 3 can be stated as follows:

LEMMA 3a. *Let G have a (M.s.g.). Then $x\in G$ can be extended to a (M.s.g.) of G iff $h(x)=0$.*

Definition. Let G be called a *starred* group if it satisfies any one of the eight equivalent conditions of Theorem 1. The following corollaries are obtained from Theorem 1 using well-known properties of abelian groups.

Corollary 1. *A primary divisible group G is not a starred group.*

Corollary 2. *Let B be basic in G, and $H \supset B$ be a subgroup of G. Then if $|H| = |B|$, H is starred. Further, if H is pure in G, then H is starred only if $|H| = |B|$. In particular, if G itself is starred then any subgroup of G containing B is starred.*

Corollary 3. *There exists a primary group without elements of infinite height which is not starred.*

Corollary 4. *Any countable reduced primary group is starred.*

Corollary 5. *Let G be a reduced primary group and B be basic in G. Assume $|B| = |B|^{\aleph_0}$. Then G is starred.*

Corollary 6. *Let G be reduced, and such that $n^{\aleph_0} < |G|$ for any cardinal n less than $|G|$. Then G is starred.*

Corollary 7. *Let G be infinite. Then G is starred iff the reduced part of G is the direct sum of $|G|$ nontrivial subgroups of G.*

L. Fuchs asks the following question—see [1, Problem 18, p. 144]:

"Which are the cardinals m such that there exist no m-indecomposable reduced p-groups (i.e., primary groups) of power m?"

We have

Theorem 2a. *There exists no infinite m-indecomposable reduced primary group of power m if and only if either $m = \aleph_0$ or for every cardinal $n < m$ we have $n^{\aleph_0} < m$.*

Since by Corollary 7 for reduced groups of infinite cardinal m, m-decomposability is equivalent to the property of being starred, we may prove instead the following version of Theorem 2a:

Theorem 2. *There exists an unstarred reduced group G of infinite cardinal m if and only if there exists an infinite cardinal $n < m$ such that $n^{\aleph_0} \geqq m$.*

Proof. If $m = \aleph_0$, Corollary 4 implies that G is starred. If $m > \aleph_0$ and for no infinite cardinal $n < m$ is $n^{\aleph_0} \geqq m$, then Corollary 6 implies that G is starred.

Next assume m is a cardinal for which there exists an infinite cardinal $n < m$ such that $n^{\aleph_0} \geqq m$. Then $m > \aleph_0$. Let B_i be the direct sum of n cyclic groups of order p^i. Let $\overline{B}$ be the closure of $B = \oplus_{i=1}^{\infty} B_i$. Then $|B| = n\aleph_0$, $\overline{B}$ is reduced (and even has no elements of infinite height) and $|\overline{B}| = n^{\aleph_0} \geqq m > n\aleph_0$. Also $\overline{B}/B$ is divisible and of cardinal $\geqq m$ since $|B| \leqq n\aleph_0$. Then there exists a pure subgroup B^*/B of $\overline{B}/B$ (where $B^* \supset B$) of cardinal m. Then $|B^*| = m$, and B is basic in B^*, which by IV of Theorem 1 means B^* is not starred.

This concludes the proof of Theorem 2.

Problem 14 in L. Fuchs' book *Abelian groups* reads as follows: "Under what conditions can every element of a p-group be embedded in a direct summand of power $\leqq m$, m an infinite cardinal?"

In this connection we have

THEOREM 3. *Let G be an infinite starred group, and let $x\in G$ be contained in some pure subgroup of G having no elements of infinite height. Then x can be embedded in a direct summand H of G of power m for any m satisfying $m\leqq |G|$.*

Proof. It is known that such an x can be embedded in a finite direct summand K of G. Then let $G=K\oplus L$, and let B be a basic subgroup of L. Then $|B|=|L|$, and by IV of Theorem 1 L is starred. By V of Theorem 1, $L=L_1\oplus L_2$ where $|L_1|=|L|=|G|$ and L_1 is a direct sum of cyclic groups. If $|G|\geqq m$, then clearly L_1 has a direct summand L_3 satisfying $|L_3|=m$. We may then set $H=K\oplus L_3$.

REMARK. If any $x\in G$ which is contained in some pure subgroup of G having no elements of infinite height is also contained in some direct summand of G having power m for any $m\leqq |G|$, then it does not follow that G is starred.

To see this we let G be the direct sum of a countable number of groups each isomorphic to $\overline{B}$ of the proof of Theorem 2 with $n=1$.

The following theorem is included for the sake of completeness:

THEOREM 4. *Let G be a primary p-group. Then G is a direct sum of cyclic groups of bounded order if and only if there is some basis of G/pG for which any system of representatives by elements of G generates G. If every system of representatives of some basis of G/pG generates G, then every system of representatives of every basis of G/pG generates G.*

Proof. Assume first that $G=\oplus_{i=1}^{n} B_i$ where each B_i is a direct sum of cyclic groups $\{v_{a_i}\}_{a_i\in A_i}$ of order p^i, and n is an integer. For any primary p-group H, let $S(H)$ denote the subgroup of H consisting of those elements of H whose orders are p or 1. For any subgroup K of G let $K^i=p^iK$. Then $0=G^n\subset G^{n-1}\subset\cdots\subset G^0=G$, and $G^i=B^i_{i+1}\oplus\cdots\oplus B^i_n$. Also let σ_i be the natural homomorphism of G onto G/G^i. Then $(\sigma_1(v_{a_i})$, all $a_i\in A_i$ and $i=1,\cdots,n)$ is a basis of G/pG, and any system of representatives of it is of the form $L=(v_{a_i}+p(g_{a_i})$, all $a_i\in A_i$ and $i=1,\cdots,n$, and where g_{a_i} is some element of $G)$. We will show that $\{L\}\supset S(G)=G^{n-1}=B_n^{n-1}$. Let $x\in B_n^{n-1}$; then $\exists v_{a_1n},\cdots,v_{a_mn}$ and integers $l_1,\cdots,l_m$ such that $x=l_1v_{a_1n}+\cdots+l_mv_{a_mn}$ where p^{n-1} divides each l_i. Then also

$$x = l_1(v_{a_1n}+p(g_{a_1n}))+\cdots+l_m(v_{a_mn}+p(g_{a_mn}))$$

since $p^ny=0$ for any $y\in G$. Thus $x\in\{L\}$. Thus $\{L\}\supset B_n^{n-1}=G^{n-1}$. Similarly $\{\sigma_{n-1}(L)\}\supset S(G/G^{n-1})$. Thus $\{L\}\supset G^{n-2}$. In the same way, having shown that $\{L\} \supset G^m$, it follows that $\{\sigma_m(L)\} \supset S(G/G^m)$, and hence that $\{L\} \supset G^{m-1}$. After a finite number of steps we arrive at $\{L\} \supset G^0 = G$. Thus if G is a bounded direct sum of cyclic groups, then there is a basis of G/pG such that any system of representatives of this basis generates G.

Next we assume that there is a basis $(\bar{v}_a)_{a\in A}$ of G/pG for which any system of representatives generates G, and we prove that if $(\bar{w}_b)_{b\in B}$ is any other basis of G/pG then any system of representatives $(w_b)_{b\in B}$ of it generates G. If $\bar{v}_a = l_n^a \bar{w}_{b_1} + \cdots + l_n^a \bar{w}_{b_n}$, then $l_1^a w_{b_1} + \cdots + l_n^a w_{b_n}$ is certainly a representative of $\bar{v}_a$. Then the set $(l_1^a w_{b_1} + \cdots + l_n^a w_{b_n})_{a\in A}$ generates G. Hence $(w_b)_{b\in B}$ also generates G.

Now we prove that if there is some basis of G/pG for which any system of representatives generates G, then G is a bounded direct sum of cyclic groups. By what we have just proved, any system of representatives of any basis of G/pG generates G. Let $B = \oplus_{a\in A}\{v_a\}$ be a basic subgroup of G. Then it is well known that if σ is the natural homomorphism of G onto G/pG, $(\sigma(v_a))_{a\in A}$ is a basis of G/pG. If we let v_a represent $\sigma(v_a)$ we obtain that $G = B$, or that G is a direct sum of cyclic groups. We must still show that G is bounded. If G were not bounded, there would exist $a_1, a_2, a_3, \cdots \in A$ such that order $v_{a_i} = p^{n_i}$, where $n_{i+1} > n_i$ for $i = 1, 2, 3, \cdots$. Then $(v_a, a\in A, a \neq a_i$ for any $i)$ $\cup (v_{a_i} + p^{n_{i+1}-n_i} v_{a_{i+1}}, i = 1, 2, 3, \cdots)$ certainly represents a basis of G/pG, and as is well known—this is also easy to see—does not generate G. The proof of Theorem 4 is now complete.

II. **On torsion groups.**

NOTATION. Let P denote the set of primes.

DEFINITION. If T is an Abelian torsion group and $p \in P$, let T_p be the p-component of T. A subgroup B of T is said to be a *basic subgroup* of T if and only if B_p is a basic subgroup of T_p for every $p \in P$.

THEOREM 5. *Let T be an Abelian torsion group. Then the following statements are equivalent.*

(I) *T has a minimal system of generators.*

(II) *T has the same power as a basic subgroup of itself.*

(III) *T has a direct summand which is a direct sum of cyclic groups and has the same power as T.*

Proof. First we prove that I implies II. So let $B = \oplus_{a\in A}\{v_a\}$ be a basic subgroup of T, and recall that all basic subgroups of a group are isomorphic. Let S be a (m.s.g.) of T. With each v_a associate a finite subset S_a of S such that $v_a \in \{S_a\}$. If we now assume that $|B| < |T|$, we obtain that $B \subset \{\cup_{a\in A} S_a\} < T$. Then if σ is the natural homomorphism of T onto $T/\{\cup_{a\in A} S_a\}$, $\sigma(S \backslash \cup_{a\in A} S_a)$ is a (m.s.g.) of $T/\{\cup_{a\in A} S_a\}$ which is divisible since it is a homomorphic image of T/B, and T/B is divisible since each T_p/B_p is. Thus it remains to show that no (non-zero) divisible torsion group D has a (m.s.g.) Q. Let $x \in Q$; then $D > \{Q\backslash(x)\}$, and $\sigma(x)$ is a (m.s.g.) of $D/\{Q\backslash(x)\}$ where σ

is the natural homomorphism of D onto $D/\{Q\backslash(x)\}$. But $D/\{Q\backslash(x)\}$ cannot be generated by one element since being a homomorphic image of D it is itself divisible. Thus I implies II.

Trivially III implies II.

Next we assume that $|T| = |B|$ for some basic subgroup B of T and show that T has a (m.s.g.), showing in the process that II implies III, i.e., we go from II to I via III.

Case 1. For each prime p, B_p is finite. If $B_p \neq 0$, for only a finite number of primes $p_1, \cdots, p_n$, then $T = \oplus_{i=1}^{n} B_{p_i}$, and the result follows. In the other case, $T_p = B_p \oplus D_p$ where D_p is divisible. Then necessarily $\aleph_0 = |\oplus_{p \in P} B_p| \geqq |\oplus_{p \in P} D_p|$. Let $\oplus_{p \in P} B_p = \oplus_{a \in A}\{v_a\}$, where the v_a's have prime power order. Then since for each $p \in P$, $|B_p| < \aleph_0$, we may map the elements of prime power order of $\oplus_{p \in P} D_p$ in a one-one fashion on a subset $(v_a)_{a \in B \subset A}$ of $(v_a)_{a \in A}$ such that no element of D_p is mapped on an element of B_p. For a justification of this see the proof of case 3. Then the set $(v_a + w_a)_{a \in A}$ where w_a is the element mapped on v_a if there is such a one, and w_a is zero otherwise is easily seen to be a (m.s.g.) of T.

Case 2. For some $p_0 \in P$, there is an infinite B_{p_0} with $|B_{p_0}| = |T|$. In this case $|B_{p_0}| = |T_{p_0}|$, and by V of Theorem 1, $T_{p_0} = T^1 \oplus T^2$ where T^1 is an infinite direct sum of cyclic groups $\oplus_{a \in A}\{v_a\}$, and $|T| = |T^1| \geqq |T^2|$. Then we may write A as the union of two disjoint subsets A_1 and A_2 with $|A_1| = |A_2| = |A| = |T|$. By V of Theorem 1, $T^2 \oplus (\oplus_{a \in A_1}\{v_a\})$ has a (m.s.g.) S. Then we may map the elements of $\oplus_{p_0 \neq p \in P} T_p$ in a one-one fashion on a subset of $(v_a)_{a \in A_2}$. Then the set $S \cup (v_a + w_a)_{a \in A_2}$ where w_a is the element mapped on v_a if there is such a one and w_a is zero otherwise is easily seen to be a (m.s.g.) of T, and even a (M.s.g.) of T.

Case 3. Neither Case 1 nor Case 2 holds. Then there is no greatest $|B_p|$. For each $p \in P$, let $B_p = \oplus_{i=1}^{\infty} B_p^i$, where B_p^i is a direct sum of cyclic groups of order p^i. Then one can verify that there is a subsequence $p_1, p_2, p_3, \cdots$ of the sequence of primes and integers $i_1, i_2, i_3, \cdots, i_j$ depending on P_j such that $\sum_{j=1}^{\infty} |B_{p_j}^{i_j}| = |T|$, and $|B_{p_j}^{i_j}| < |B_{p_{j+1}}^{i_{j+1}}|$ for $j = 1, 2, 3, \cdots$. Since each B_p^i is a direct summand of T_p, we have $T = T^1 \oplus \oplus_{j=1}^{\infty} B_{p_j}^{i_j}$. Let $\oplus_{j=1}^{\infty} B_{p_j}^{i_j} = \oplus_{a \in A}\{v_a\}$, where each v_a belongs to some B_p. Divide A into a sequence of disjoint subsets $A_1, A_2, A_3, \cdots$ such that $|A_k| = |A| = |T|$ for each integer k. Then for each integer k, $|(v_a)_{a \in A_k} \cap T_{p_k}| < |T|$. Thus if $q_1, q_2, q_3, \cdots$ is the sequence of primes, we may map the elements of $T_{q_l}^1$ in a one-one fashion onto a subset of $((v_a)_{a \in A_l} \backslash T_{q_l})$. Then the set $(v_a + w_a)_{a \in A}$ where w_a is the element mapped on v_a if there is such a one and w_a is zero otherwise can be verified to be a (m.s.g.) of T.

This completes the proof of Theorem 5.

THEOREM 6. *Let T be an Abelian torsion group. Then the following statements are equivalent*:

(I) *T has a minimum system of generators.*

(II) *T is finite or some basic subgroup of some primary component of T has the same power as T.*

(III) *T is finite or T has a direct summand of the same power as T which is a direct sum of cyclic groups whose orders are powers of the same prime.*

Proof. The theorem is easily verified for finite groups. Thus we may assume in the following that T is infinite. First we prove that I implies II. So assume that for no $p \in P$ is $|B_p| = T$, and let S be a (M.s.g.) of T. Let Q be the set of primes such that $|T_q| < |T|$ for each $q \in Q$. For each $x \in T$, let $x = x_{p_1} + \cdots + x_{p_n}$, where $x_{p_i} \in T_{p_i}$, the p_i's being different primes, and let $P_x = (p_1, \cdots, p_n)$. Then we assume that $|(x \in S \text{ such that } p \in P_x)| \leqq |T_p|$ for each $p \in Q$. Now let $x \in S$ be written as $x = x_{p_1} + \cdots + x_{p_n}$ as above. Then we may assume that $(P_1, \cdots, P_m) \subset P \backslash Q$, and $(P_{m+1}, \cdots, P_n) \subset Q$ for some integer m. Now since for P_{m+j}, $j = 1, \cdots, n-m$, $|T_{p_{m+j}}| < |T|$, there exist $n-m$ different elements $y_{(m+j)} \in S$ with $y_{p_{m+j}} = 0$, where the index $(m+j)$ in brackets does not mean that $y_{(m+j)} \in T_{m+j}$. Let $\bar{y}_{(m+j)} = y_{(m+j)} + x_{p_{m+j}}$. Moreover, since for $i = 1, \cdots, m$, $|B_{p_i}| < |T|$, and $|T_{p_i}| = |T|$, by Theorem 1 T_{p_i} has no (m.s.g.). But $T_{p_i} \subset \{(x_{p_i})_{x \in S}\}$. Thus there exist m different elements $y_{(p_i)} \in (S \backslash (y_{(m+1)}, \cdots, y_{(n)}, x))$ such that for $i = 1, \cdots, m$, $T_{p_i} \subset \{(x_{p_i})_{y_{(p_i)} \neq x \in S}\}$, where the index in brackets of $y_{(p_i)}$ does not mean that $y_{(p_i)} \in T_{p_i}$. Then if $y_{(p_i)} = y_{p_1} + \cdots + y_{p_i} + \cdots + y_{p_k}$, let $y_{(i)} = y_{p_1} + \cdots + y_{p_{i-1}} + x_{p_i} + y_{p_{i+1}} + \cdots + y_{p_k}$.

Then the set

$$[(y_{(1)}, \cdots, y_{(m)}, \bar{y}_{(m+1)}, \cdots, \bar{y}_{(n)}) \cup (S \backslash (x, y_{(p_1)}, \cdots, y_{(p_m)}, y_{(m+1)}, \cdots, y_{(n)}))]$$

is easily seen to generate G. But this set was obtained from S by replacing $n+1$ elements of S by n elements of G. This contradicts the minimum property of S. Thus I implies II. The proof of II implies III follows easily from Theorem 1. The proof of III implies I is contained in Case 2 of the proof of Theorem 5—see the last sentence of the proof of that case.

The following theorem is partly included in results obtained by W. R. Scott, [5, p. 19–22], and is included here as an illustration of the applicability of Theorems 5 and 6.

THEOREM 7. *Let T be an Abelian torsion group having a minimal system of generators and having one infinite primary component. Then T has $2^{|T|}$ different subgroups all isomorphic to T; for every cardinal $n \leqq |T|$, T has 2^n different isomorphic subgroups of cardinal n; T has $2^{|T|}$ different automorphisms mapping some fixed basic subgroup of T onto itself, and T has $2^{|T|}$ different direct summands.*

Proof. By III of Theorem 5, $T = H \oplus K$ where $|H| = |T|$, and H is a direct sum of cyclic groups.

To prove the first part, it is easy to establish, because of the particularly simple structure of H, that for any $n \leqq |T|$, H has 2^n different isomorphic subgroups $(H_a)_{a\in A}$ of power n, all isomorphic to H if $n=|T|$. If $n=|T|$, then $(H_a+K)_{a\in A}$ are the desired $2^{|T|}$ different subgroups isomorphic to T.

To prove the second part, let B be a basic subgroup of K. Again because of the simple structure of H, it is easy to establish that H has $2^{|T|}$ different automorphisms $(\sigma_a)_{a\in A}$. Let $\bar{\sigma}_a$ be the automorphism of T which coincides with σ_a on H and maps K identically on itself. Then $(\bar{\sigma}_a)_{a\in A}$ is a set of $2^{|T|}$ automorphisms of T all of which map $H\oplus B$ which is a basic subgroup of T onto itself. The proof of the last statement is obvious.

THEOREM 8. *Let G be an infinite reduced primary group, and B be a basic subgroup of G. Then the power of the automorphism group of G is $2^{|B|}$.*

Proof. E. Walker proved this result in the case $|B|<|G|$; see [4, p. 867]. If $|B|=|G|$, then Theorem 7 applies.

THEOREM 9. *Let T be an Abelian torsion group. Then T can be embedded as a direct summand in an Abelian torsion group H with a minimum system of generators (which is a fortiori a minimal system of generators) satisfying $|H| =|T|$.*

Proof. If T is finite, set $H=T$. If T is infinite, then let K be a direct sum of $|T|$ finite cyclic groups whose orders are powers of the same prime. Set $H=T\oplus K$. Theorem 6 completes the proof.

THEOREM 10. *Let T be an Abelian torsion group. Then T can be embedded as a fully invariant subgroup in a group H having a minimum system of generators (which is a fortiori a minimal system of generators) and having the same cardinal as T; more precisely, if T is infinite, then T can be embedded as a fully invariant subgroup in a group H having a minimum system of generators such that $T=nH$ for any prescribed integer $n>1$.*

Proof. If T is finite set $H=T$. If T is infinite we proceed as follows: There exists a free Abelian group $F=\oplus_{a\in A}\{v_a\}$ with $|F|=|T|$ and a subgroup K of F such that $F/K\cong T$. Consider another set $(w_a)_{a\in A}$ of free generators. Identify each v_a with nw_a, and thus F with $n\{(w_a)_{a\in A}\}$. Let $L=\{(w_a)_{a\in A}\}/K$. Then clearly $nL=T$, and $|L|=|T|$. Now let p be a prime dividing n, and let Z be a direct sum of $|T|$ cyclic groups of order p. Set $H=Z\oplus L$. Then $|H|=|T|$, and by Theorem 6 H has a (M.s.g.). Then also $nH=nZ\oplus nL =nL=T$, which implies the full invariance of T in H.

COROLLARY 8. *Let G be a primary group. Then G can be embedded as a direct summand in a starred (primary) group H satisfying $|H|=|T|$.*

COROLLARY 9. *Let G be a p-group. Then G can be embedded as a fully invariant subgroup in a starred group H of the same cardinal, more precisely; if G is* *infinite then G can be embedded as a fully invariant subgroup in a starred group H of the same cardinal such that $G=pH$.*

REMARK. If in Corollary 8, $H=K\oplus G$ where K is a direct sum of cyclic groups of order p, then this embedding coincides with that of Corollary 9 in the sense that $pH=G$ if and only if G is divisible, for $G=pH=pK\oplus pG=pG$, implies that G is divisible.

REMARK. W. R. Scott suggested generalizing the above results to modules over countable rings. It turns out that most of the above theorems and in particular Theorem 1 hold for torsion modules over a principal ideal ring or over a complete discrete valuation ring, after the customary changes in terminology—such as substituting finitely generated for finite, etc.—are made; see [1, p. 29].

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