EXAMPLES OF NON-GAUSSIAN QUASI-INVARIANT DISTRIBUTIONS IN HILBERT SPACE

BY

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1. **Introduction.** Distributions on infinite-dimensional real linear spaces were studied in [5] by I. E. Segal, in connection with quantum statistics. The quasi-invariant (Definition 4) distributions in real Hilbert space can be used to get representations of the commutation relations for dynamical systems with infinitely many variables, as described in [6]. Irreducible representations are obtained if and only if the distribution is ergodic, in the sense of Definition 5, and equivalent representations are obtained if and only if the representations are equivalent in the sense of absolute continuity, Definition 2. Certain known ergodic quasi-invariant distributions are the Gaussian ones. In the present paper are constructed classes of distributions which are *not* equivalent, in the above sense, to Gaussian ones. This can be done quite easily if one does not require that the distribution be ergodic (see Example 1 in §3), and with more difficulty if ergodicity is demanded (Example 2).

The author is indebted to L. LeCam for referring him to the theorem used to get Lemma 6.

2. **Definitions and general remarks.** Let S be a set, $\mathcal{S}$ a Borel field of subsets. For some probability measure σ on $\mathcal{S}$, let $\mathbf{A}$ denote the algebra of $\mathcal{S}$-measurable real-valued functions on S, modulo σ-null functions. After [4], we call such equivalence classes *measuroids*. Observe that there is an identification between the nonnegative countably additive functions on the idempotents of $\mathbf{A}$ which send the identity to 1 and vanish on no nonzero idempotent, and the probability measures on $\mathcal{S}$ which are equivalent to σ. Such measures we then call *suitable to* $\mathbf{A}$. An algebra such as $\mathbf{A}$ we call a *measuroid algebra*. Observe that $\mathbf{A}$ comes equipped with a topology, induced by almost everywhere convergence of functions; and this topology is independent of the particular $(S, \mathcal{S}, \sigma)$ from which $\mathbf{A}$ arose. Indeed, it can be described purely in terms of the ordering in $\mathbf{A}$. For details, see [4].

DEFINITION 1. A *predistribution* is a linear map $\bar{m}$ from a real linear space X to a measuroid algebra $\mathbf{A}$ which *generates* $\mathbf{A}$, in the sense that the polynomials in $\{\bar{m}(x) \mid x \in X\}$ are dense in $\mathbf{A}$. A *distribution* m is a predistribution $\bar{m}$ together with a choice of a suitable measure for $\mathbf{A}$.

Thus a predistribution induces, in general, many distributions.

DEFINITION 2. Two predistributions $\bar{m}: X \to \mathbf{A}$ and $\bar{m}': X \to \mathbf{A}'$ will be called *equivalent*, symbolically $\bar{m} \sim \bar{m}'$, if there is an algebraic isomorphism ϕ from $\mathbf{A}$ onto $\mathbf{A}'$ such that $\phi(\bar{m}(x)) = \bar{m}'(x)$ for all $x \in X$. Two distributions will be called *mutually absolutely continuous*, symbolically $m \sim m'$, if their predistributions are equivalent.

It should be noted that ϕ is then unique, since any isomorphism of measuroid algebras is a homeomorphism, and since ϕ is determined on a dense subset.

DEFINITION 3. Two distributions m, m' will be called *strongly equivalent*, symbolically $m \approx m'$, if

(1) $m \sim m'$,

(2) the ϕ inducing the correspondence (1) enjoys also the following property: for each bounded Baire function f of k real variables and each $A_1, \cdots,$ $A_k \in \mathbf{A}$, we have $\int f(A_1, \cdots, A_k) d\sigma = \int f(\phi(A_1), \cdots, \phi(A_k)) d\sigma$.

NOTE. Baire functions of measuroids can be defined by first taking Baire functions of measurable functions. This can be done by taking polynomials, and then almost everywhere limits. Thus, $f(A_1, \cdots, A_k)$ has an invariant meaning with respect to $\mathbf{A}$. Further, the integral has a meaning, obtainable by first integrating a representative measurable function; and this is again invariant (i.e. independent of how $\mathbf{A}$ arose as a measuroid algebra, but depending only on the value of σ at idempotents of $\mathbf{A}$). Facts and conventions of this nature will hereafter be used without further explanation.

The following two lemmas are common knowledge and are included for reference, together with a brief indication of proof.

LEMMA 1. *Necessary and sufficient for* $m: X \to (\mathbf{A}, \sigma) \approx m': X \to (\mathbf{A}', \sigma')$ *is that, for each Baire set* Δ *in Euclidean k-space, and each* $x_1, \cdots, x_k \in X$,

$$\sigma(\{s \mid (m(x_1)(s), \cdots, m(x_k)(s)) \in \Delta\})$$
$$= \sigma'(\{s' \mid (m'(x_1)(s'), \cdots, m'(x_k)(s')) \in \Delta\}).$$

Proof. Necessity is shown by choosing the function f in Definition 3 to be the characteristic function of Δ. For sufficiency, ϕ must be constructed. This is done by first defining it on the characteristic function of $\{s \mid (m(x_1)(s), \cdots, m(x_k)(s)) \in \Delta\}$; this is sent to the characteristic function of $\{s' \mid (m'(x_1)(s'), \cdots, m'(x_k)(s')) \in \Delta\}$. This correspondence sends σ to σ', and extends uniquely to an algebraic isomorphism from $\mathbf{A}$ onto $\mathbf{A}'$.

LEMMA 2. $\bar{m} \sim \bar{m}'$ *if and only if, for every* $\mathbf{A}'$-*suitable measure* σ' *there is an* $\mathbf{A}$-*suitable measure* σ *such that* $(\bar{m}, \sigma) \approx (\bar{m}', \sigma')$.

Proof. Necessity is shown by defining $\sigma(T) = \int \phi(\chi_T) d\sigma'$, when χ_T is the characteristic function of T. Since ϕ induces a lattice-isomorphism between the idempotents of $\mathbf{A}$ and those of $\mathbf{A}'$, σ is again a measure.

Sufficiency follows from Lemma 1.

Hereafter, let the domain X of our distributions and predistributions be a real Hilbert space. In the ensuing definitions, when an adjective is used to describe a property of a predistribution, the same adjective will be used for any distribution arising from that predistribution.

DEFINITION 4. $\bar{m}$ will be called *quasi-invariant* if, for each $y \in X$, the new predistribution $\bar{m}_y$ obtained by setting $\bar{m}_y(x) = \bar{m}(x) + (x, y)$ is equivalent to $\bar{m}$.

REMARK 1. If m is a quasi-invariant distribution, and X_0 is a k-dimensional linear subspace of X, then $m \mid X_0$ induces a measure on the Baire sets of Euclidean n-space, as follows: choose a basis $x_1, \cdots, x_k$ for X_0 and a coordinate system in n-space, and assign to the Baire set Δ in n-space the measure

$$\sigma(\{s \mid (m(x_1)(s), \cdots, m(x_n)(s)) \in \Delta\}).$$

Quasi-invariance of m implies quasi-invariance of this n-dimensional measure under translation. Thus, by a well-known theorem (see [3]), the measure is equivalent to n-dimensional Lebesgue measure.

DEFINITION 5. $\bar{m}$ will be called *ergodic* if it is quasi-invariant, and if every $A \in \mathbf{A}$ for which $\phi_y(A) = A$ for all y (ϕ_y being the isomorphism in the weak equivalence between $\bar{m}$ and $\bar{m}_y$) is a constant.

DEFINITION 6. A distribution m is called *Gaussian* if, for each $x \in X$, $m(x)$ has a Gaussian "distribution function" in the sense of probability theory (any nonnegative variance, any mean; variance 0 means $m(x)$ is constant, and will also be permitted).

DEFINITION 7. m is called normal with variance c if it is Gaussian, and $\int m(x) m(y) d\sigma = c(x, y)$.

DEFINITION 8. m will be called *completely factorable* if there is an orthonormal basis $\{x_j\}$ for X such that the $m(x_j)$ are independent random variables, in the sense of probability theory. It was shown in [5] that a completely factorable quasi-invariant distribution is ergodic.

3. Examples.

LEMMA 3. *Let σ_1, σ_2 be mutually singular probability measures on $\mathbf{S}$. Let $\tau(T) = (1/2)(\sigma_1(T) + \sigma_2(T))$. Let $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}$ be the measuroid algebras obtained from σ_1, σ_2, τ respectively. Let π_j be the map which assigns to an element of $\mathbf{B}$ its coarser equivalence class in $\mathbf{A}_j$. Let π be the map $A \to \pi_1(A) \oplus \pi_2(A) \in \mathbf{A}_1 \oplus \mathbf{A}_2$.*

(1) π is an isomorphism from $\mathbf{B}$ onto $\mathbf{A}_1 \oplus \mathbf{A}_2$.

(2) The subsets $\mathbf{S}_j$ of $\mathbf{A}_j$ are both dense if and only if $\pi^{-1}(\mathbf{S}_1 \oplus \mathbf{S}_2)$ is dense in $\mathbf{B}$.

Proof. By mutual singularity, there are sets T_1, T_2 in $\mathbf{S}$ such that $\sigma_i(T_j) = \delta_{ij}$, $T_1 \cap T_2 = \varnothing$, $T_1 \cup T_2 = S$. Let P_j be the τ-equivalence class of the characteristic function of T_j. Then P_1 and P_2 are disjoint idempotents whose sum is the identity, and π_j is one to one from $\mathbf{B} P_j$ onto $\mathbf{A}_j$. Since π_j is a homomorphism, this shows that π is an isomorphism. Since τ-almost everywhere convergence holds for $\mathbf{S}$-measurable functions if and only if σ_j-convergence holds for both σ_1 and σ_2, (2) now follows.

We retain this notation in the following two corollaries.

COROLLARY 1. *Let $\bar{n}$ be a linear map from X to* **B**. *Then $\pi_j \circ \bar{n}$, $j = 1, 2$, are both predistributions if and only if $\bar{n}$ is a predistribution.*

Proof. The set of polynomials in the $\bar{n}(x)$ has as its image under π precisely the direct sum of the sets of polynomials in the $\pi_1 \circ \bar{n}(x)$ and the $\pi_2 \circ \bar{n}(x)$, so we can apply (2) of Lemma 3.

COROLLARY 2. *If $\bar{n}$ is a predistribution, and $\pi_1 \circ \bar{n}$, $\pi_2 \circ \bar{n}$ are quasi-invariant, then so is $\bar{n}$.*

Proof. For $y \in X$, let ϕ_j be the automorphism of $\mathbf{A}_j$ sending $\pi_j \circ \bar{n}(x)$ to $\pi_j \circ \bar{n}(x) + (x, y)$. Then

$$A \to \pi^{-1}(\phi_1(\pi_1(A)) \oplus \phi_2(\pi_2(A)))$$

is an automorphism of **B** sending $\bar{n}(x)$ to $\bar{n}(x) + (x, y)$.

Now, let Λ be the real line, S the set of all functions from an index set J to Λ. To avoid fuss we make J countable, although this is not essential. Let $\mathbb{S}$ be the Borel field of sets generated by those of the form

$$T = \left\{ s \mid s(j_1) \in \Delta_1, \cdots, s(j_k) \in \Delta_k \right\},$$

$\Delta_1, \cdots, \Delta_k$ being Baire subsets of Λ. Let σ^0 be a nonnegative probability measure on the Baire subsets of Λ, and let σ be the measure induced on $\mathbb{S}$ by setting $\sigma(T) = \sigma^0(\Delta_1) \cdots \sigma^0(\Delta_k)$. Assume $\int \lambda d\sigma^0(\lambda) = 0$, and $\int \lambda^2 d\sigma^0(\lambda) = c$, c being a fixed positive number. Let $\lambda_j(s) = s(j)$, so that λ_j is, for each $j \in J$, an $\mathbb{S}$-measurable function, mean 0, variance c (with respect to σ measure), and λ_j is independent of $\lambda_{j'}$ for $j \neq j'$. We also use λ_j to denote the corresponding measuroid, so that $\left\{ c^{-1/2}\lambda_j \mid j \in J \right\}$ is an orthonormal set in $\mathfrak{L}_2(\sigma)$.

Now let X be a real Hilbert space with an orthonormal basis $\left\{ x_j \mid j \in J \right\}$, and set $m(x) = \sum_{j \in J} (x, x_j)\lambda_j$. The sum converges in $\mathfrak{L}_2(\sigma)$, and m is thus a distribution, with $\int m(x)m(y)d\sigma = c(x, y)$.

EXAMPLE 1. Takes as σ_c^0 the measure on Λ for which

$$\frac{d\sigma_c^0(\lambda)}{d\lambda} = \frac{1}{(2\pi c)^{1/2}} e^{-\lambda^2/2c}.$$

Then σ_c is the normal distribution with variance c. If $c \neq d$, then, by [1], σ_c and σ_d are mutually singular. Let $\tau = (\sigma_c + \sigma_d)/2$, as in Lemma 3. Then the distributions m_c, m_d obtained from σ_c and σ_d as above give a distribution $m(x) = \pi^{-1}(m_c(x) \oplus m_d(x))$ as in Lemma 3. m is quasi-invariant, but clearly not ergodic, since the automorphism sending $m(x)$ to $m(x) + (x, y)$ leaves the projections P_1 and P_2 invariant (cf. the proof of Lemma 3). One can see directly that for no Gaussian n is $n \sim m$, since, by [1], τ would then have to be either singular to σ_c or equivalent to it.

EXAMPLE 2. The notation immediately preceding Example 1 is retained, with $c = 1$. Suppose the completely factorable m constructed there is quasi-invariant, and suppose $m \sim m'$ for some Gaussian distribution m'. By Lemma 1, there is a measure σ_1 on $\mathcal{S}$ equivalent to σ such that the distribution m_1 gotten by combining the predistribution of m with the measure σ_1 is strongly equivalent to m'.

LEMMA 4. *The map* $x \to \int m_1(x) d\sigma_1$ *is continuous.*

Proof. Suppose $x^{(k)} \to 0$. Then $m(x^{(k)}) \to 0$ in $\mathcal{L}_2(\sigma)$. Thus, for some subsequence $\{k_j\}$ we have $m(x^{(k_j)}) \to 0$ almost everywhere. So $m_1(x^{(k_j)}) \to 0$ almost everywhere, since m and m_1 come from the same predistribution. In [1, Lemma 6], it is shown that then $\int m_1(x^{(k_j)}) d\sigma_1 \to 0$. So there is some $y \in X$ such that $\int m_1(x) d\sigma_1 = (x, y)$.

LEMMA 5. *The map* $x \to m_1(x)$ *is continuous from* X *to* $\mathcal{L}_2(\sigma_1)$.

Proof. The map $x \to m(x) - (x, y)$ is continuous. But $m_1(x) - (x, y)$ is Gaussian, mean zero, so that convergence of x implies almost everywhere convergence of $m_1(x) - (x, y)$ (see proof of Lemma 4), which implies $\mathcal{L}_2(\sigma_1)$-convergence of $m_1(x) - (x, y)$, as shown in the discussion of Example 1 in [5, p. 14]. Thus, the map $x \to (m_1(x) - (x, y)) + (x, y) = m_1(x)$ is continuous.

Now let $m_0(x) = m_1(x) - (x, y)$. Then m_0 is Gaussian, mean zero, and since m_1 is quasi-invariant we have $m_0 \sim m_1 \sim m$.

LEMMA 6. *Let* f_l^k, $k = 1, 2, \cdots$; $l = 1, \cdots, l(k)$ *be identically distributed, nonconstant, independent measurable functions on the probability space* $(R, \mathcal{R}, \rho)$. *Let* a_l^k, c^k *be numbers such that* $\sum_l a_l^k f_l^k + c^k \to 0$ *in measure as* $k \to \infty$. *Then* $\sum_l a_l^k f^k + c^k \to 0$.

Proof. Since the f_l^k are nonconstant there are numbers $\gamma > 0$, $\delta > 0$ and β such that for all k, l we have

$$\rho\{r \mid f_l^k(r) > \beta + \gamma\} \quad \text{and} \quad \rho\{r \mid f_l^k(r) < \beta\} \text{ are both} \geq \delta.$$

Now, $\sum_l a_l^k f_l^k + c_l^k$ can be written $a_{l_0}^k f_{l_0}^k + g_{l_0}^k$, where $g_{l_0}^k$ and $f_{l_0}^k$ are independent. Then simple considerations of independence show that if $a_{l_0}^k f_{l_0}^k + g_{l_0}^k$ fluctuates by less than $|a_l^k \gamma|$ on a set, this set must have measure $< 1 - \delta$. Thus $a_{l_0}^k \to 0$ uniformly on l_0.

Now, Remark 3 in [2, p. 134] gives conditions for convergence to 0. Condition (1) there translates into

$$\sum_l \rho\{\mid a_l^k f_l^k \mid > \epsilon\} + d^k \to 0 \qquad \text{for each } \epsilon > 0,$$

where

$$d^k = \rho \left\{ |c^k| > \epsilon \right\} \begin{matrix} = 1 \\ = 0 \end{matrix} \qquad \begin{matrix} \text{if } |c^k| > \epsilon, \\ \text{if } |c^k| \leqq \epsilon. \end{matrix}$$

Then $c^k \to 0$. Condition (2) gives

$$\sum_l a_l^k \int_{\Delta_l^k} f_l^k d\rho \to 0.$$

But

$$\int_{\Delta_l^k} f_l^k d\rho \to 1 \quad (\text{where } \Delta_l^k = \{ r \mid a_l^k f_l^k(r) < \epsilon \})$$

uniformly in l, since $a_l^k \to 0$ uniformly in l. Thus $\sum_l a_l^k \to 0$.

COROLLARY. *Let h_k be a finite sum $\sum v_{ij} g_i g_j + w$, for certain measurable functions g_j with jointly Gaussian distributions of mean 0 and certain numbers v_{ij}, w (actually depending on k, of course, but this is suppressed in the notation). Let $h_k \to 0$ in measure. Then $\int h_k d\rho \to 0$.*

Proof. Choose an orthonormal basis for the linear subspace of $\mathfrak{L}_2(\rho)$ spanned by the g_j. Then the g_j are finite linear combinations of these basis elements, and the effect on the expression for h_k is that one can assume without loss of generality that the g_j are orthonormal. And, of course, one can let $v_{ij} = v_{ji}$. Now choose an orthogonal matrix $U = (u_{ij})$ such that $U(v_{ij})U^t$ is diagonal. Then, letting $\bar{g}_i = \sum_j u_{ij} g_j$, we get h_k in the form $\sum a_j \bar{g}_j^2 + w$.

Now Lemma 6 can be applied, since $\int h_k d\rho$ is exactly $\sum a_j + w$.

Next, let $\{ c_{ij} \mid i, j \in J \}$ be a square-summable set of real numbers, and consider the sum

$$\sum_i c_{ii}(m(x_i)^2 - 1) + \sum_{i \neq j} c_{ij} m(x_i) m(x_j).$$

This converges in $\mathfrak{L}_2(\sigma)$, since

$$\{ m(x_i)^2 - 1 \mid i \in J \} \cup \{ m(x_i) m(x_j) \mid i \neq j; i, j \in J \}$$

forms an orthogonal set in $\mathfrak{L}_2(\sigma)$: not normal, indeed, but with exactly two nonzero norms occurring, which is equally good for our purpose. Thus there is an increasing sequence J_k of finite subsets of J, with $\bigcup_k J_k = J$, and such that the partial sums

$$s_k = \sum_{j \in J_k} c_{jj}(m(x_j)^2 - 1) + \sum_{i,j \in J_k} c_{ij} m(x_i) m(x_j)$$

are a Cauchy sequence in σ-measure, and therefore also in σ_1-measure. By the corollary to Lemma 6, we then have $\int (s_k - s_l) d\sigma_1 \to 0$, so that

$$\sum_{i \in J_k} c_{ii} \left(\int m_0(x_i)^2 d\sigma_1 - 1 \right) + \sum_{i,j \in J_k; \, i \neq j} c_{ij} \int m_0(x_i) m_0(x_j) d\sigma,$$

converges as $k \to \infty$. Since the signs of the c_{ij} may be changed at will, the series

$$\sum_{i \in J} c_{ii} \left(\int m_0(x_i)^2 d\sigma_1 - 1 \right) + \sum_{i,j \in J} c_{ij} \int m_0(x_i) m_0(x_j) d\sigma_1$$

actually converges unconditionally. Then

$$\sum_{i \in J} \left| \int m_0(x_i)^2 d\sigma_1 - 1 \right|^2 + \sum_{i,j \in J} \left| \int m_0(x_i) m_0(x_j) d\sigma_1 \right|^2 < \infty.$$

This says that the operator m_0 from X to $\mathcal{L}_2(\sigma_1)$ has the property that $m_0^* m_0 - I$ is a Hilbert-Schmidt operator. By [5], this implies that $m_0 \sim n$, where n is a normal distribution on X with variance 1.

Then $m \sim n$, and so there is a measure τ on $\mathcal{S}$ which is equivalent to σ, and for which the coordinate functions λ_j are independent, mean 0, variance 1, with Gaussian distribution functions. But a simple consequence of Kakutani's criterion for equivalence of infinite product measures says that can happen only if λ_j has the same distribution with respect to σ, i.e. if $\tau = \sigma$.

All that remains is to construct σ^0 on Λ such that $\int d\sigma^0 = 1$, $\int \lambda d\sigma^0(\lambda) = 0$, $\int \lambda^2 d\sigma^0(\lambda) = 1$, and such that the σ obtained from it is quasi-invariant. A necessary condition (cf. Remark 1) is that $d\sigma^0(\lambda) = g^2(\lambda) d\lambda$ for some positive Lebesgue square-integrable function g, with

$$(1) \qquad\qquad \int g^2(\lambda) d\lambda = 1,$$

$$(2) \qquad\qquad \int \lambda g^2(\lambda) d\lambda = 0,$$

$$(3) \qquad\qquad \int \lambda^2 g^2(\lambda) d\lambda = 1.$$

(2) can be guaranteed by making g even, together with (3). Let $G(\mu) = (2\pi)^{-1/2} \int e^{i\lambda\mu} g(\mu) d\mu$. Let us assume further that

$$(4) \qquad\qquad \int \mu^2 |G(\mu)|^2 d\mu < \infty.$$

We now find the condition for quasi-invariance to be (using Lemma 2 and Kakutani's criterion)

$$(*) \qquad\qquad \prod_{j \in J} \left(\int g(\lambda + c_j) g(\lambda) d\lambda \right) > 0 \qquad \text{if } \sum |c_j|^2 < \infty.$$

Plancherel's theorem says

$$\int g(\lambda + c)g(\lambda)d\lambda = \int e^{i\mu c}\big|\, G^2(\mu)\,\big|\, d\mu,$$

so (*) becomes

$$\prod_{j \in J} \int e^{i\mu c_j}\big|\, G^2(\mu)\,\big|\, d\mu > 0 \qquad \text{if } \sum_j \big|\, c_j\,\big|^2 < \infty.$$

Note that each factor is positive. We rewrite the product as

$$\prod_j \left(1 - 2\int \sin^2\left(\frac{\mu c_j}{2}\right)\big|\, G(\mu)\,\big|^2 d\mu\right).$$

Now,

$$2\int \sin^2\left(\frac{\mu c}{2}\right)\big|\, G(\mu)\,\big|^2 d\mu \leqq \frac{c^2}{2}\int \mu^2\big|\, G(\mu)\,\big|^2 d\mu,$$

so the infinite product converges to a nonzero limit if $\sum \big|\, c_j\,\big|^2 < \infty$.

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