A SURFACE IS TAME IF ITS COMPLEMENT IS 1-ULC(1)

BY

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1. **Introduction.** In this paper we show that a 2-manifold M in Euclidean 3-space E^3 is tame if E^3-M is uniformly locally simply connected.

A closed subset X of a triangulated manifold Y is *tame* if there is a homeomorphism of Y onto itself taking X onto a polyhedron (geometric complex) in Y. If there is no such homeomorphism, X is called ***wild***. Examples of 2-manifolds wildly embedded in E^3 are found in [1; 8; 6].

An *n-manifold* is a separable metric space each of whose points lies in a neighborhood homeomorphic to Euclidean n-space. An ***n-manifold-with-boundary*** is a separable metric space each of whose points lies in a neighborhood whose closure is a topological n-cell. If M is an n-manifold-with-boundary, we use Int M to denote the set of points of M with neighborhoods homeomorphic to Euclidean n-space and Bd D to denote $M-\text{Int } M$. For example, if D is a disk, Bd D is a simple closed curve which is the rim of the disk. If we have a manifold embedded in a larger space and treat the manifold as a subset rather than a space, we insist that it be closed. If S is a 2-sphere embedded in E^3, we use Int S and Ext S to denote the bounded and unbounded components of E^3-S. The double meaning of the symbol Int should not lead to confusion.

A subset X of a manifold-with-boundary is *locally tame* at a point p of X if there is a neighborhood N of p and a homeomorphism of $\overline{N}$ (the closure of N) onto a cell that takes $X\cdot\overline{N}$ onto a polyhedron. If the manifold-with-boundary is triangulated, we say that X is locally polyhedral at p if there is a neighborhood N of p such that $X\cdot\overline{N}$ is a polyhedron.

Suppose D is a disk. We say that *a map of* Bd D *into a set* Y *can be shrunk to a constant in* Y if the map can be extended to take D into Y. If each map of Bd D into Y can be shrunk to a constant in Y, we say that Y is *simply connected*. Also, Y is *locally simply connected* at a point p of $\overline{Y}$ if for each neighborhood U of p there is a neighborhood V of p such that each map of Bd D into $V\cdot Y$ can be shrunk to a point in $U\cdot Y$. A metric space Y is *uniformly locally simply connected* (or 1-ULC) if for each $\epsilon>0$ there is a $\delta>0$ such that each map of Bd D into a δ subset of Y can be shrunk to a point on an ϵ subset of Y. If $\overline{Y}$ is a compact subset of a metric space, it can be shown that Y is 1-ULC if it is locally simply connected at each point of $\overline{Y}$.

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The surfaces studied in solid geometry are tame. It is useful to have criteria for determining which surfaces are tame. We mention three such criteria.

1. A closed subset of a triangulated 3-manifold-with-boundary is tame if and only if it is locally tame at each of its points.

2. A 2-sphere in E^3 is tame if and only if it can be homeomorphically approximated from both sides.

3. A 2-manifold M in a triangulated 3-manifold M_3 is tame if and only if $M_3 - M$ is locally simply connected at each point of M.

Criterion 1 is proved in [2] and [10]. Criterion 2 is proved in [5] and restated below as Theorem 0. Criterion 3 is Theorem 7 of the present paper.

The distance function is denoted by ρ. We shall make use of the following approximation theorem proved in [3].

APPROXIMATION THEOREM. *For each 2-manifold M in a triangulated 3-manifold-with-boundary and each non-negative continuous function f defined on M, there is a 2-manifold M' and a homeomorphism h of M onto M' such that*

$$\rho(x, h(x)) \leqq f(x) \qquad (x \in M)$$

and M' is locally polyhedral at $h(x)$ if $f(x) > 0$.

Another theorem that we shall use is the Side Approximation Theorem for 2-Spheres. It is proved by the same methods as the Approximation Theorem and its proof will be given in another paper [7].

SIDE APPROXIMATION THEOREM FOR 2-SPHERES. *Each 2-sphere S in E^3 can be polyhedrally approximated almost from either side—that is for each $\epsilon > 0$ and each component U of $E^3 - S$ there is a homeomorphism h of S onto a polyhedral 2-sphere such that*

h moves no point more than ϵ and

$h(S)$ contains a finite collection of mutually exclusive disks each of diameter less than ϵ such that $h(S)$ minus the sum of the disks lies in U.

If A and B are two sets and there is a homeomorphism of A onto B that moves no point by more than ϵ, we write

$$H(A, B) \leqq \epsilon.$$

The following theorem is proved in [5].

THEOREM 0. *A 2-sphere S in E^3 is tame if it can be homeomorphically approximated from both sides—that is, for each $\epsilon > 0$ and each component U of $E^3 - S$, there is a 2-sphere S' in U such that*

$$H(S, S') < \epsilon.$$

To show that a 2-sphere is tame, a first step might be to show that the hypothesis of Theorem 0 is met. As a step toward proving that a 2-sphere in E^3 is tame if its complement is 1-ULC, we prove the following.

THEOREM 1. *If S is a 2-sphere in E^3 such that* Int S *is* 1-ULC, *then for each* $\epsilon > 0$, *S can be homeomorphically approximated from* Int S—*that is, for each* $\epsilon > 0$ *there is a 2-sphere S' in* Int S *such that*

$$H(S, S') < \epsilon.$$

Our plan for proving Theorem 1 is to get a special cellular decomposition T of S, get a homeomorphism of S onto itself that pulls the boundaries of the cells of the decomposition T into Int S, and finally pull the cells themselves into Int S.

It is hoped that the serious reader will understand the why of the attack as well as the details. Hence, we give our over-all plan of attack first and reserve epsilontics to last so that the reader can see why these particular ϵ's are used. When we need a close approximation, we let it be an ϵ_i approximation and decide later how small the ϵ_i would need to be to make the details work.

2. **Proof of Theorem** 1. a. **Special cellular decomposition** T **of** S. We need a cellular decomposition T of S with the following properties.

The mesh of T is less than ϵ_1 (ϵ_1 is a small number whose size is to be described later).

The collection of 2-cells of T is the sum of three subcollections A_1, A_2, A_3 such that no two elements of A_i ($i=1, 2, 3$) intersect each other.

That for each $\epsilon_1 > 0$ there is such a cellular decomposition T of S follows from a consideration of a triangulation T' of S of mesh less than $\epsilon_1/2$. The vertices of T' are swelled into 2-cells and become the elements of A_1. The parts of the 1-simplexes of T' not in elements of A_1 are expanded into the elements of A_2. The closures of the parts of the 2-simplexes of T' not in elements of A_1 or A_2 are the elements of A_3.

b. **Pulling elements of** T **partially into** S. Suppose T is a fixed special cellular decomposition of S such as mentioned in the preceding section. The 1-skeleton of T is the sum of the boundaries of the 2-cells in T and is denoted by K_1. We select a small number ϵ_2 whose size is described later. Then there is a polyhedral 2-sphere S_1 and a homeomorphism h_1 of S onto S_1 such that

h_1 moves no point more than ϵ_2,
$h_1(K_1) \subset$ Int S, and
S_1 contains a finite collection of mutually exclusive ϵ_2 disks such that S_1 minus the sum of the interiors of these disks lies in Int S.

That for each ϵ_2 there are such an S_1 and an h_1 follows from the Side Approximation Theorem of 2-Spheres. We let $\epsilon_2/2$ be the ϵ in the statement of that theorem and S_1 be the S' guaranteed by the conclusion of that theorem. The homeomorphism h_1 is the homeomorphism h guaranteed by that theorem followed by a homeomorphism of S_1 onto itself that moves no point by more than $\epsilon_2/2$ but pulls the image of K_1 off of the disks.

c. **The next approximations to elements of** T. For each 2-cell D of T, $h_1(D)$ is a first approximation to D. We note that $h_1(D)$ is homeomorphically close to D and $h_1(\text{Int } D)$ contains a finite collection $E_1, E_2, \cdots, E_n$ of mutually exclusive ϵ_2 disks such that $h_1(D)-\sum \text{Int } E_i \subset \text{Int } S$. We suppose that $h_1(\text{Bd } D)$ is a polygon.

The second approximation $h_2(D)$ to D may not be quite as close homeomorphically to D as is $h_1(D)$ but it will still be close. However, it will have the advantage that the components of $S \cdot h_2(D)$ will have diameters much smaller than ϵ_2 (which is an upper bound on the diameters of $E_1, E_2, \cdots, E_n$) and simple closed curves in Int S near the components of $S\cdot h_2(D)$ can be shrunk to points in Int S without hitting $h_1(K_1)$.

Let ϵ_3 be a very small positive number selected in a fashion to be described later and S' be a polyhedral 2-sphere which is homeomorphically within ϵ_3 of S and which contains a finite collection of mutually exclusive ϵ_3 disks such that S' minus the sum of the interiors of these disks is contained in Int S. We select ϵ_3 so that $S'\cdot h_1(D) \subset \sum \text{Int } E_i$ and suppose that $S'\cdot h_1(D)$ is the sum of a finite collection of mutually exclusive simple closed curves $J_1, J_2, \cdots, J_r$.

The ϵ_2 and ϵ_3 were selected so that each J_i bounds a disk F_i on S' of small diameter. We suppose that these disks F_i are ordered by size with the small ones first so that no F_i contains an F_{i+j}.

The disk in $h_1(D)$ bounded by J_1 is first replaced by F_1 and then pushed slightly to one side of S' so as to reduce the number of components with the intersection with S'. The process is continued by replacing disks in the adjusted $h_1(D)$ by F_i's and then pushing slightly so as to get a polyhedral disk $h_2(D)$ which is close to D homeomorphically and which lies on Int S'. We select h_2 so that it agrees with h_1 in a neighborhood of Bd D and such that the components of $S\cdot h_2(D)$ are not much bigger than those of $S'\cdot h_1(D)$. By selecting ϵ_3 very small, we can insure that the components of $h_2(D)\cdot S$ are very small—in fact of diameter less than some preselected positive number ϵ_4.

Although we could have chosen the sum of the $h_2(D)$'s to be a 2-sphere, we did not insist on this since at the third approximation of D, there seems to be no easy way to prevent the approximating disks from intersecting at interior points of each.

d. **Third approximation to** D. Since each component of $S\cdot h_2(D)$ is of diameter less than ϵ_4, $h_2(\text{Int } D)$ contains a collection of mutually exclusive disks $E_1', E_2', \cdots, E_m'$ which cover $S\cdot h_2(D)$ such that each Bd E_i' lies close to S and is of diameter less than ϵ_4. Each Bd E_i' lies in Int S since Bd D does.

Each Bd E_i' is of such small diameter that it can be shrunk to a point on a small subset of Int S where ϵ_4 and ϵ_3 have been selected so that this subset will not intersect $h_1(K_1)=h_2(K_1)$. Hence there is a map g of $h_2(D)$ into Int S

such that $gh_2(D)$ lies close to $h_2(D)$, and g is the identity in a neighborhood of $h_2(\text{Bd } D)$, $gh_2(D)$ intersects $h_2(K_1)$ only in $h_2(\text{Bd } D)$, and $gh_2(D)$ has no singular points near $gh_2(\text{Bd } D)$. It follows from Dehn's lemma as proved by Papakyriokopoulos [11] that for each neighborhood U of the set of singular points of $gh_2(D)$, there is a homeomorphism h_3 of D onto a polyhedral disk $h_3(D)$ in $gh_2(D)+U$ that agrees with h_2 in a neighborhood of Bd D. The third approximation to D is $h_3(D)$. The advantage it has over $h_2(D)$ is that it lies in Int S. The only thing that makes $h_3(D)$ and D homeomorphically close is that each is of small diameter and their boundaries are homeomorphically close.

e. **The fourth approximation to** D. Our task is now to untangle the $h_3(D)$'s so that their sum forms a 2-sphere in Int S. We recall that the collection of 2-cells of T is the sum of three subcollections A_1, A_2, A_3 so that no two elements of any A_i have a point in common. We will have enlarged the $h_1(D)$'s so little as we changed them to $h_2(D)$'s and then to $h_3(D)$'s that two $h_3(D)$'s will not intersect if the corresponding D's do not intersect.

For each element D of A_1, the fourth approximation $h(D)$ to D is $h_3(D)$.

We may suppose that for each element D of A_1, the intersection of $h_3(\text{Int } D)=h(\text{Int } D)$ and the sum of the images under h_3 of the elements of A_2 is the sum of a finite collection of mutually exclusive simple closed curves $J_1, J_2, \cdots, J_s$ such that J_i bounds a disk G_i in Int $h(D)$ and the J_i's are ordered according to the size of the disks G_i they bound in $h(D)$, with the small ones coming first.

If J_1 lies in an $h_3(D')$ for an element D' of A_2, we replace the disk in $h_3(D')$ by G_1 and shove this replaced disk slightly to one side of $h_3(D)$. This process is continued until for each element D' of A_2, there is an h defined on D' so that $h(D)$ and the $h(D')$'s fit together only along their boundaries as they should.

Finally we turn to the elements of A_3. We suppose that for each element D of A_1+A_2, $h(\text{Int } D)$ intersects the sum of the images under h_3 of the elements of A_3 in the sum of a collection of mutually exclusive simple closed curves. These simple closed curves are eliminated one by one, starting at the inside in a manner already described. For each element D' of A_3, the resulting adjustment of $h_3(D')$ is called $h(D')$.

The sum of the $h(D)$'s is a 2-sphere in Int S. We note that $h=h_1=h_2=h_3$ on K_1. We will cause h to be near the identity by picking the ϵ_i's so that the D's and the $h(D)$'s are small and h is near the identity on K_1.

f. **Epsilontics.** In this section we explain the sizes for ϵ_1, ϵ_2, ϵ_3, ϵ_4 and the reasons for these selections. We recall that

ϵ_1 limits the mesh of T,

ϵ_2 limits h_1 to be near the identity and limits the sizes of the E_i's,

ϵ_3 limits S' to be near S and limits the sizes of the disks in S' such that S' minus these disks lies in Int S, and

ϵ_4 limits the sizes of the components of $h_2(D)\cdot S$.

We choose

$$\epsilon_1 < \epsilon/8,$$

where ϵ is the number mentioned in the statement of Theorem 1. This is the only restriction we place on ϵ_1.

Let δ_1 be a positive number so small that the distance between two elements of T without a common point is more than δ_1.

Our goal is to choose ϵ_2, ϵ_3, ϵ_4 so that for each 2-cell D of T, the following conditions are satisfied.

$$h_1(D) \text{ lies in a } \delta_1/6 \text{ neighborhood of } D.$$

$$h_2(D) \text{ lies in a } \delta_1/6 \text{ neighborhood of } h_1(D).$$

$$h_3(D) \text{ lies in a } \delta_1/6 \text{ neighborhood of } h_2(D).$$

This will insure that if D_1, D_2 are two 2-cells of T without a common point, then $h_3(D_1)\cdot h_3(D_2)=0$.

Since each point of $h_3(D)$ lies within $3\delta_1/6$ of a point of D and $\delta_1<\epsilon_1$,

$$\text{diameter } h_3(D) < \epsilon_1 + \delta_1 < 2\epsilon_1.$$

Since $h=h_3$ on elements of A_1,

$$\text{diameter } h(D) < 2\epsilon_1 \text{ if } D \text{ is an element of } A_1.$$

In changing from h_3 to h on elements D of A_2, the adjustment was made so that $h(D)$ lies within $2\epsilon_1$ of $h_3(D)$. Hence,

$$\text{diameter } h(D) < 6\epsilon_1 \text{ if } D \text{ is an element of } A_2.$$

For each element D of A_3,

$$h(D) \text{ lies within } 6\epsilon_1 \text{ of } h_3(D).$$

Since each point of $h(D)$ lies within $6\epsilon_1$ of a point of $h_3(D)$, this point in turn lies within $3\delta_1/6$ of a point of D, and diameter $D<\epsilon_1$, we find that h moves no point as much as $8\epsilon_1$. This is the reason we selected $\epsilon_1<\epsilon/8$.

Let ϵ_2 be a positive number so small that each subset of S of diameter $3\epsilon_2$ lies in a disk on S of diameter less than $\delta_1/10$. We note that

$$\epsilon_2 < \delta_1/30 < \delta_1/6.$$

Since $\epsilon_2<\delta_1/6$, for each 2-cell D of T, $h_1(D)$ lies in a $\delta_1/6$ neighborhood of D. The more stringent condition that $\epsilon_2<\delta_1/30$ is used later to help insure that $h_2(D)$ lies in a $\delta_1/6$ neighborhood of $h_1(D)$.

Let δ_2 be a positive number so small that for each 2-cell D of T, δ_2 is less than the distance between S and $h_1(D)-\sum E_i$. We note that

$$\delta_2 < \epsilon_2.$$

Although we shall place more stringent conditions on ϵ_3, we first consider the sizes of the components of $S' \cdot h_1(D)$ if we merely suppose

$$\epsilon_3 < \delta_2.$$

Let X be a component of $S' \cdot h_1(D)$ and h' be a homeomorphism of S onto S' that moves no point more than ϵ_3. Then X lies in an E_i (which is of diameter less than ϵ_2), $h'^{-1}(X)$ is of diameter less than $\epsilon_2 + 2\epsilon_3 < 3\epsilon_2$, $h'^{-1}(X)$ lies in a disk of diameter less than $\delta_1/10$, and the image of this disk under h' is a disk in S' of diameter less than $\delta_1/10 + 2\epsilon_3 < \delta_1/10 + 2\delta_1/30 = \delta_1/6$. Hence, the restrictions we have placed on ϵ_2, ϵ_3 are enough to insure that we can select h_2 so that $h_2(D)$ lies in a $\delta_1/6$ neighborhood of $h_1(D)$.

Let ϵ_4 be a positive number so small that each simple closed curve in $E^3 - S$ of diameter less than ϵ_4 can be shrunk to a point on a subset of $E^3 - S$ of diameter less than the minimum of $\delta_1/6$ and $\delta_2/2$. Note that ϵ_4 does not depend on ϵ_3. The Bd E_i''s are selected to lie within $\epsilon_4 < \delta_2/2$ of S. Also, the Bd E_i''s have diameters less than ϵ_4 so that $gh_2(D)$ intersects $h_2(K_1)$ only in $h_2(\text{Bd } D)$. Furthermore, $h_3(D)$ lies in a $\delta_1/6$ neighborhood of $h_2(D)$.

The final restriction we place on ϵ_3 is to insure that each component of $S \cdot h_2(D)$ is of diameter less than ϵ_4. Let δ_3 be a number so small that each δ_3 subset of S lies in a disk in S of diameter less than ϵ_4. We suppose

$$\epsilon_3 < \delta_3.$$

Each component X of $S \cdot h_2(D)$ lies on Int S' and is separated from the big component of $S - S'$ by a disk in S' of diameter less than $\epsilon_3 < \delta_3$. Hence, diameter $X < \epsilon_4$.

3. **Conditions under which a 2-sphere is tame.** Tame 2-spheres in E^3 have uniformly locally simply connected complements. A converse is as follows.

THEOREM 2. *A 2-sphere S in E^3 is tame if each component of $E^3 - S$ is* 1-ULC.

Proof. It follows from Theorem 1 that for each positive number ϵ there is a 2-sphere S' in Int S such that

$$H(S, S') < \epsilon.$$

Similarly, it follows that there is a 2-sphere S'' in Ext S such that

$$H(S, S'') < \epsilon.$$

That S is tame then follows from Theorem 0.

COROLLARY. *A 2-sphere S in E^3 is tame if $E^3 - S$ is locally simply connected at each point of S.*

4. **Retaining simple connectivity.** We shall be applying the Approximation Theorem to a 2-manifold to make part of the 2-manifold locally poly-

hedral. We are interested in seeing what this does to the local simple connectivity of the complement. First we examine the effect of throwing away part of the 2-manifold.

THEOREM 3. *Suppose M is a 2-manifold in E^3, D is a disk in M, and p is a point of D at which E^3-M is locally simply connected. Then E^3-D is locally simply connected at p.*

Proof. We only consider the case where $p\in \text{Bd } D$. Suppose U is a given neighborhood of p. Let V' be a neighborhood of p such that each map of a circle into $V'\cdot(E^3-M)$ can be shrunk to a point in $U\cdot(E^3-M)$ and V be a neighborhood of p such that each pair of points of $V\cdot(M-D)$ lie in an arc in $V'\cdot(M-D)$. We show that if E is a plane circular disk and f is a map of Bd E into $V\cdot(E^3-D)$, then there is a map of E into $U\cdot(E^3-D)$ that agrees with f on Bd E.

We want to simplify f so that $M\cdot f(\text{Bd } E)$ does not have infinitely many components. Suppose aa' is an arc on Bd E and bb' is an arc on $V\cdot(M-D)$ such that $f(a)=b$, $f(a')=b'$, and $bb'+f(aa')$ lies in a convex subset of $V-D$. Then there is a homotopy F_t on aa' such that $F_0=f$, F_t is constant on a and a', $F_1(aa')=bb'$, and each $F_t(aa')$ lies in $V-D$. Hence, we suppose with no loss of generality that $f^{-1}(M\cdot f(\text{Bd } E))$ is the sum of a finite number of arcs $a_1a_2, a_3a_4, \cdots, a_{2n-1}a_{2n}$ on Bd E ordered so that there is no a_j between any a_i and a_{i+1}.

Let q be the center of E and $f(q)$ be any point of $V\cdot(M-D)$. Extend f to map the radius qa_i of E onto an arc in $V'\cdot(M-D)$ from $f(q)$ to $f(a_i)$. Since each component of M divides E^3 into two pieces and any arc in M can be approximated by arcs on either side of M, the map f on the boundary of the sector a_iqa_{i+1} of E can be extended to map an annulus ring in the sector one of whose boundary components is the boundary of the sector into $V'\cdot(E^3-D)$ so that the image of the other boundary component of the annulus misses M. The map f can be further extended to take the rest of the sector into $U\cdot(E^3-M)$.

THEOREM 4. *Suppose M is a 2-manifold in E^3, D is a disk in M, p is a point of M at which E^3-M is locally simply connected, and h is a homeomorphism of M into E^3 such that h is the identity on D and $h(M)$ is locally polyhedral at each point of $h(M)-D$. Then $E^3-h(M)$ is locally simply connected at p.*

Proof. The only case we consider is the one in which p is a point of Bd D. Suppose U is a given neighborhood of p. Let V' be a neighborhood of p such that each closed set in $V'\cdot(h(M)-D)$ lies in a disk in $U\cdot(h(M)-D)$. It follows from Theorem 3 that there is a neighborhood V of p such that each map of a circle into $V\cdot(E^3-D)$ can be shrunk to a point in $V'\cdot(E^3-D)$. We show that if E is a disk and f is a map of Bd E into $V\cdot(E^3-h(M))$, then f can be extended to map E into $U\cdot(E^3-h(M))$.

Let f' be a map of E into $V'\cdot(E^3-D)$ that is an extension of f on Bd E. Let E' be the component of $E-f'^{-1}(h(M)\cdot f(E))$ containing Bd E. With no loss of generality we suppose that E' is E minus a finite collection of mutually exclusive subdisks $E_1, E_2, \cdots, E_n$ of E.

Since $f'(\text{Bd } E_i)$ lies on the interior of a polyhedral disk in $U\cdot(h(M)-D)$, it is possible to adjust f' on disks in E slightly larger than the E_i's so as to take the larger disks slightly to one side of $h(M)$. The adjusted f' is f and takes E into $U\cdot(E^3-h(M))$.

QUESTION. Would Theorem 4 be true if we supposed that D were merely a closed subset of M with only nondegenerate components rather than actually a disk in M?

5. **Enlarging a disk to a 2-sphere.** Not each disk in E^3 lies on a 2-sphere. An example of such a disk is obtained by taking a horizontal disk D in E^3; removing two circular holes from Int D; adding tubes from the holes, one tube going up and the other down and around; and finally adding hooked disks as shown in [1]. The disk does not lie on a 2-sphere since its boundary cannot be shrunk to a point in the complement of the disk. Although it does not lie on a 2-sphere, it does lie on a torus as was pointed out to me by David Gillman. If instead of removing a pair of holes from the horizontal disk and replacing the holes with hooked wild disks, one had removed an infinite collection of pairs of holes converging to a boundary point of D and replaced each pair of holes with wild disks hooked over the boundary of D, there would have resulted a wild disk in E^3 that does not lie on any 2-manifold in E^3.

The following result shows that each disk contains many small disks each of which lies on a 2-sphere.

THEOREM 5. *Suppose M is a 2-manifold in E^3, p is a point of M, and U is a neighborhood of p. Then there is a disk D in $M\cdot U$ and a 2-sphere S in U such that $p\in \text{Int } D\subset S$ and S is locally polyhedral at each point of $S-D$.*

Proof. Let E be a disk in M such that $p\in\text{Int } E$ and C be a cube in U whose interior contains p and whose exterior contains Bd E. Let D be a disk in $M\cdot\text{Int } C$ such that $p\in\text{Int } D$. It follows from the Approximation Theorem that there is a homeomorphism h of E into E^3 such that h is the identity on D, h takes Bd E into Ext C, and $h(E)$ is locally polyhedral at each point of $h(E)-D$. We suppose with no loss of generality that Bd $C\cdot h(E)$ is the sum of a finite collection of mutually exclusive polygons.

Let E' be the component of $h(E)-\text{Bd } C$ containing D. It is topologically equivalent to a 2-sphere minus the sum of a finite collection of mutually exclusive disks. By adding polygonal disks in $U\cdot(C+\text{Ext } C)$ to E', one obtains the required 2-sphere S.

6. **Locally tame subsets of 2-manifolds.** A 2-sphere S in E^3 may not be

locally tame at a point p even if E^3-S is locally simply connected at p as shown by the following example.

EXAMPLE. Consider a spherical 2-sphere S' and a sequence of mutually exclusive spherical disks $E_1, E_2, \cdots$ in S' converging to a point p of S'. Fox and Artin have described [8] a wild arc which is locally polyhedral except at one end point. For each E_i let A_i be such an arc reaching out from the center of E_i such that the arc is of diameter less than the radius of E_i and such that A_i intersects S only at the polyhedral end of A_i. By replacing each E_i in S by a disk obtained by swelling up A_i as done in [7] one can obtain a 2-sphere S such that E^3-S is locally simply connected at p even though S is not locally tame at p.

THEOREM 6. *Suppose M_2 is a 2-manifold embedded in a 3-manifold M_3 and U is an open subset of M_2 such that M_3-M_2 is locally simply connected at each point of U. Then M_2 is locally tame at each point of U.*

Proof. Since local tameness is only a local property, we suppose that M_3 is E^3 and U is all of M_2. If this were not the case already we would take a homeomorphism h of a neighborhood of p in M_3 into E^3 such that this neighborhood did not intersect M_2-U.

It follows from Theorem 5 that there is a disk D in M_2 and a 2-sphere S such that $p\in \operatorname{Int} D\subset S$ and S is locally polyhedral at each point of $S-D$. It follows from Theorem 4 that E^3-S is locally simply connected at each point of S. Since S is compact, E^3-S is 1-ULC. Theorem 2 implies that S is tame. Since S is tame, M_2 is locally tame at p.

Since a closed set in a 3-manifold is tame if it is locally tame [2; 10] we have the following result.

THEOREM 7. *A 2-manifold M_2 in a triangulated 3-manifold M_3 is tame if and only if M_3-M_2 is locally simply connected at each point of M_2.*

COROLLARY. *A 2-manifold M_2 in a 3-manifold M_3 is tame if M_3-M_2 is* 1-ULC.

7. **Tame 2-manifolds-with-boundaries.** In this section we extend our results about 2-manifolds to 2-manifolds-with-boundaries.

THEOREM 8. *Suppose M_2 is a 2-manifold-with-boundary embedded in a 3-manifold M_3 and U is an open subset of M_2 such that M_3-M_2 is locally simply connected at each point of U. Then M_2 is locally tame at each point of U.*

Proof. Since we only look at M_3 locally, we suppose that M_3 is E^3 and U is M_2. Since Theorem 6 takes care of points of Int M_2, we only show that M_2 is locally tame at a point p of Bd M_2.

Let D be a disk in M_2 such that Bd $D\cdot$Bd M_2 is an arc containing p as a non end point. An argument like that used in the proof of Theorem 3 shows that E^3-D is locally simply connected at each point of D. We finish the proof of Theorem 8 by showing that D is tame.

Since Theorem 6 shows that D is locally tame at each point of Int D, it follows from [4; 9] that there is a homeomorphism h of E^3 onto itself such that $h(D)$ is locally polyhedral at each point of $h(\text{Int } D)$. Hence, we suppose with no loss of generality that D is locally polyhedral at each point of Int D.

By pushing D to one side at points of Int D, one obtains a disk D' such that Bd $D=$Bd D', D' is locally polyhedral at points of Int D', and $D+D'$ bounds a topological cube C. Since E^3-Bd C is locally simply connected at each point of Bd C, it follows from either Theorem 2 or 6 that Bd C is tame. Hence D is tame and M_2 is locally tame at p.

8. **Extensions of Theorem** 0. As pointed out in [5], we can use Theorem 6 to extend Theorem 0 as follows.

THEOREM 9. *A 2-manifold M_2 in a 3-manifold M_3 is locally tame at a point p of M_2 if there is a disk D with $p\in\text{Int } D\subset M_2$ such that for each positive number ϵ, there are disks D', D'' on opposite sides of M_2 such that*

$$H(D, D') < \epsilon, \qquad H(D, D'') < \epsilon.$$

Proof. Theorem 9 follows from Theorem 6 when we show that M_3-M_2 is locally simply connected at each point of Int D.

Suppose E is a disk, q is a point of Int D, and f is a map of Bd E into a small subset of M_3-M_2 near q. Suppose f is extended to map E into a small subset of M_3. Let D', D'' be disks on opposite sides of M_2 which are homeomorphically close to D and whose sum separates $f(\text{Bd } E)$ from $M_2\cdot f(E)$ in $f(E)$. If E' is the component of $E-f^{-1}(D'+D'')$ containing Bd E, f on E' can be extended to map E into a small subset of $f(E')+D'+D''$. Hence, M_3-M_2 is locally simply connected at q.

Since locally tame closed subsets are tame in triangulated 3-manifolds [2; 10], we have the following as a corollary of Theorem 9.

COROLLARY. *A 2-manifold M_2 in a triangulated 3-manifold M_3 is tame if and only if for each positive number ϵ the appropriate one of the following conditions is satisfied*:

CASE 1. *If M_2 is two sided in some neighborhood N of M_2 there are 2-manifolds M', M'' on opposite sides of M_2 in N such that*

$$H(M_2, M') < \epsilon, \qquad H(M_2, M'') < \epsilon.$$

CASE 2. *If M_2 is one sided in each neighborhood of M_2 there is a connected double covering M' of M_2 with projection map π and a homeomorphism h' of M' into M_3-M_2 such that*

$$\rho(h(x), \pi(x)) < \epsilon, \qquad x \in M'.$$

Neither Theorem 9 nor its corollary can be extended to 2-manifolds-with-boundaries. Besides having to speak with care about the two sides of a 2-manifold-with-boundary in a 3-manifold, one would have to contend with the example of Stallings [12] in which he describes an uncountable family of mutually exclusive wild disks in E^3. It would follow from an application of Theorem 9 that most of these disks are locally tame except on their boundaries.

References

1. J. W. Alexander, *An example of a simply connected surface bounding a region which is not simply connected*, Proc. Nat. Acad. Sci. U.S.A. vol. 10 (1924) pp. 8–10.

2. R. H. Bing, *Locally tame sets are tame*, Ann. of Math. vol. 59 (1954) pp. 145–158.

3. ———, *Approximating surfaces with polyhedral ones*, Ann. of Math. vol. 61 (1957) pp. 456–483.

4. ———, *An alternative proof that 3-manifolds can be triangulated*, Ann. of Math. vol. 69 (1959) pp. 37–65.

5. ———, *Conditions under which a surface in E^3 is tame*, Fund. Math. vol. 47 (1959) pp. 105–139.

6. ———, *A wild sphere each of whose arcs is tame*, Duke Math. J.

7. ———, *Side approximations of 2-spheres*, submitted to Annals of Math.

8. R. H. Fox and E. Artin, *Some wild cells and spheres in three-dimensional space*, Ann. of Math. vol. 49 (1948) pp. 979–990.

9. E. E. Moise, *Affine structures in 3-manifolds*. IV. *Piecewise linear approximations of homeomorphisms*, Ann. of Math. vol. 55 (1952) pp. 215–222.

10. ———, *Affine structures in 3-manifolds*, VIII. *Invariance of the knot-type; local tame imbedding*, Ann. of Math. vol. 59 (1954) pp. 159–170.

11. C. D. Papakyriokopoulos, *On Dehn's lemma and the asphericity of knots*, Ann. of Math. vol. 66 (1957) pp. 1–26.

12. J. R. Stallings, *Uncountably many wild disks*, Ann. of Math. vol. 71 (1960) pp. 185–186.

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