EXTREMALS ON COMPACT E-SURFACES

BY

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1. **Introduction.** Compact two-dimensional manifolds without conjugate points have been studied extensively in the Riemannian case and also without the assumption of differentiability hypotheses. The universal covering space $\bar{R}$ of the compact surface R may be realized as a remetrization of the euclidean plane or of the hyperbolic plane, accordingly as the genus of R is one or greater than one. Such a realization is called a Poincaré model of $\bar{R}$.

In the Riemannian case, if the genus of the surface is one, the principal results include that (1) given a euclidean straight line of the model, there is a geodesic of the universal covering space which remains within a fixed distance of the given straight line, (2) a unique geodesic segment lies within a constant distance (fixed for the space) of the euclidean straight line segment joining its endpoints, (3) the metric is flat if no point has a conjugate point. For flatness, the Riemannian character of the metric is essential. In the non-Riemannian case, the parallel axiom holds for the geodesics but the metric need not be Minkowskian which is the non-Riemannian analog of flatness (see Busemann [1; 3] and Green [5]).

If the genus is greater than one, still assuming the Riemannian character of the metric, the principal results include that (1′) given a hyperbolic straight line of the model, there is a geodesic which remains within a fixed distance of the straight line, (2′) a straight geodesic remains within a constant (fixed for the space) distance of some suitable hyperbolic straight line, (3′) the space possesses transitive geodesics.

In this paper we will be guided by the work of Morse and Hedlund (see [6; 7; 8], where additional references will be found) and carry over their principal results (1)–(2), (1′)–(3′) to non-symmetric Finsler spaces, called E-spaces. In our treatment, we make systematic use of the notions of axis and axial motion. The use of these notions, which are in fact implicit in the work of Morse, permits the simplification of several of the proofs.

We start our discussion (§2) by agreeing on matters of notation and establishing several general results. In §3, we prove (1)–(2) for E-spaces. A finite model of the euclidean plane is used here, which has the advantage of displaying more clearly the effect of genus on the behavior of the extremals. We next prove (1′)–(2′) in §4, using the familiar Poincaré model of the universal covering space.

We do not presuppose straightness of the universal covering space (i.e. absence of conjugate points) *throughout the paper*, since it is one of our main

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purposes to prove that *the universal covering space of a compact E-surface which possesses the Divergence Property is straight* (§5). This is a result which is new also for the symmetric G-spaces. In the Riemannian case the converse also holds and it is known that in a general non-Riemannian space straightness need not imply the Divergence Property, see Green [5]. The interesting and apparently difficult question of whether the Divergence Property holds in a straight space which is covering space of a compact E-surface remains open.

Our last section is devoted to the question of the existence of transitive extremals on compact surfaces of higher genus. Under suitable hypotheses on the Poincaré model of the universal covering space, the essential ideas of the original arguments of Nielsen [9] carry over to our case. The main problem here is therefore to find conditions having intrinsic geometrical significance. We discuss conditions equivalent to the Divergence Property in a space with negative curvature and establish the existence of transitive extremals on compact E-surfaces of genus $\gamma \geqq 2$ satisfying these conditions. Busemann had already carried this out for G-spaces (see [1, §34]); what is new here is the removal of the hypothesis that the distance be symmetric.

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2. **Generalities on compact E-surfaces and their universal covering spaces.** Recall that an E-space is a metric space with not necessarily symmetric distance which is finitely compact and satisfies additional hypotheses which guarantee the existence of locally unique extremals. For the precise definition see [10], the terminology and results of which will be freely used.

Let R be a compact, orientable E-surface (i.e. two-dimensional E-space). We denote the points of R by x', y', $\cdots$ and, following the method of [1, §30], we remetrize R as a surface of constant curvature R^*. The metric of R^* is to be locally euclidean if the genus γ of R is one and locally hyperbolic if γ is greater than one. Denote this new distance by $d(x', y')$.

The universal covering space of R^* is then either the euclidean plane E or the hyperbolic plane H which, in either case, will be considered to be realized in the interior D of the unit circle C of a euclidean plane E^2 with distance $\varepsilon(x, y)$. Call it P^* and denote noneuclidean distances in P^* by $d(x, y)$. If the metric of P^* is to be hyperbolic, we use the Poincaré model of H in which the hyperbolic straight lines, which we will call *H-lines*, are realized as the intersections with D of the circles orthogonal to C. If the metric of P^* is to be euclidean, we may obtain a model of E by taking as the euclidean straight lines, which we will call *E-lines*, the intersections with D of the circles (in the sense of $\varepsilon(x, y)$) which pass through pairs of points diametrically opposite on C. That this system of curves yields a model of the euclidean plane is easily seen by mapping E^2 onto D by means of a projection from the center of a sphere of radius 1/2 tangent to E^2 at the origin followed by a stereographic projection.

Denote the elements of the fundamental group $\mathfrak{F}^*$ of R^*, realized as the group of motions of P^* lying over the identity motion of R^* by $\Xi_1 = E$, Ξ_2, Ξ_3, $\cdots$ and the corresponding locally isometric map of P^* on R^* by Ω.

We now introduce a lifted metric xy, by means of Ω^{-1}, in the interior of C which, with this distance, becomes the universal covering space P of our given space R. By [2, (12.18)] the Ξ_k are motions of P and, since $\Xi_k\Omega$ is the identity motion of R, the Ξ_k also represent the fundamental group $\mathfrak{F}$ of R as group of motions of P lying over the identity motion of R. We will refer to this realization of P as a *Poincaré model* of P.

Because R is compact a given class of freely homotopic curves, which are not contractible to a point, contains a shortest curve K which is a closed extremal. We wish to show that the extremals in P lying over K are straight lines and, therefore, that the corresponding motion Ξ in $\mathfrak{F}$ is axial as a motion of P. Since the motions in $\mathfrak{F}$ are orientation preserving it suffices to have the following result:

(2.1) THEOREM. *Let Φ be an orientation preserving motion of an E-plane (i.e. an E-surface homeomorphic to a plane) S and p be a point for which $0 < pp\Phi = \inf_{x\in S} xx\Phi$. Then, if $T = T(p, p\Phi)$ is any segment from p to $p^1 = p\Phi$, the curve $E = \bigcup_{\nu=-\infty}^{\infty} T\Phi^\nu$ is an extremal and, in fact, a straight line. Φ is axial with axis E.*

Proof. Following the proof of Busemann and Pedersen (see [4, (2.4) and (2.6)]) for the symmetric case, we first observe that E is an extremal.

To show that the segment T is unique requires more care here since two extremals may very well intersect without crossing and we do not know in advance that, if S and T are two segments from p to $p\Phi$, $S\cup T$ cannot intersect $(S\cup T)\Phi$. For an indirect proof, let S be a second segment from p to $p^1 = p\Phi$ so that $G = \bigcup_{\nu=-\infty}^{\infty} S\Phi^\nu$ is also an extremal. We know that S and T do not meet except at p, $p\Phi$ so that $S\cup T$ bounds a disc M. Now, $S\cap S\Phi = p^1$. S cannot meet $S\Phi$ in $p = p^2$ since Φ^2 has no fixed points and also not in an interior point x of S. For, if $x\Phi$ follows x on $S\Phi$, then $xx\Phi = pp\Phi - p\Phi x - xp\Phi < pp\Phi$ would deny the definition of p. If x follows $x\Phi$ on $S\Phi$, $pp\Phi \leqq xx\Phi \leqq xp\Phi + p\Phi x\Phi = px + xp\Phi = pp\Phi$ implies $(xx\Phi x\Phi^2)$ (compare [10, (11.1)]). Then, $2xx\Phi = xx\Phi + x\Phi x\Phi^2 = xx\Phi^2 \leqq xp\Phi^2 + p\Phi^2 x\Phi^2 = xp\Phi^2 + px < pp\Phi + pp\Phi$ is again a contradiction. Similarly, S cannot meet $T\Phi$ in an interior point of S.

An argument based on case distinctions now easily shows that either $M \subset M\Phi^2$ or $M\Phi^2 \subset M$ and, therefore, Brouwer's Theorem yields the contradiction that either Φ^2 or Φ^{-2} has a fixed point.

We now consider two cases accordingly as the curve S followed by $T\Phi$ does, or does not, separate points on T, $S\Phi$ near p^1. In the first case, the argument of [10, p. 18] shows that E and G must cross at p^1 which contradicts the hypothesis that Φ preserves orientation. In the second case, since Φ preserves orientation, we must have $M \subset M\Phi$ or $M\Phi \subset M$ and either Φ or Φ^{-1} has a fixed point.

With the uniqueness of the segment $T(p, p\Phi)$ now established, an inductive argument completes the proof as in the symmetric case.

We will need the following form of [1, (34.1)], the proof of which follows from the same arguments as for the symmetric case using the set D of [10, (11.8)]:

(2.2) *Let $\sigma(x, y) = \max(xy, yx)$. Then there exist positive functions f, g, F, G such that*

$$\sigma(x, y) \leqq \lambda \quad \textit{implies that} \quad d(x, y) \leqq F(\lambda),$$
$$d(x, y) \leqq \lambda \quad \textit{implies that} \quad \sigma(x, y) \leqq G(\lambda),$$
$$\sigma(x, y) \geqq \lambda \quad \textit{implies that} \quad d(x, y) \geqq f(\lambda),$$
$$d(x, y) \geqq \lambda \quad \textit{implies that} \quad \sigma(x, y) \geqq g(\lambda),$$

where $d(x, y)$ denotes distance in the sense of P^*.

We will also need the following general proposition:

(2.3) *Let $\{E_\nu^+\}$ be a sequence of straight lines converging to an extremal E^+. Then E^+ is a straight line.*

Proof. If $x_\nu^+(\tau)$ represents E_ν^+, then the betweenness relation $(x_\nu^+(\tau_1)x_\nu^+(\tau_2)x_\nu^+(\tau_3))$ holds for each fixed ν and arbitrary $\tau_1 < \tau_2 < \tau_3$ and this relation must hold in the limit.

3. **The E-torus.** Throughout this section we will assume that R is a compact, orientable E-surface of genus one. We denote the locally euclidean remetrization of R by R_e with distance $d(x', y')$. The universal covering space of R_e is the euclidean plane E with distance $e(x, y)$ realized as the Poincaré model described in §2. The universal covering space of R obtained by remetrization of E is denoted by P with distance xy.

Because an E-line L_e^+ in E is carried by an arc of a circle, a point traversing L_e^+ in the positive [negative] sense tends to a definite point of C which we will call the *positive* [*negative*] *endpoint* of L_e^+. We will also see that a point traversing a straight extremal E^+ of P tends to a point of C which will be called the *positive* [*negative*] *endpoint* of E^+.

(3.1) *Given an oriented euclidean straight line L^+ which is axis of $\Xi_k \neq E$, there exists a straight line E^+ in the sense of P which is axis of Ξ_k. If the euclidean axis L^+ has u_k^-, u_k^+ as positive and negative endpoints, then E^+ also has u_k^-, u_k^+ as positive and negative endpoints. Such a line E^+ can be found passing through each given fundamental region F.*

Proof. Choose a fundamental region F and let p_0 be a point for which $p_0p_0\Xi_k = \sup_{p \in F} pp\Xi_k$. Then $E^+ = \bigcup_{\nu=-\infty}^{\infty} T(p_0\Xi_k^\nu, p_0\Xi_k^{\nu+1})$ is an extremal and, in fact, a straight line by (2.1). Since $T(p_0, p_0\Xi_k) \subset S_e(L^+, \rho)$ for some $\rho > 0$, we have $E^+ \subset S_e(L^+, \rho)$, where $S_e(L^+, \rho) = \{x \mid e(L^+, x) < \rho\}$. Our statement on the endpoints now follows, since the points x with $e(L^+, x) = \rho$ form two convex (in the euclidean sense of the model) curves with u_k^-, u_k^+ as endpoints.

The argument of [1, (34.3)] can be adopted to show:

(3.2) *If two proper subarcs J^- and J^+ of C contain antipodal points p^-, p^+ respectively in their interiors, then there is a $\Xi_k \neq E$ such that the positive endpoint u_k^+ of an axis of Ξ_k lies in J^+ and the negative endpoint u_k^- lies in J^-.*

(3.3) *Any ray R^+ [or R^-] in P possesses an endpoint on C.*

Proof. Let $x(\tau)$, $\tau \geqq 0$ represent R^+. Then $x(\tau)$ tends, for $\tau \to \infty$, to a point of C in the sense of $\varepsilon(x, y)$. If sequences $\tau_\nu \to \infty$, $\tau_\nu' \to \infty$ exist for which $x(\tau_\nu) \to a \in C$ and $x(\tau_\nu') \to a' \neq a$, then there are, by (3.2), antipodal points u_k^-, u_k^+ which separate a and a' on C and are endpoints of an axis E^+ intersecting the segment $T(x(\tau_\nu), x(\tau_\nu'))$ if $\tau_\nu' > \tau_\nu$ or the segment $T(x(\tau_\nu'), x(\tau_\nu))$ if $\tau_\nu > \tau_\nu'$ for all large ν.

(3.4) *A straight line E^+ in the sense of P possesses exactly two endpoints and these are antipodal on C.*

Proof. It is clear from (3.3) that E^+ possesses at least two endpoints. Suppose, for an indirect proof, that u^-, u^+ are two endpoints of E^+ which are not antipodal on C. Then there is a diameter L of C which carries a euclidean axis L^+ of some Ξ_k with the same orientation as E^+ such that both u^-, u^+ lie on the same side of L^+. Now, let F be a fundamental region in the opposite side of E^+ from the endpoints of L^+. Then there is an axis of Ξ_k in the sense of P, intersecting F, having the same endpoints as L^+ and, therefore, intersecting E^+ twice. It now follows that E^+ has at most two endpoints.

(3.5) *Let $p_\nu \to p \in P$, $a_\nu \to a \in C$ and $T(p_\nu, a_\nu)$ converge to a ray R^+. Then R^+ has a as positive endpoint.*

Proof. Assume, for an indirect proof, that R^+ has endpoint $b \neq a$. Let U be a neighborhood $\varepsilon(a, x) < \epsilon$ of a and choose a pair of axes L^+, L^- with the same endpoints and opposite orientations which separate p, p_ν with large ν, and U from b. If the endpoints are chosen to separate b from U on C, it is clear from (3.2), (3.1) that this can be done. Choose τ_0 so large that, if $x^+(\tau)$ represents R^+, $x^+(\tau_0)$ lies near b separated from U, p, p_ν by L^+, L^-.

Next, choose ν so large that $\bar{a}_\nu$ on $T(p_\nu, a_\nu)$ with $p\bar{a}_\nu = \tau_0$ lies near $x^+(\tau_0)$ separated from p, U and also $a_\nu \in U$. But then $T(p_\nu, a_\nu)$ must cross both $L,^+$ L^- at least twice.

(3.6) *Given points $p \in P$ and $a \in C$, there is a ray S^+ with origin p and endpoint a. A coray R^+ to S^+ has endpoint a.*

(3.7) Theorem. *Given two antipodal points u^-, u^+ on C, there is a line E^+ in the sense of P having the given points as endpoints.*

Proof. By (3.2) we may select sequences of endpoints of axes $u_\nu^- \to u^-$, $u_\nu^+ \to u^+$. Choose a definite fundamental region F and denote by E_ν^+ the axis with endpoints u_ν^-, u_ν^+ whose existence is guaranteed by (3.1). A suitable subsequence E_μ^+, $\{\mu\} \subset \{\nu\}$, converges to an extremal E^+, compare [2, (6.6)]. To complete our proof, we can now refer to (2.3).

(3.8) *There is a number $\eta > 0$ such that for any point x the set $S(x, \eta)$ contains a fundamental region.*

Proof. Let $S(y_1, \eta_1)$ contain a fundamental region F. Choose η_2 so that each $S(p, \eta_2)$ contains a point $y_k = y_1\Xi_k$. If $\eta = \eta_1 + \eta_2$, then $S(x, \eta)$ will contain a fundamental region because of the relations

$$S(y_k, \eta_1) \subset S(y_k, \eta - \sigma(x, y_k)) \subset S(x, \eta).$$

Let E^+ be an axis of Ξ in the sense of P with endpoints a^-, a^+. Let L^+ be a euclidean straight line with endpoints a^-, a^+ containing a point p of E^+. L^+ is an axis of Ξ as a motion of E. For Ξ has an axis L_1^+ in the sense of E and this axis has endpoints a^-, a^+ by (3.1). But then L^+ is parallel to L_1^+ since it has the same endpoints and is therefore an axis by [1, (32.13)]. Now let u, v be the points of $T(p, p\Xi)$ on the left and right sides respectively of L^+ and at a maximum distance from L^+. Euclidean parallels T_L, T_R through u, v bound a strip containing E^+ and are also axes of Ξ. The euclidean lines T_L, T_R will be called the *bounding tangents* of the axis E^+.

We now state a fundamental result of Hedlund [6, Theorem VII], the proof of which carries over almost unchanged to our case.

(3.9) THEOREM. *The euclidean distance between the bounding tangents of an axis in the sense of P cannot exceed a constant k, fixed for the space.*

The u- and v-coordinate axes, in a suitable cartesian coordinate system in E, are euclidean axes of motions Ξ_u, Ξ_v. There are axes of Ξ_u, Ξ_v in the sense of P which intersect at a certain point q. Using the segments $T(q, q\Xi_u)$, $T(q, q\Xi_v)$ we may obtain a covering of E by congruent regions bounded by the translates of these segments under the motions of $\mathfrak{F}$. We then apply Hedlund's arguments using these regions.

(3.10) THEOREM. *There are fixed constants β, $\bar\beta$ such that whenever $T^+ = T(p, q)$ is an extremal segment and $S_e^+ = S(p, q)$ is the E-line segment joining its endpoints, then*

(a) $T^+ \subset S_e(S_e^+, \beta)$ *and* (b) $S_e^+ \subset S(T^+, \bar\beta)$.

Proof. T^+ and S_e^+ bound together one or several domains and we start by showing that none of these domains can contain a disc $S_e(p, \xi)$, $\xi = \eta + k + d$, where η, k are the constants of (3.8), (3.9) and $d = \sup_{x,y \in F} e(x, y)$ for a fundamental region F. For if it did, $S_e(p, \eta)$ would contain a fundamental region F and we could find a euclidean axis L_e^+ of a motion Ξ with endpoints so near those of the E-line carrying S_e^+ that the distance from S_e^+ to L_e^+ is not less than $k+d$. Now, let E^+ be an axis of Ξ in the sense of P and passing through F. The bounding tangents of E^+ are parallel to L_e^+ and lie at a euclidean distance from L_e^+ less than $k+d$. Consequently, E^+ must intersect T^+ at least twice.

Now, let $x_e^+(\tau)$ represent the line carrying S_e^+ with $x_e^+(0)=p$ and denote by p_ν the point $x_e^+(2\nu\xi)$. Each disc $S_e(p_{2\nu}, 2\xi)$, $1\leqq\nu\leqq n$, where $n>0$ is the smallest integer such that $e(p, q)<(2n+3)\xi$, contains a point a_ν of T^+. For otherwise, one of the semicircles of the $S_e(p_{2\nu}, 2\xi)$ bounded by S_e^+ would contain a disc $S_e(p, \xi)$ contradicting the first part of this proof. Put $\beta=F[3G(4\xi)]$. Since $T^+=T(p, a_1)\cup\bigcup_{\nu=1}^{n-1} T(a_\nu, a_{\nu+1})\cup T(a_n, q)$, it suffices to show that $T(a_\nu, a_{\nu+1})\subset S_e(S_e^+, \beta)$. $a_\nu, a_{\nu+1}\in S_e(p_{2\nu+1}, 4\xi)$ so that $a_\nu, a_{\nu+1}\in S(p_{2\nu+1}, G(4\xi))$ and, therefore, $T(a_\nu, a_{\nu+1})\subset S(p_{2\nu+1}, 3G(4\xi))\subset S_e(S_e^+, \beta)$.

Put $\bar\beta=G(6\xi)$. For a point x of $T_e(p_{2\nu}, p_{2\nu+2})$, where $T_e(p, q)$ denotes the E-line segment from p to q, we have $e(a_\nu, x)<6\xi$ and, therefore, $\sigma(a_\nu, x)<G(6\xi)$. Consequently, since $S_e^+=\bigcup_{\nu=0}^{n} T_e(p_{2\nu}, p_{2\nu+2})\cup T_e(p_{2n}, q)$, we have $T_e(p_{2\nu}, p_{2\nu+2})\subset S(a_\nu, G(6\xi))\subset S(T^+, \bar\beta)$.

4. E-**surfaces of genus** $\gamma\geqq 2$. In this section we assume that R is a compact, orientable E-surface of genus greater than one and denote the locally hyperbolic remetrization of R by R_h with distance $d(x', y')$. The universal covering space of R_h is the hyperbolic plane with distance $h(x, y)$ realized as Poincaré model. The universal covering space of R, obtained by remetrizing H by means of Ω^{-1}, is denoted by P with distance xy.

It follows from [10, (11.7)] and (2.2), exactly as before, that the endpoints of axes in the senses of P and of H coincide and, therefore, depend only upon the Ξ_k. The proof of [1, (34.3)], since it employs only the properties of H, shows that:

(4.1) *The endpoints of the axes of the Ξ_k are dense on C in the sense that if J_1 and J_2 are any two proper subarcs of C then there is a $\Xi_k\neq E$ which has an axis with positive and negative endpoints lying in J_1 and J_2, respectively.*

Recall that a *diverging pentagon* is a pentagon for which nonadjacent sides do not intersect in P and no two sides have common points on C (see [1, p. 229]). We now construct triples of diverging pentagons with the same endpoints on C: one hyperbolic pentagon π_h and two extremal pentagons π^+, π^- with opposite orientations.

The argument for (3.8) shows that:

(4.2) *There is a number $\eta>0$ such that for any x the set $D(x)=S_h(x, \eta)\cap S(x, \eta)$ contains a triple of diverging pentagons.*

(4.3) *Let the oriented hyperbolic line L_h^+ and the extremal E^+ have the same endpoints a^-, a^+ with $a^-\neq a^+$. Then, $L_h^+\cup E^+$ bounds one or several domains in D. None of these domains can contain a triple of diverging pentagons and, therefore, also not a disc $S(x, \eta)$ or $S_h(x, \eta)$.*

We now come to one of the principal results of Morse [7, Theorem 2]:

(4.4) THEOREM. *There are fixed constants β and $\bar{\beta}$ such that whenever E^+ and L_h^+ are an extremal straight line and a hyperbolic line with the same endpoints a^-, a^+ then*

(a) $E^+ \subset S_h(L_h^+, \beta)$ *and* (b) $L_h^+ \subset S(E^+, \bar{\beta})$.

Proof. Let $x_h^+(\tau)$ represent L_h^+ in terms of hyperbolic arclength and denote by q_ν the point $x_h^+(2\nu\eta)$. Then the discs $S_h(q_{2\nu}, 2\eta)$, $-\infty < \nu < \infty$, are all disjoint and each contains a point a_ν of E^+. For otherwise, one of the hyperbolic semicircles of the $S_h(q_{2\nu}, 2\eta)$ bounded by L_h^+ would lie in a domain bounded by E^+ and L_h^+ and this domain would contain a disc $S_h(x, \eta)$ contradicting (4.3). We use the fact that $E^+ = \bigcup_{\nu=-\infty}^{\infty} T(a_\nu, a_{\nu+1})$ but make no claim as to the order in which the a_ν lie on E^+. Now, put $\beta = F[3G(4\eta)]$. We wish to show that $T(a_\nu, a_{\nu+1}) \subset S_h(L_h^+, \beta)$. Since $a_\nu, a_{\nu+1} \in S_h(q_{2\nu+1}, 4\eta)$ we know that also $a_\nu, a_{\nu+1} \in S(q_{2\nu+1}, G(4\eta))$. But then

$$T(a_\nu, a_{\nu+1}) \subset S(q_{2\nu+1}, 3G(4\eta)) \subset S_h(q_{2\nu+1}, F[3G(4\eta)]) \subset S_h(L_h^+, \beta).$$

For a proof of the second statement, put $\bar{\beta} = G(6\eta)$ and notice that $L_h^+ = \bigcup_{\nu=-\infty}^{\infty} T_h(q_{2\nu}, q_{2\nu+2})$. For a point x of the H-line segment $T_h(q_{2\nu}, q_{2\nu+2})$ we have $h(a_\nu, x) < 6\eta$ and, hence, $\sigma(a_\nu, x) < G(6\eta)$. Consequently $T_h(q_{2\nu}, q_{2\nu+2}) \subset S(a_\nu, G(6\eta)) \subset S(E^+, G(6\eta))$.

(4.5) *Given $\epsilon > 0$, there is a $\delta > 0$ such that if the endpoints a^-, a^+ of E^+ lie in $\varepsilon(p, x) < \delta$, $p \in C$, then E^+ lies in $\varepsilon(p, x) < \epsilon$.*

This is a corollary of (4.4) since the corresponding statement for hyperbolic lines is trivial.

(4.6) *Any ray $x^+(\tau)$, $\tau \geqq 0$, or $x^-(\tau)$, $\tau \leqq 0$, in P possesses an endpoint on C.*

Proof. This means that if $x^+(\tau)$, $\tau \geqq 0$, represents a ray R^+ then the point $x^+(\tau)$ tends for $\tau \to \infty$ to a point a of C in the euclidean sense. If there are two sequences $\tau_\nu \to \infty$, $\tau'_\nu \to \infty$ such that $x^+(\tau_\nu) \to a$, $x^+(\tau'_\nu) \to a' \neq a$, then by (4.1) there is an axis E^+ whose endpoints u^-, u^+ separate a from a' on C. But then $T(x^+(\tau_\nu), x^+(\tau'_\nu))$ would intersect E^+ for all sufficiently large ν.

(4.7) *The endpoints of a straight line in P are distinct.*

Proof. Let E be a straight line whose endpoints coincide at $a \in C$ and choose a point $p \in E$. By (4.5) we can find an axis E_k^+ with endpoints u_k^-, u_k^+ which separates a from p. But then, after perhaps interchanging the roles of u_k^-, u_k^+ and choosing an oppositely oriented axis, E and E_k^+ will intersect at least twice.

(4.8) *Let $p_\nu \to p \in P$, $a_\nu \to a \in C$ and $T(p_\nu, a_\nu)$ converge to the ray S^+. Then S^+ has a as positive endpoint.*

Proof. Assume, for an indirect proof, that S^+ has endpoint $b \neq a$. We take two pairs of axes in the sense of P, A_1^+, A_1^- and A_2^+, A_2^- with endpoints u_1, v_1 and u_2, v_2, respectively. We choose the endpoints so that u_1, v_1 lie in the interior of one and u_2, v_2 in the interior of the other of the two subarcs into which a, b separate C and also so that the intersections H_i of the pairs of half-planes H_i^+, H_i^- bounded by the A_i^+, A_i^- and not containing a, b also do not contain p. We can do this because of (4.5). Now let $\rho > \max(pH_1, pH_2)$ and let U be a neighborhood $\varepsilon(a, x) < \delta$ of a which is disjoint from $S'(p, \rho+1)$ and also from $\overline{H}_1$, $\overline{H}_2$.

Since S^+ has endpoint b and $T(p_\nu, a_\nu) \to S^+$, we can choose $\tau_0 > \rho+1$ so large that, if $x^+(\tau)$ represents S^+, $x^+(\tau_0)$ lies outside of $S'(p, \rho+1)$, $\overline{H}_1$, $\overline{H}_2$, and U and in fact close to b. We now choose ν so large that $p_\nu p < 1$, $a_\nu \in U$, $p_\nu a_\nu > \tau_0$ and the point $\bar{a}_\nu$ of $T(p_\nu, a_\nu)$ with $p\bar{a}_\nu = \tau_0$ therefore lies close to $x^+(\tau_0)$.

Now, the part of $T(p_\nu, a_\nu)$ following $\bar{a}_\nu$ lies outside $S'(p_\nu, \rho+1) \supset S'(p, \rho)$. Since it connects a_ν to $\bar{a}_\nu$ it must intersect one of the A_i^+, A_i^- at least twice.

We will need the following corollary to (4.8):

(4.9) *Given points $p \in P$ and $a \in C$, there is a ray S^+ with origin p and endpoint a. A coray R^+ to S^+ ends at a.*

For a coray R^+ to S^+ is by definition a ray R^+ obtainable as a limit of segments $T(q_\nu, a_\nu)$ with $q_\nu \to q$ and $a_\nu \in S^+$, $pa_\nu \to \infty$, i.e. $a_\nu \to a \in C$.

(4.10) THEOREM. *Given two distinct points a^-, a^+ of C, there is a line E^+ in the sense of P with a^-, a^+ as endpoints.*

5. **The Divergence Property and straightness of the universal covering space.** An E-space, *not assumed to be straight here*, will be said to possess the *Divergence Property* if, for any two distinct rays R^+, S^+ with the same origin, the distance $x^+(\tau)S^+$ (where $x^+(\tau)$ represents R^+) tends, for $\tau \to +\infty$, to infinity.

(5.1) *Let the Divergence Property hold in P and assume that R^+, S^+ are two distinct rays with origin p. Then each of the domains in P bounded by $R^+ \cup S^+$ contains discs $S(x, \eta)$ with arbitrarily large η.*

Proof. Let D be one of the domains. Let $\{C_\nu\}$ be a sequence of curves in D from $r_\nu \in R^+$ to $s_\nu \in S^+$, with $pr_\nu = ps_\nu = \nu$, such that C_ν tends to a (proper or improper) subarc of C. Then each C_ν contains a point q_ν such that $R^+q_\nu = q_\nu S^+$. Let f_ν be a foot on R^+ toward q_ν and select a subsequence $\{\mu\} \subset \{\nu\}$ for which either f_μ converges to a point f or $pf_\mu \to \infty$. If $pf_\mu < M$, then $pq_\mu \leq pf_\mu + f_\mu q_\mu$ shows that $f_\mu q_\mu \to \infty$, since $pq_\mu \to \infty$. If $pf_\mu \to \infty$ then $f_\mu S^+ \to \infty$ by hypothesis.

Now, let g_μ denote a foot on S^+ from q_μ. Then $f_\mu S^+ \leqq f_\mu g_\mu \leqq f_\mu q_\mu + q_\mu g_\mu = 2R^+q_\mu = 2q_\mu S^+$ shows that both $R^+q_\mu \to \infty$ and $q_\mu S^+ \to \infty$. The assertion follows now since $S(q_\mu, \rho) \subset D$ for $\rho = \min\{\sigma^*(q_\mu, s_\mu), \sigma^*(q_\mu, r_\mu)\}$.

(5.2) *If the Divergence Property holds in P, then the ray from a given point $p \in P$ to a given point $a \in C$ is unique.*

Proof. *Case* 1. $\gamma = 1$. Let R^+, S^+ be two distinct rays with origin p and endpoint $a^+ \in C$. There is a number $\eta > 0$ (compare (4.2)) such that for any x the sphere $S(x, \eta)$ contains the union of four fundamental regions with a common boundary point. By (5.1), the domain bounded by $R^+ \cup S^+$ contains the union $F = \bigcup_{i=1}^4 F_i$, of four such fundamental regions.

Consider a sequence of positive endpoints of axes $\{a_\nu^+\}$ such that $a_\nu^+ \to a^+$ and a^+, a^- separate a_ν^+, $a_{\nu+1}^+$ on C for all ν. Here, a^- denotes the antipodal point of a^+ on C. Then, for a suitable subsequence $\{\mu\}$ of $\{\nu\}$, there is a sequence $\{E_\mu^{1+}\}$ of axes in the sense of P, meeting the fundamental region F_1 and converging to a straight line E^{1+} in P with positive endpoint a^+.

If E^{1+} intersects $R^+ \cup S^+$ in a point $q \neq p$, then E_μ^{1+} will intersect R^+ or S^+ in a point distinct from p for all sufficiently large μ. Therefore, E_μ^{1+} will intersect R^+ or S^+ twice for some large μ.

If E^{1+} meets $R^+ \cup S^+$ only in p, we choose a second fundamental region F_2 and consider a sequence of axes $\{E_\nu^{2+}\}$, in the sense of P, meeting F_2 and having the same sequence $\{a_\nu^+\}$ of positive endpoints. Again, a suitable subsequence $\{E_\mu^{2+}\}$ converges to a straight line E^{2+}. Clearly, we can select F_2 so that $E^{2+} \neq E^{1+}$. Now, if E^{2+} intersects $R^+ \cup S^+$ in a point $q \neq p$, we are finished. Otherwise, an easy continuity argument, based on case distinctions, shows that E_μ^{2+} must intersect E^{1+} twice for some large μ.

Case 2. $\gamma \geqq 2$. This is a corollary of (4.3) and (5.1).

(5.3) THEOREM. *If the Divergence Property holds in the universal covering space P of a compact orientable E-surface, then P is straight.*

Proof. We show that given any two distinct points $p, q \in P$, there exists a ray with origin p containing q. For otherwise, there would be a point $b \in C$ such that if X_ν^+, Y_ν^+ are rays with origin p and endpoints a_ν, c_ν near b, with a_ν, b, c_ν following in that order on C, then q lies in the interior of a domain bounded by X_ν^+, Y_ν^+ and an arc of C. If $a_\nu \to b$, $c_\nu \to b$ the rays X_ν^+, Y_ν^+ converge to distinct rays X^+, Y^+ with origin p and endpoint b contradicting (5.2).

We state two further consequences of the Divergence Property.

(5.4) *If the Divergence Property holds, then the asymptote relation is symmetric and transitive.*

(5.5) *If the Divergence Property holds, then two lines are parallel if, and only if, they have the same endpoints.*

6. **Transitive extremals on E-surfaces.** The existence of transitive geodesics has attracted much attention in part because of the relevance of this question for dynamics and ergodic theory, but also because of its intrinsic geometric interest. The question has been treated by many mathematicians including Artin, G. D. Birkhoff, Busemann, L. W. Green, Hedlund, E. Hopf, Koebe, Löbell, Morse and Nielsen through whose work the existence of transitive geodesics is now established for a very wide class of surfaces.

Following Busemann's proof for G-spaces [1, (34.13)], we complete these results by giving a version of Nielsen's Theorem for the case of nonsymmetric distance. In terms of the Poincaré model of the universal covering space, it is not difficult to formulate sufficient conditions for the existence of transitive extremals in our case. It suffices, in order to use these original arguments in the form of (6.4), that (1) an axis varies continuously with its endpoints and (2) the pairs of endpoints of axes be dense on $C\times C$.

We begin by considering conditions equivalent to (1) in the case that R is a compact, orientable E-surface of genus $\gamma>1$.

(6.1) *The endpoints of a straight line, in a Poincaré model of P, determine the line uniquely if, and only if, one of the following conditions is satisfied:*

C1. *P has the Divergence Property and there are no pairs of parallel lines.*

C2. *A domain bounded by two nonintersecting, equally oriented straight lines contains discs of arbitrarily large radii.*

Proof. (5.5) shows that C1 is sufficient. By (4.3), C2 is sufficient.

C2 *is necessary*. For, if no two distinct equally oriented lines have the same endpoints, then two nonintersecting equally oriented lines have corresponding endpoints a, a' which are distinct. Hence, we can find an axis A^+ between them which bounds a half-plane between the lines and this half-plane contains discs of arbitrarily large radii.

C1 *is necessary*. Assume that rays X^+, Y^+ with origin p do not satisfy the Divergence Property. Then there is a sequence $\{x_\nu\}$, $x_\nu\in X^+$ and $px_\nu\to\infty$ while $x_\nu Y^+<M<\infty$. If y_ν is a foot on Y^+ from x_ν, then $px_\nu\leqq py_\nu+y_\nu x_\nu$ shows $py_\nu\to\infty$ since $x_\nu y_\nu$ is bounded if, and only if, $y_\nu x_\nu$ is bounded. Consequently X^+, Y^+ have the same positive endpoint $a^+\in C$.

Denote by B the closed domain of E^2 bounded by X^+ and Y^+. Orient C and take a sequence $\{u_\nu^+\}$ of points following a^+ on C and such that $u_\nu^+\to a^+$. For all sufficiently large ν, we construct a line L_ν^+ with u_ν^+ as positive endpoint and containing p. This can be done by considering the line L with positive endpoint u_ν^+ and negative endpoint u^- following a^+, u_ν^+ on C. If u^- is sufficiently close to u_ν^+, L does not meet B. Now, let u^- traverse C in the positive sense. There is a first position of u^-, namely u_ν^-, for which the line $L_\nu^+ =L^*(u_\nu^-, u_\nu^+)$ meets B. $L\cap B$ obviously contains no point of Y^+ other than p.

Since $p\in L_\nu^+$ for each ν, L_μ^+ converges for a suitable subsequence $\{\mu\}\subset\{\nu\}$, to a straight line L^+ with negative endpoint a^-. Call the side of L^+ on which B lies σ and take an axis A^+ in σ with the same orientation as L^+ and with endpoints s^-, s^+. If a_ν^-, a_ν^+ are endpoints of an axis H_ν^+ such that a_ν^- lies between s^-, a^- and a_ν^+ lies between s^+, a^+ and also $a_\nu^-\to a^-$, $a_\nu^+\to a^+$, then H_μ^+ for a suitable subsequence converges to a line with endpoints a^-, a^+ which must, by hypothesis, be L^+. But B lies for large μ between H_μ^+ and L^+.

Finally, it is clear from (5.5) that there are no pairs of parallel lines.

If the conditions of (6.1) are satisfied, an extremal depends continuously on its endpoints. This means that if $\epsilon > 0$, $N > 0$ are given and $x(\tau)$ represents an extremal E^+ with endpoints u^-, u^+ then there exist intervals J^-, J^+ on C such that any extremal having its negative endpoint in J^- and its positive endpoint in J^+ has a representation $y(\tau)$ for which

$$\sigma(x(\tau), y(\tau)) < \epsilon \quad \text{if} \quad |\tau| < N.$$

From (4.1) we see that, under the conditions of (6.1), the axes of the Ξ_k are dense among all extremals in P and, therefore:

(6.2) *The closed extremals in R are dense among all extremals in R if P satisfies the conditions of* (6.1).

We need the following fact concerning the mappings of C induced by the Ξ_k.

(6.3) *If J_1 and J_2 are two proper subarcs of C, then there is a Ξ_k for which $J_2\Xi_k$ lies in the interior of J_1.*

(6.4) THEOREM. *Let R be a compact E-surface of genus greater than one whose universal covering space satisfies the conditions of* (6.1). *Then R possesses a transitive extremal, i.e. an extremal with a representation $y'(\tau)$ having the property*:

Let an extremal curve $x'(\tau)$, $0 \leqq \tau \leqq \lambda$, and numbers $\epsilon > 0$, $N > 0$ be given. Then there is a number $\alpha = \alpha(x', \epsilon, N) > N$ such that $\sigma(y'(\tau), x'(\tau - \alpha)) < \epsilon$ for $\alpha \leqq \tau \leqq \alpha + \lambda$.

Proof. In the proof, we may assume that R is orientable. For a nonorientable compact surface of genus $\gamma > 2$ has a compact orientable surface of genus $\gamma > 1$ as covering space and the theorem for this space yields our theorem for the given surface.

Denote the axis of Ξ_i, $i > 1$, in the sense of P by G_i^+ and represent it by $x_i(\tau)$. If $\gamma(\Xi_i) = \min_{x \in P} xx\Xi_i$, denote the segment $x_i(\tau)$, $0 \leqq \tau \leqq \gamma(\Xi_i)$ of G_i^+ by T_i. There are disjoint arcs B_i^- and B_i^+ on C, containing the endpoints u_i^-, u_i^+ of G_i^+, such that any extremal K^+ from a point of B_i^- to a point of B_i^+ has a representation $z(\tau)$ which satisfies $\sigma(x_i(\tau), z(\tau)) < 1/i$ for $0 \leqq \tau \leqq \gamma(\Xi_i)$. In this case we say that K^+ approximates T_i within $1/i$.

We now choose $\Xi_2' = \Xi_2, \Xi_3', \Xi_4', \cdots$ in $\mathfrak{F}$ so that an extremal from a point of B_2^- to a point of $B_i^+\Xi_i'$ approximates $T_i\Xi_i'$ within $1/i$. If we put $B_{i+1} = C - \mathrm{Cl}[B_i^-]$, then we choose Ξ_{i+1}' by (6.3) so that it carries B_{i+1} and, therefore, also B_{i+1}^+ into the interior of B_i^+. $B_i^-\Xi_i'$ will then contain B_2^- in its interior. The arcs $B_i^+\Xi_i'$ will shrink to a point e^+.

Any extremal H^+ from a point e^- in B_2^- to e^+ will satisfy the assertion. Let the segment T^+ over the given extremal curve be represented by $x(\tau)$, $0 \leqq \tau \leqq \lambda$, and let $\epsilon > 0$ and $N > 0$ be given. Because of (6.2) there is a G_j^+ with a representation $v(\tau)$ such that $\sigma(v(\tau), x(\tau)) < \epsilon/2$ for $0 \leqq \tau \leqq \lambda$ or

$\sigma(v(\tau-\alpha), x(\tau-\alpha))<\epsilon/2$ for $\alpha\leqq\tau\leqq\alpha+\lambda$. Denote by S^+ the segment $v(\tau)$, $0\leqq\tau\leqq\lambda$.

The powers Ξ_j^k occur among the Ξ_ν. Put $\Xi_{i_k}=\Xi_j^k$ so that $i_k\to\infty$ and $\gamma(\Xi_{i_k})=k\cdot\gamma(\Xi_j)\to\infty$. Consequently, for sufficiently large k and suitable n satisfying the relation $k\cdot\gamma(\Xi_j)>\lambda+n\cdot\gamma(\Xi_j)$, the segments $S^+\Xi_j^n$ will be contained in T_{i_k}. Now, $S^+\Xi_j^n\Xi'_{i_k}=S^+\Xi_m$ is approximated within $1/i_k$ by H^+ because $T_{i_k}\Xi'_{i_k}$ is. This means that there is a representation $z(\tau)$ of $S^+\Xi_m$ such that $\sigma(y(\tau), z(\tau))<1/i_k$ for $\alpha_k\leqq\tau\leqq\alpha_k+\lambda$. As in the proof for the symmetric case, we may choose k so large that both $1/i_k<\epsilon/2$ and $\alpha_k>N$.

Now, $v(\tau-\alpha)\Xi_m=z(\tau)$, $\alpha\leqq\tau\leqq\alpha+\lambda$, since $v(\tau)\Xi_m$ represents $S^+\Xi_m$. Consequently, we have

$$\begin{aligned}\sigma(y(\tau), x(\tau-\alpha)\Xi_m) &\leqq \sigma(y(\tau), v(\tau-\alpha)\Xi_m)+\sigma(v(\tau-\alpha)\Xi_m, x(\tau-\alpha)\Xi_m)\\ &= \sigma(y(\tau), v(\tau-\alpha)\Xi_m)+\sigma(v(\tau-\alpha), x(\tau-\alpha))<\epsilon\end{aligned}$$

for $N<\alpha\leqq\tau\leqq\alpha+\lambda$.

Because Ξ_m lies over the identity motion of R, this implies that the image $H^+\Omega$ of H^+ in R satisfies our assertion.

Because of [10, (7.5), (16.11), (16.13)] and (6.4) we also have:

(6.5) THEOREM. *On a compact E-surface with negative curvature there is an extremal* $y(\tau)$ *with the property: Given any extremal curve* $x(\tau)$, $0\leqq\tau\leqq\lambda$, *any* $\epsilon>0$ *and any* $N>0$, *there exists an* $\alpha=\alpha(x, \epsilon, N)>N$ *such that* $\sigma(y(\tau), x(\tau-\alpha))<\epsilon$ *for* $\alpha\leqq\tau\leqq\alpha+\lambda$.

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