CONSTRUCTION OF AUTOMORPHIC FORMS ON *H*-GROUPS AND SUPPLEMENTARY FOURIER SERIES(1)

BY

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I. **Introduction.** 1. In several previous papers [1; 3], the author has discussed the problem of deriving the transformation properties of automorphic forms directly from their Fourier series expansions. The automorphic forms considered in [1] and [3] are entire (that is, analytic functions regular in the upper half plane), of integral dimension $r \geqq 0$, and connected with the modular group and several other closely related groups.

In the present paper these considerations are extended to entire automorphic forms, of integral dimension $r > 0$, connected with a rather wide class of groups which Lehner [5] has called *H*-groups. The Fourier series expansions are given in [5] for such automorphic forms, *but without the assumption that r is an integer.*

In §III we consider these Fourier series in the case when *r is a positive integer*, and show that although not all such series represent automorphic forms, they do all represent functions with transformation properties very much like the transformation properties of automorphic forms (see Theorem (3.3)). Using this result it is then a simple matter to construct entire automorphic forms of all positive integral dimensions.

In §IV we introduce new Fourier series which we called the *supplementary Fourier series* to the series discussed in §III and we obtain the following result (Theorem (4.9)).

A Fourier series of the type of §III *is an automorphic form if and only if its supplementary series reduces to a constant.*

We also show that these supplementary series are related to "expansions of zero," of the type obtained by Rademacher [8] in connection with the partition function.

2. A group Γ of linear fractional transformations of a complex variable τ is an *H-group* provided Γ satisfies the following conditions (see [6]).

(i) Γ is discontinuous in the half plane $\mathscr{I}(\tau) > 0$, but is not discontinuous at any point of the real axis.

(ii) Every transformation of Γ preserves the half plane $\mathscr{I}(\tau) > 0$.

(iii) Γ contains translations.

(iv) Γ is finitely generated.

Received by the editors June 12, 1961.

(1) Research supported in part by National Science Foundation Grant No. G-14362.

For the sake of simplicity we follow the practice of [5] in imposing the additional condition that *all of the parabolic cusps of* Γ *are equivalent* to ∞. This restriction is not essential; we could drop the restriction by considering the Fourier series given in [6] rather than those given in [5].

With the transformation $V \in \Gamma$ given by

$$V\tau = \frac{a\tau + b}{c\tau + d}$$

we associate the two matrices

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad -V = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix},$$

which we can assume to be unimodular. Γ can be thought of as a matrix group provided V and $-V$ are identified. Notice that a, b, c and d can be chosen real because of (ii).

Let r be real and let $F(\tau)$ be an analytic function, regular in $\mathscr{I}(\tau) > 0$, which there satisfies the functional equation

$$F(V\tau) = \epsilon(V)(-i(c\tau + d))^{-r}F(\tau), \tag{1.1}$$

for every

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Here $\epsilon(V)$ is a complex number independent of τ such that $|\epsilon(V)| = 1$ for all $V \in \Gamma$. Let

$$S = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \qquad \lambda > 0,$$

generate the group of translations of Γ. Then by (1.1) we have

$$F(S\tau) = \epsilon(S)(-i)^{-r}F(\tau) = e^{2\pi i\kappa}F(\tau), \tag{1.2}$$

where we choose κ so that $0 \leq \kappa < 1$. It follows from (1.2) that $F(\tau)$ has a Fourier expansion

$$F(\tau) = \sum_{m=-\infty}^{\infty} a_m \exp\left\{\frac{2\pi i}{\lambda}(m + \kappa)\tau\right\}.$$

If in addition this expansion has only a finite number of terms with negative exponents, that is, if

$$F(\tau) = \sum_{m=-\mu}^{\infty} a_m \exp\left\{\frac{2\pi i}{\lambda}(m + \kappa)\tau\right\},$$

then we say that $F(\tau)$ is an *entire automorphic form of dimension* r *on* Γ.

When such a function F exists, it follows that if

$$V_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma, \qquad V_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \Gamma,$$

then

$$\epsilon(V_1V_2)(-i(c_3\tau + d_3))^{-r} = \epsilon(V_1)\epsilon(V_2)(-i(c_1V_2\tau + d_1))^{-r}(-i(c_2\tau + d_2))^{-r}, \tag{1.3}$$

where

$$V_1V_2 = \begin{pmatrix} * & * \\ c_3 & d_3 \end{pmatrix}.$$

Also, since we have identified V and $-V$,

$$\epsilon(-V)(-i(-c\tau - d))^{-r} = \epsilon(V)(-i(c\tau + d))^{-r}, \tag{1.4}$$

where

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is an immediate consequence of (1.3) that

$$\epsilon(VS^q) = \epsilon(S^qV) = e^{2\pi i q\kappa}\epsilon(V), \tag{1.5}$$

for q an integer. A function $\epsilon(V)$ on Γ is said to form a *multiplier system for* Γ *corresponding to the dimension* r, provided $\epsilon(V)$ is complex-valued, $|\epsilon(V)| = 1$ for all $V \in \Gamma$, and $\epsilon(V)$ satisfies (1.3).

We will denote the vector space of automorphic forms of dimension r, with multiplier system ϵ, on Γ by $\{\Gamma, r, \epsilon\}$.

II. **Some preliminaries.** 1. In this section we state some results of which we will have need later. Proofs will be omitted since these results are all either well known or fairly straightforward generalizations of results given in [1; 2; 3].

We begin by introducing some of the notation employed in [5, §2]. By C we denote the set of lower left-hand entries in the matrices of Γ. That is,

$$C = \left\{ c \mid \exists \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \Gamma \right\}.$$

Similarly

$$D = \left\{ d \mid \exists \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in \Gamma \right\}.$$

Also, we put

$$D_c = \left\{ d \,\middle|\, \exists \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma, \text{ with } 0 \leqq -d < c\lambda \right\}$$

and

$$D^c = \left\{ d \,\middle|\, \exists \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma \right\}.$$

Note that D_c is empty if $c \leqq 0$.

Following [5, pp. 268–269] we regard Γ as being fixed and normalized in such a manner that $|c| \geqq 1$ if $c \neq 0$. Let r be a *positive integer*. This assumption on r will be made throughout the remainder of the paper. Suppose $\epsilon(V)$ is a multiplier system for Γ corresponding to the dimension r. Let $V_{c,d} \in \Gamma$ be given by

$$V_{c,d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and assume that ν is a positive integer. With m a nonnegative integer put

$$a_m(\nu, r, \epsilon) = \frac{2\pi}{\lambda} \sum_{c \in C;\, c>0} c^{-1} A_{c,\nu}(m, \epsilon) L_c(m, \nu, r, \kappa), \tag{2.1}$$

where

$$A_{c,\nu}(m, \epsilon) = \sum_{d \in D_c} \epsilon^{-1}(V_{c,d}) \exp\left[\frac{2\pi i}{c\lambda} \{ (m + \kappa)d - (\nu - \kappa)a \} \right], \tag{2.2}$$

and

$$\begin{aligned} L_c(m, \nu, r, \kappa) &= \left(\frac{\nu - \kappa}{m + \kappa} \right)^{(r+1)/2} I_{r+1}\left[\frac{4\pi}{c\lambda} (\nu - \kappa)^{1/2} (m + \kappa)^{1/2} \right], && \\ & && \text{if } m + \kappa > 0, \\ &= \left(\frac{2\pi\nu}{c\lambda} \right)^{r+1} \frac{1}{(r+1)!}, && \text{if } m = \kappa = 0. \end{aligned} \tag{2.3}$$

Here κ is connected with $\epsilon(V)$ and defined by (1.2). Also, I_{r+1} is the modified Bessel function of the first kind which can be defined by means of the power series

$$I_{r+1}(x) = \sum_{p=0}^{\infty} \frac{(x/2)^{p+r+1}}{p!(p + r + 1)!}, \tag{2.4}$$

since r is an integer. From the definition it appears that $A_{c,\nu}(m, \epsilon)$ depends upon the choice of a and b in $V_{c,d}$. Actually it follows in a simple way from (1.5) that

$$\epsilon^{-1}(V_{c,d}) \exp\left[\frac{2\pi i}{c\lambda}(\kappa - \nu)\, a\right]$$

is independent of this choice, as long as a and b are chosen so that $V_{c,d} \in \Gamma$.

2. We will make use of a result due to Poincaré which can be found in [6, p. 191]. Namely, if $r > 0$, then

$$\sum_{c \in C} \sum_{d \in D_c} |c\tau + d|^{-2-r}$$

converges. Now if $c > 0$ and $d \in D_c$, then $|d| = -d < c\lambda$ and therefore

$$|c\tau + d| \leqq c|\tau| + |d| < c(|\tau| + \lambda).$$

Hence

$$\sum_{c \in C;\, c>0} \sum_{d \in D_c} c^{-r-2} < \sum_{c \in C;\, c>0} \sum_{d \in D_c} \frac{(|\tau| + \lambda)^{r+2}}{|c\tau + d|^{r+2}},$$

and we may conclude that

$$\sum_{c \in C;\, c>0} \sum_{d \in D_c} c^{-r-2}$$

converges. The following result is a straightforward consequence of this fact.

LEMMA (2.5). *If $A_{c,\nu}(m, \epsilon)$ is defined and $r > 0$, then the sum*

$$\sum_{c \in C;\, c>0} |A_{c,\nu}(m, \epsilon)|\, c^{-r-2}$$

converges.

Making use of this lemma, we can modify the method of [1, pp. 278–279] to obtain

LEMMA (2.6). *Let $a_m(\nu, r, \epsilon)$ be defined by* (2.1). *Then, as $m \to \infty$,*

$$a_m(\nu, r, \epsilon) = O\left((m + \kappa)^{-3/4-r/2} \exp\left\{\frac{4\pi}{\lambda}(\nu - \kappa)^{1/2}(m + \kappa)^{1/2}\right\}\right). \tag{2.7}$$

REMARKS. It should be pointed out that in order to obtain (2.7) in this form we have to make use of the previously mentioned normalization of Γ. This implies that the smallest value of c occurring in the infinite sum (2.1) is $c = 1$. Another proof of this lemma is given in [6, pp. 190–192].

Some of our succeeding calculations and results will be split into the cases $\kappa > 0$ and $\kappa = 0$. We now state the Lipschitz summation formula [7] for these two cases. If $0 < \kappa < 1$, $\Re(t) > 0$, and $p > -1$, then

$$\sum_{n=0}^{\infty} (n + \kappa)^p e^{-2\pi t(n+\kappa)} = \frac{\Gamma(p + 1)}{(2\pi)^{p+1}} \sum_{q=-\infty}^{\infty} e^{2\pi i q\kappa}(t + qi)^{-p-1}. \tag{2.8}$$

For $\kappa=0$ the formula is similar; we have

$$\begin{aligned}\sum_{n=1}^{\infty} n^p e^{-2\pi t n} &= \frac{\Gamma(p+1)}{(2\pi)^{p+1}} \sum_{q=-\infty}^{\infty} (t+qi)^{-p-1}, && \text{for } p>0, \\ &= -\frac{1}{2} + \frac{1}{2\pi} \lim_{N\to\infty} \sum_{q=-N}^{N} (t+qi)^{-1}, && \text{for } p=0\end{aligned} \tag{2.9}$$

where $\Re(t)>0$, and Γ here denotes the gamma function.

The principal analytic tool in [1; 2; 3] is a lemma in which the terms of a certain conditionally convergent double series are rearranged. The original result in this direction is due to Rademacher [9]. We state below, without proof, the several versions of this lemma that we shall use here.

LEMMA (2.10). *Let $\tau=iy$, with $y>0$, and let r and ν be positive integers. Let*

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$$

with

$$\alpha < 0, \qquad \beta < 0, \qquad \delta > \gamma > 0, \tag{2.11}$$

and put $t=(\gamma-\beta/2)\delta^{-1}>0$. Let $\mathfrak{J}_V(K)$ be the trapezoid in the c-d plane bounded by the lines

$$c = 0, \qquad \alpha c + \gamma d = tK, \qquad \delta d + \beta c = \pm K.$$

Then,

$$\begin{aligned}&\sum_{c\in C;\, c>0} \lim_{N\to\infty} \sum_{d\in D^c;\, |d|\le N} \frac{\epsilon^{-1}(V_{c,d}) \exp\{-2\pi i(\nu-\kappa)a/c\lambda\}}{c^{r+1}(c\tau+d)} \\ &\quad = \lim_{K\to\infty} \sum_{c\in C;\, c>0} \sum_{d\in D^c;\, (c,d)\in \mathfrak{J}_V(K)} \frac{\epsilon^{-1}(V_{c,d}) \exp\{-2\pi i(\nu-\kappa)a/c\lambda\}}{c^{r+1}(c\tau+d)}.\end{aligned} \tag{2.12}$$

LEMMA (2.13). *Let τ, y, r and ν be as above. Let ρ be any positive real number. Then,*

$$\begin{aligned}&\sum_{c\in C;\, c>0} \lim_{N\to\infty} \sum_{d\in D^c;\, |d|\le N} \frac{\epsilon^{-1}(V_{c,d}) \exp\{-2\pi i(\nu-\kappa)a/c\lambda\}}{c^{r+1}(c\tau+d)} \\ &\quad = \lim_{K\to\infty} \sum_{c\in C;\, 0<c\le\rho K} \sum_{d\in D^c;\, |d|\le K} \frac{\epsilon^{-1}(V_{c,d}) \exp\{-2\pi i(\nu-\kappa)a/c\lambda\}}{c^{r+1}(c\tau+d)}.\end{aligned} \tag{2.14}$$

REMARKS. The proofs of these lemmas depend in an essential way upon Lemma (2.5). In the case when Γ is the modular group, the exponential sum $A_{c,\nu}(m, \epsilon)$ can be reduced to the classical Kloosterman sum and as a result can be shown to have the nontrivial estimate, $A_{c,\nu}(m, \epsilon)=O(c^{2/3+\epsilon})$, as $c\to\infty$.

In the case of the modular group, therefore, Lemma (2.5) holds when $r=0$ and, as a matter of fact, all of our results carry through for $r=0$ (cf. [1]). However, when Γ is a general H-group no such estimate is known for the exponential sums that occur and our theory is restricted to $r>0$.

When $\kappa>0$, the inner summation on the left-hand sides of both (2.12) and (2.14) can be replaced by $\sum_{-\infty;d\in D^c}^{\infty}$. This follows from the fact that the Lipschitz summation formula, which is used to prove the convergence of the left-hand side, has the form (2.8) when $\kappa>0$.

III. **Construction of automorphic forms.** 1. In [5] it is shown that if $F(\tau)\in\{\Gamma, r, \epsilon\}$ with $r>0$, then the Fourier coefficients in the expansion

$$F(\tau) = \sum_{\nu=1}^{\mu} b_\nu e^{2\pi i(-\nu+\kappa)\tau/\lambda} + \sum_{m=0}^{\infty} a_m e^{2\pi i(m+\kappa)\tau/\lambda} \tag{3.1}$$

are given by

$$a_m = \sum_{\nu=1}^{\mu} b_\nu a_m(\nu, r, \epsilon), \qquad \text{for } m \geqq 0, \tag{3.2}$$

where $a_m(\nu, r, \epsilon)$ is defined in (2.1). For convenience we have relabelled the coefficients of the terms with negative exponents. The converse to this result does not hold. If we *define* $F(\tau)$ by means of (3.1)–(3.2) with arbitrary $\mu\geqq 1$, $r>0$, and $b_1, \cdots, b_\mu$, $F(\tau)$ may or may not be in $\{\Gamma, r, \epsilon\}$. Even under the assumption that r is an integer, the converse is not true. However in this case we obtain the following "weak converse."

THEOREM (3.3). *Let $F(\tau)$ be defined by means of* (3.1)–(3.2) *with r and μ positive integers and $b_1, \cdots, b_\mu$ any constants. Then, in $\mathfrak{I}(\tau)>0$, $F(\tau)$ is regular and satisfies*

$$F(\tau) - \epsilon^{-1}(V)(-i(\gamma\tau+\delta))^r F(V\tau) = p_V(\tau), \tag{3.4}$$

for all

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma,$$

where $p_V(\tau)$ is a polynomial in τ of degree at most r.

REMARK. It is clear from (1.1) that $F(\tau)\in\{\Gamma, r, \epsilon\}$ if and only if $p_V(\tau)\equiv 0$, for all $V\in\Gamma$. We will make use of this later in constructing automorphic forms on Γ.

Instead of proving (3.4) directly, we will first recast it slightly. With ν a positive integer let $F_\nu(\tau)$ be defined by

$$F_\nu(\tau) = e^{2\pi i(-\nu+\kappa)\tau/\lambda} + \sum_{m=0}^{\infty} a_m(\nu, r, \epsilon) e^{2\pi i(m+\kappa)\tau/\lambda}. \tag{3.5}$$

With $F(\tau)$ defined by (3.1), we have $F(\tau)=\sum_{\nu=1}^{\mu} b_\nu F_\nu(\tau)$, and therefore it is sufficient to prove Theorem (3.3) for $F_\nu(\tau)$. Lemma (2.6) shows that $F_\nu(\tau)$ is regular for $\mathfrak{I}(\tau)>0$, so that if we derive (3.4) for $F_\nu(\tau)$ with τ on the positive imaginary axis, the result follows immediately for $\mathfrak{I}(\tau)>0$ by analytic continuation. Hence in what follows we assume that $\tau=iy$, with $y>0$.

2. In order to avoid lengthy repetition of calculations already to be found in [1; 2; 3], we omit the details of the first part of the proof. We begin by transforming (3.5) as follows. The series (2.1) for $a_m(\nu, r, \epsilon)$ is inserted into (3.5) and Lemma (2.6) is used to justify an interchange in the order of summation. The result is

$$\begin{aligned} F_\nu(\tau) &= e^{-2\pi i(\nu-\kappa)\tau/\lambda} \\ &\quad + \frac{2\pi}{\lambda} \sum_{c\in C;\, c>0} c^{-1} \sum_{d\in D_c} \epsilon^{-1}(V_{c,d}) e^{-2\pi i(\nu-\kappa)a/c\lambda} \Phi_c\left(\exp\left\{\frac{2\pi i}{\lambda}\left(\tau+\frac{d}{c}\right)\right\}\right) \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} \Phi_c(z) &= \sum_{m=0}^{\infty} g_c(m) z^{m+\kappa}, && \text{if } \kappa>0 \\ &= \left(\frac{2\pi\nu}{c\lambda}\right)^{r+1} \frac{1}{(r+1)!} + \sum_{m=1}^{\infty} g_c(m) z^m, && \text{if } \kappa=0. \end{aligned} \tag{3.7}$$

Here we have put

$$g_c(m) = \left(\frac{\nu-\kappa}{m+\kappa}\right)^{(r+1)/2} I_{r+1}\left\{\frac{4\pi}{c\lambda}(\nu-\kappa)^{1/2}(m+\kappa)^{1/2}\right\}. \tag{3.8}$$

Next I_{r+1} is replaced by the power series (2.4), another interchange of summation is performed, and use is made of the Lipschitz formula (2.8) if $\kappa>0$, (2.9) if $\kappa=0$. If $\kappa>0$ we obtain

$$\begin{aligned} F_\nu(\tau) &= e^{-2\pi i(\nu-\kappa)\tau/\lambda} \\ &\quad + \sum_{c\in C;\, c>0} \sum_{d\in D_c} \epsilon^{-1}(V_{c,d}) e^{-2\pi i(\nu-\kappa)a/c\lambda} \\ &\quad \cdot \sum_{q=-\infty}^{\infty} e^{2\pi i\kappa q}(-i(c\tau+d-c\lambda q))^r \sum_{p=r+1}^{\infty} \frac{1}{p!}\left\{\frac{2\pi i(\nu-\kappa)}{c\lambda(c\tau+d-c\lambda q)}\right\}^p. \end{aligned} \tag{3.9}$$

When $\kappa=0$, we have

$$\begin{aligned} F_\nu(\tau) &= e^{-2\pi i\nu\tau/\lambda} + \frac{1}{2} a_0(\nu, r, \epsilon) \\ &\quad + \sum_{c\in C;\, c>0} \sum_{d\in D_c} \epsilon^{-1}(V_{c,d}) e^{-2\pi i\nu a/c\lambda} \\ &\quad \cdot \lim_{N\to\infty} \sum_{q=-N}^{N} (-i(c\tau+d-c\lambda q))^r \sum_{p=r+1}^{\infty} \frac{1}{p!}\left\{\frac{2\pi i\nu}{c\lambda(c\tau+d-c\lambda q)}\right\}^p. \end{aligned} \tag{3.10}$$

In order to avoid separate treatment of the two cases, we introduce the function

$$
(3.11)\qquad \begin{aligned} G(\tau) &= F_\nu(\tau), && \text{if } \kappa > 0,\\ &= F_\nu(\tau) - \frac{1}{2}\, a_0(\nu, r, \epsilon), && \text{if } \kappa = 0. \end{aligned}
$$

Now (3.9) and (3.10) can be rewritten somewhat so that in either case we have

$$
(3.12)\qquad \begin{aligned} G(\tau) &= e^{-2\pi i(\nu-\kappa)\tau/\lambda} \\ &\quad + \sum_{c\in C;\, c>0} \sum_{d\in D_c} \epsilon^{-1}(V_{c,d}) e^{-2\pi i(\nu-\kappa)a/c\lambda} \\ &\quad \cdot \lim_{N\to\infty} \sum_{q=-N}^{N} e^{2\pi i\kappa q}(-i(c\tau + d - c\lambda q))^r \sum_{p=r+1}^{\infty} \frac{1}{p!} \left\{ \frac{2\pi i(\nu-\kappa)}{c\lambda(c\tau + d - c\lambda q)} \right\}^p . \end{aligned}
$$

A simple calculation now shows that as q runs through the integers and d through the set D_c, $d' = d - c\lambda q$ assumes, exactly once, each value $d' \in D^c$. Furthermore $V_{c,d} = V_{c,d'}S^q$ and therefore by (1.5),

$$
\epsilon^{-1}(V_{c,d}) = e^{-2\pi i q\kappa} \epsilon^{-1}(V_{c,d'}).
$$

Hence substituting $d' = d - c\lambda q$ in (3.12), and then dropping the prime ($'$), we obtain

$$
(3.13)\qquad \begin{aligned} G(\tau) &= e^{-2\pi i(\nu-\kappa)\tau/\lambda} \\ &\quad + \sum_{c\in C;\, c>0} \lim_{N\to\infty} \sum_{d\in D^c;\, |d|\le N} \epsilon^{-1}(V_{c,d}) e^{-2\pi i(\nu-\kappa)a/c\lambda} (-i(c\tau+d))^r \\ &\quad \cdot \sum_{p=r+1}^{\infty} \frac{1}{p!} \left\{ \frac{2\pi i(\nu-\kappa)}{c\lambda(c\tau+d)} \right\}^p . \end{aligned}
$$

Now (3.13) can be written

$$
\begin{aligned} G(\tau) &= e^{-2\pi i(\nu-\kappa)\tau/\lambda} \\ &\quad + \sum_{c\in C;\, c>0} \lim_{N\to\infty} \sum_{d\in D^c;\, |d|\le N} \epsilon^{-1}(V_{c,d}) e^{-2\pi i(\nu-\kappa)a/c\lambda} \\ &\quad \cdot \frac{(-i(c\tau+d))^r}{(r+1)!} \left\{ \frac{2\pi i(\nu-\kappa)}{c\lambda(c\tau+d)} \right\}^{r+1} \\ &\quad + \sum_{c\in C;\, c>0} \lim_{N\to\infty} \sum_{d\in D^c;\, |d|\le N} \epsilon^{-1}(V_{c,d}) e^{-2\pi i(\nu-\kappa)a/c\lambda} (-i(c\tau+d))^r \\ &\quad \cdot \sum_{p=r+2}^{\infty} \frac{1}{p!} \left\{ \frac{2\pi i(\nu-\kappa)}{c\lambda(c\tau+d)} \right\}^p . \end{aligned}
$$

This separation into two sums is justified since the first converges by Lemma (2.10) and the second is an absolutely convergent triple sum. The latter fact can be proved by using Lemma (2.5). Note that Lemma (2.10) is applicable here since we assumed at the outset that $\tau = iy$, with $y > 0$.

The second sum can therefore be rearranged in any manner whatsoever. If we use this fact and apply Lemma (2.10) to the first sum, with

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$$

chosen to satisfy conditions (2.11), and $t = (\gamma - \beta/2)\delta^{-1}$, we obtain

$$\begin{aligned} G(\tau) = {} & e^{-2\pi i(\nu-\kappa)\tau/\lambda} \\ & + \lim_{K\to\infty} \sum_{c\in C;\, c>0} \sum_{d\in D^c;\, (c,d)\in \mathfrak{Z}_V(K)} \epsilon^{-1}(V_{c,d}) e^{-2\pi i(\nu-\kappa)a/c\lambda} \\ & \cdot \frac{(-i(c\tau+d))^r}{(r+1)!} \left\{ \frac{2\pi i(\nu-\kappa)}{c\lambda(c\tau+d)} \right\}^{r+1} \\ & + \lim_{K\to\infty} \sum_{c\in C;\, c>0} \sum_{d\in D^c;\, (c,d)\in \mathfrak{Z}_V(K)} \epsilon^{-1}(V_{c,d}) e^{-2\pi i(\nu-\kappa)a/c\lambda} (-i(c\tau+d))^r \\ & \cdot \sum_{p=r+2}^{\infty} \frac{1}{p!} \left\{ \frac{2\pi i(\nu-\kappa)}{c\lambda(c\tau+d)} \right\}^p \end{aligned}$$

or,

$$\begin{aligned} G(\tau) = {} & e^{-2\pi i(\nu-\kappa)\tau/\lambda} \\ & + \lim_{K\to\infty} \sum_{c\in C;\, c>0} \sum_{d\in D^c;\, (c,d)\in \mathfrak{Z}_V(K)} \epsilon^{-1}(V_{c,d}) e^{-2\pi i(\nu-\kappa)a/c\lambda} (-i(c\tau+d))^r \\ & \cdot \left\{ \exp\left(\frac{2\pi i(\nu-\kappa)}{c\lambda(c\tau+d)} \right) - \sum_{p=0}^{r} \frac{1}{p!} \left(\frac{2\pi i(\nu-\kappa)}{c\lambda(c\tau+d)} \right)^p \right\}. \end{aligned} \tag{3.14}$$

Now let

$$\begin{aligned} S_K(\tau) = {} & e^{-2\pi i(\nu-\kappa)\tau/\lambda} \\ & + \sum_{c\in C;\, c>0} \sum_{d\in D^c;\, (c,d)\in \mathfrak{Z}_V(K)} \epsilon^{-1}(V_{c,d}) (-i(c\tau+d))^r e^{-2\pi i(\nu-\kappa)a/c\lambda} \\ & \cdot \exp\left(\frac{2\pi i(\nu-\kappa)}{c\lambda(c\tau+d)} \right). \end{aligned}$$

Using the fact that

$$V_{c,d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has determinant one, a short calculation shows that

$$e^{-2\pi i(\nu-\kappa)a/c\lambda} \exp\left(\frac{2\pi i(\nu-\kappa)}{c\lambda(c\tau+d)} \right) = \exp\left(\frac{-2\pi i(\nu-\kappa)}{\lambda} V_{c,d}\tau \right)$$

and therefore

$$
(3.15) \qquad \begin{aligned} S_K(\tau) &= e^{-2\pi i(\nu-\kappa)\tau/\lambda} \\ &\quad + \sum_{c\in C;\, c>0} \ \sum_{d\in D^c;\, (c,d)\in \mathfrak{Z}_V(K)} \epsilon^{-1}(V_{c,d})(-i(c\tau+d))^r \\ &\quad \cdot \exp\left(\frac{-2\pi i(\nu-\kappa)}{\lambda} V_{c,d}\tau\right). \end{aligned}
$$

In order to include the term $e^{-2\pi i(\nu-\kappa)\tau/\lambda}$ in the double sum we observe that (1.3) implies that if

$$
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
$$

then $\epsilon(I)(-i)^{-r}=1$. Hence

$$
e^{-2\pi i(\nu-\kappa)\tau/\lambda} = \epsilon(I)(-i)^{-r} \exp\left(\frac{-2\pi i(\nu-\kappa)}{\lambda} I(\tau)\right).
$$

On the other hand, it is shown in [5, p. 267] that if $V_{c,d}\in\Gamma$ and $c=0$, then $d=\pm 1$. Hence if we remove the summation condition $c>0$ we introduce only the two new pairs $(c, d)=(0, \pm 1)$. Therefore, (3.15) becomes

$$
(3.16) \qquad \begin{aligned} S_K(\tau) = \sum_{\substack{c\in C \\ (c,d)\in\mathfrak{Z}_V(K);\,(c,d)\neq(0,-1)}} \ \sum_{d\in D^c} & \epsilon^{-1}(V_{c,d})(-i(c\tau+d))^r \\ & \cdot \exp\left(\frac{-2\pi i(\nu-\kappa)}{\lambda} V_{c,d}\tau\right). \end{aligned}
$$

We now extend the region of summation in $S_K(\tau)$ to the parallelogram $\mathcal{P}_V(K)$ obtained by reflecting $\mathfrak{Z}_V(K)$ through the origin. This is accomplished by including with

$$
V_{c,d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
$$

the matrix

$$
-V_{c,d} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}.
$$

But by (1.4) we have

$$
\begin{aligned} &\epsilon^{-1}(-V_{c,d})(-i(-c\tau-d))^r \exp\left(\frac{-2\pi i(\nu-\kappa)}{\lambda}(-V_{c,d}\tau)\right) \\ &\qquad = \epsilon^{-1}(V_{c,d})(-i(c\tau+d))^r \exp\left(\frac{-2\pi i(\nu-\kappa)}{\lambda} V_{c,d}\tau\right), \end{aligned}
$$

so that in the extended region $\mathcal{P}_V(K)$ each term of $S_K(\tau)$ occurs twice. Therefore,

$$(3.17)\quad S_K(\tau) = \frac{1}{2}\sum_{\substack{c\in C \\ (c,d)\in\mathcal{P}_V(K)}}\sum_{d\in D^c} \epsilon^{-1}(V_{c,d})(-i(c\tau+d))^r \exp\left(\frac{-2\pi i(\nu-\kappa)}{\lambda} V_{c,d}\tau\right).$$

From the definition of $\mathfrak{J}_V(K)$ it follows that $\mathcal{P}_V(K)$ is bounded by the four lines

$$\alpha c + \gamma d = \pm\, tK, \qquad \beta c + \delta d = \pm\, K.$$

Now from (3.17) we see that

$$\begin{aligned}
&\epsilon^{-1}(V)(-i(\gamma\tau+\delta))^r S_K(V\tau)\\
&\quad = \frac{1}{2}\sum_{\substack{c\in C \\ (c,d)\in\mathcal{P}_V(K)}}\sum_{d\in D^c} \epsilon^{-1}(V)\epsilon^{-1}(V_{c,d})(-i(\gamma\tau+\delta))^r(-i(cV\tau+d))^r\\
&\qquad\cdot\exp\left(\frac{-2\pi i(\nu-\kappa)}{\lambda} V_{c,d}V\tau\right)\\
(3.18)\quad &\quad = \frac{1}{2}\sum_{\substack{c\in C \\ (c,d)\in\mathcal{P}_V(K)}}\sum_{d\in D^c} \epsilon^{-1}(V\cdot V_{c,d})\left\{-i((\alpha c+\gamma d)\tau + (\beta c+\delta d))\right\}^r\\
&\qquad\cdot\exp\left(\frac{-2\pi i(\nu-\kappa)}{\lambda} V_{c,d}V\tau\right),
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
V_{c,d}V &= \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}\\
&= \begin{pmatrix} * & * \\ \alpha c+\gamma d & \beta c+\delta d\end{pmatrix}
\end{aligned}$$

and applied the consistency condition (1.3).

We now perform the transformation

$$\begin{aligned} c' &= \alpha c + \gamma d,\\ d' &= \delta d + \beta c \end{aligned}$$

which maps the region $\mathcal{P}_V(K)$ of the c-d plane in a 1-1 fashion onto the rectangle

$$|c'| \leqq tK, \qquad |d'| \leqq K$$

of the c'-d' plane. Furthermore this transformation sets up a 1-1 mapping between the pairs (c, d), with $c\in C$, $d\in D^c$, and the pairs (c', d') with $c'\in C$, $d'\in D^{c'}$. Hence (3.18) becomes

$$
\begin{aligned}
&\epsilon^{-1}(V)(-i(\gamma\tau+\delta))^r S_K(V\tau)\\
&\quad = \frac{1}{2}\sum_{c'\in C;\,|c'|\leq tK}\ \sum_{d'\in D^{c'};\,|d'|\leq K} \epsilon^{-1}(V_{c',d'})(-i(c'\tau+d'))^r\\
&\qquad \cdot\exp\left(\frac{-2\pi i(\nu-\kappa)}{\lambda}V_{c',d'}\tau\right)\\
&\quad = e^{-2\pi i(\nu-\kappa)\tau/\lambda}\\
&\qquad + \sum_{c\in C;\,0<c\leq tK}\ \sum_{d\in D^c;\,|d|\leq K} \epsilon^{-1}(V_{c,d})(-i(c\tau+d))^r\\
&\qquad \cdot\exp\left(\frac{-2\pi i(\nu-\kappa)}{\lambda}V_{c,d}\tau\right),
\end{aligned}
\tag{3.19}
$$

where we have separated out the terms with $(c', d')=(0, \pm 1)$, applied the same reasoning as that preceding (3.17), and dropped the primes from c' and d'. Therefore, it follows from (3.14) that

$$
\begin{aligned}
&\epsilon^{-1}(V)(-i(\gamma\tau+\delta))^r G(V\tau)\\
&\quad = \lim_{K\to\infty}\Bigg\{\epsilon^{-1}(V)(-i(\gamma\tau+\delta))^r S_K(V\tau)\\
&\qquad - \sum_{c\in C;\,c>0}\ \sum_{d\in D^c;\,(c,d)\in\mathfrak{J}_V(K)} \epsilon^{-1}(V)\epsilon^{-1}(V_{c,d})e^{-2\pi i(\nu-\kappa)a/c\lambda}\\
&\qquad \cdot(-i(\gamma\tau+\delta))^r(-i(cV\tau+d))^r\sum_{p=0}^{r}\frac{1}{p!}\left(\frac{2\pi i(\nu-\kappa)}{c\lambda(cV\tau+d)}\right)^p\Bigg\}\\
&\quad = e^{-2\pi i(\nu-\kappa)\tau/\lambda}\\
&\qquad + \lim_{K\to\infty}\Bigg\{\sum_{c\in C;\,0<c\leq tK}\ \sum_{d\in D^c;\,|d|\leq K} \epsilon^{-1}(V_{c,d})(-i(c\tau+d))^r\\
&\qquad\qquad \cdot\exp\left(\frac{-2\pi i(\nu-\kappa)}{\lambda}V_{c,d}\tau\right)\\
&\qquad - \sum_{c\in C;\,c>0}\ \sum_{d\in D^c;\,(c,d)\in\mathfrak{J}_V(K)} \epsilon^{-1}(V)\epsilon^{-1}(V_{c,d})e^{-2\pi i(\nu-\kappa)a/c\lambda}(-i(\gamma\tau+\delta))^r\\
&\qquad\qquad \cdot(-i(cV\tau+d))^r\sum_{p=0}^{r}\frac{1}{p!}\left(\frac{2\pi i(\nu-\kappa)}{c\lambda(cV\tau+d)}\right)^p\Bigg\}.
\end{aligned}
\tag{3.20}
$$

We now return to (3.13) and proceed as in the proof of (3.14), this time applying Lemma (2.13) with $\rho=t$. The result is

$$
\begin{aligned}
G(\tau) &= e^{-2\pi i(\nu-\kappa)\tau/\lambda} \\
&\quad + \lim_{K\to\infty} \Bigg\{ \sum_{c\in C;\,0<c\leq tK} \ \sum_{d\in D^c;\,|d|\leq K} \epsilon^{-1}(V_{c,d})(-i(c\tau+d))^r \\
&\qquad\qquad \cdot\exp\left(\frac{-2\pi i(\nu-\kappa)}{\lambda} V_{c,d}\tau\right) \\
&\quad - \sum_{c\in C;\,0<c\leq tK} \ \sum_{d\in D^c;\,|d|\leq K} \epsilon^{-1}(V_{c,d}) e^{-2\pi i(\nu-\kappa)a/c\lambda} \\
&\qquad\qquad \cdot(-i(c\tau+d))^r \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i(\nu-\kappa)}{c\lambda(c\tau+d)}\right) \Bigg\}.
\end{aligned}
$$

Comparing this with (3.20) we find that

$$
\begin{aligned}
&G(\tau) - \epsilon^{-1}(V)(-i(\gamma\tau+\delta))^r G(V\tau) \\
&= \lim_{K\to\infty} \Bigg\{ \sum_{c\in C;\,c>0} \ \sum_{d\in D^c;\,(c,d)\in\mathfrak{J}_V(K)} \epsilon^{-1}(V)\epsilon^{-1}(V_{c,d}) e^{-2\pi i(\nu-\kappa)a/c\lambda}(-i(\gamma\tau+\delta))^r \\
&\qquad\qquad \cdot(-i(cV\tau+d))^r \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i(\nu-\kappa)}{c\lambda(cV\tau+d)}\right)^p \\
&\quad - \sum_{c\in C;\,0<c\leq tK} \ \sum_{d\in D^c;\,|d|\leq K} \epsilon^{-1}(V_{c,d}) e^{-2\pi i(\nu-\kappa)a/c\lambda} \\
&\qquad\qquad \cdot(-i(c\tau+d))^r \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i(\nu-\kappa)}{c\lambda(c\tau+d)}\right)^p \Bigg\}.
\end{aligned} \tag{3.21}
$$

The factor $(\gamma\tau+\delta)^r$ combines with the denominator of $(cV\tau+d)^{r-p}$ in such a way that the expression in braces is a polynomial of degree at most r. On the other hand the limit of a sequence of polynomials of degree at most r converging at $r+1$ points is again a polynomial of degree at most r. Thus the right hand side of (3.21) is a polynomial in τ of degree at most r. We denote this polynomial by $q_V(\tau;\nu,\epsilon)$. If we now make use of (3.11) we obtain

$$
F_\nu(\tau) - \epsilon^{-1}(V)(-i(\gamma\tau+\delta))^r F_\nu(V\tau) = p_V(\tau;\nu,\epsilon), \tag{3.22}
$$

where

$$
\begin{aligned}
p_V(\tau;\nu,\epsilon) &= q_V(\tau;\nu,\epsilon), && \text{if } \kappa>0, \\
&= \frac{1}{2} a_0(\nu,r,\epsilon)\{1-\epsilon^{-1}(V)(-i(\gamma\tau+\delta))^r\} + q_V(\tau;\nu,\epsilon), && \text{if } \kappa=0.
\end{aligned} \tag{3.23}
$$

3. We have thus far derived (3.21) for those $V\in\Gamma$ which satisfy the condition (2.11). In order to remove this restriction we proceed as follows. Let

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be *any* element of Γ. If $\gamma=0$, then, as we have previously seen, $\delta=\pm 1$ and hence $V=S^q$, with q an integer. In this case the transformation equation

$$F_\nu(S^q\tau) = e^{2\pi i q\kappa}F_\nu(\tau) \tag{3.24}$$

follows directly from the definition (3.5) of $F_\nu(\tau)$. Hence we may assume that $\gamma>0$, by changing the signs of α, β, γ, and δ if necessary. Now a straightforward calculation shows that

$$V = S^m\cdot\begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix}\cdot S^n$$

where

$$\begin{aligned} \alpha^* &= \alpha - \gamma\lambda m, & \beta^* &= \beta - \delta\lambda n - \lambda n(\alpha - \gamma\lambda m), \\ \gamma^* &= \gamma, & \delta^* &= \delta - \gamma\lambda m. \end{aligned}$$

Clearly integers m and n can be chosen such that

$$V^* = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix}$$

satisfies (2.11) and thus we can apply (3.22) with V replaced by V^*.

Applying (3.22) and also making use of (3.24) and (1.5) we have

$$\begin{aligned} F_\nu(V\tau) &= F_\nu(S^mV^*S^n\tau) = e^{2\pi i m\kappa}F_\nu(V^*S^n\tau) \\ &= e^{2\pi i m\kappa}\epsilon(V^*)(-i(\gamma^*S^n\tau + \delta))^{-r}(F_\nu(S^n\tau) - p_{V^*}(S^n\tau;\nu,\epsilon)) \\ &= \epsilon(S^mV^*)(-i(\gamma\tau + \delta))^{-r}e^{2\pi i n\kappa}F_\nu(\tau) \\ &\quad - \epsilon(S^mV^*)(-i(\gamma\tau + \delta))^{-r}p_{V^*}(S^n\tau;\nu,\epsilon). \end{aligned}$$

A slight rearrangement yields

$$F_\nu(\tau) - \epsilon^{-1}(V)(-i(\gamma\tau + \delta))^rF_\nu(V\tau) = e^{-2\pi i n\kappa}p_{V^*}(S^n\tau;\nu,\epsilon) \equiv p_V(\tau;\nu,\epsilon). \tag{3.25}$$

The right-hand side of (3.25) is a polynomial in τ of degree at most r which we have again denoted by $p_V(\tau;\nu,\epsilon)$, and the proof of Theorem (3.3) is complete.

4. We here indicate how automorphic forms of dimension r on Γ may be constructed, by making use of Theorem (3.3) and the fact that Γ is finitely generated. Suppose Γ is generated by $V_1, \cdots, V_g$, with

$$V_j = \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}.$$

By Theorem (3.3) we have, for $\mathfrak{I}(\tau)>0$,

$$F_\nu(\tau) - \epsilon^{-1}(V_j)(-i(\gamma_j\tau + \delta_j))^r F_\nu(V_j\tau) = p_j(\tau; \nu) \qquad (1 \leqq j \leqq g),$$

with $p_j(\tau; \nu)$ a polynomial in τ of degree at most r. Hence if $b_1, \cdots, b_\mu$ are constants and we put

$$F(\tau) = \sum_{\nu=1}^{\mu} b_\nu F_\nu(\tau) \tag{3.26}$$

it follows that

$$F(\tau) - \epsilon^{-1}(V_j)(-i(\gamma_j\tau + \delta_j))^r F(V_j\tau) = p_j(\tau) \qquad (1 \leqq j \leqq g)$$

where $p_j(\tau) = \sum_{\nu=1}^{\mu} b_\nu p_j(\tau; \nu)$.

We therefore consider the following system of equations in the unknowns $b_1, \cdots, b_\mu$

$$\sum_{\nu=1}^{\mu} b_\nu p_j(\tau; \nu) \equiv 0 \qquad (1 \leqq j \leqq g).$$

If $\mu \geqq (r+1)g+1$, this system has a nontrivial solution. If $b_1, \cdots, b_\mu$ is chosen to be such a solution,then with $F(\tau)$ defined by (3.26),

$$\epsilon^{-1}(V_j)(-i(\gamma_j\tau + \delta_j))^r F(V_j\tau) = F(\tau), \qquad \mathfrak{I}(\tau) > 0,$$

for $j = 1, \cdots, g$ and therefore, since the V_j generate Γ,

$$\epsilon^{-1}(V)(-i(\gamma\tau + \delta))^r F(V\tau) = F(\tau), \qquad \mathfrak{I}(\tau) > 0,$$

for all

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma.$$

That is $F(\tau) \in \{\Gamma, r, \epsilon\}$.

IV. **The supplementary series.** 1. Let κ' and ν' be defined by

$$\kappa' = 1 - \kappa, \qquad \nu' = 1 - \nu, \qquad \text{if } \kappa > 0,$$
$$\kappa' = 0, \qquad \nu' = -\nu, \qquad \text{if } \kappa = 0.$$

Further, if we define ϵ' by

$$\epsilon'(V) = e^{\pi i r}\epsilon^{-1}(V), \qquad V \in \Gamma,$$

then since r is an integer and ϵ is a multiplier system for Γ corresponding to the dimension r, it follows that ϵ' is also a multiplier system for Γ corresponding to the dimension r. That is, ϵ' satisfies (1.3).

Let $a_m(\nu', r, \epsilon')$ be defined by (2.1)–(2.3) with ν, ϵ, and κ replaced by ν', ϵ', and κ', respectively. We define $\hat{F}_{\nu'}(\tau)$, *the series supplementary to* $F_\nu(\tau)$, by

$$(4.1) \qquad \hat{F}_{\nu'}(\tau) = e^{-2\pi i(\nu'-\kappa')\tau/\lambda} + \sum_{m=0}^{\infty} a_m(\nu', r, \epsilon')e^{2\pi i(m+\kappa')\tau/\lambda}.$$

A result for $a_m(\nu', r, \epsilon')$ analogous to (2.7) shows that $\hat{F}_{\nu'}(\tau)$ is regular for $\mathscr{I}(\tau) > 0$.

A careful examination of the derivation of (3.21) reveals that it does not depend upon the fact that ν is positive. Hence we can apply the same argument to $\hat{F}_{\nu'}(\tau)$. The result is as follows. Let

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$$

satisfy condition (2.11). Then,

$$(4.2) \qquad \hat{F}_{\nu'}(\tau) - \epsilon'^{-1}(V)(-i(\gamma\tau+\delta))^r\hat{F}_{\nu'}(V\tau) = \hat{p}_V(\tau; \nu', \epsilon'),$$

where

$$(4.3) \qquad \begin{aligned} \hat{p}_V(\tau; \nu', \epsilon') &= \hat{q}_V(\tau; \nu', \epsilon'), && \text{if } \kappa' > 0, \\ &= \frac{1}{2} a_0(\nu', r, \epsilon')\{1 - \epsilon'^{-1}(V)(-i(\gamma\tau+\delta))^r\} + \hat{q}_V(\tau; \nu', \epsilon'), && \\ & && \text{if } \kappa' = 0. \end{aligned}$$

Here $\hat{q}_V(\tau; \nu', \epsilon')$ is the result of replacing ν, κ, and ϵ by ν', κ', and ϵ', respectively, in the right-hand side of (3.21).

If now

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$$

does not satisfy (2.11), we proceed as in §III to obtain

$$(4.4) \qquad \hat{F}_{\nu'}(\tau) - \epsilon'^{-1}(V)(-i(\gamma\tau+\delta))^r\hat{F}_{\nu'}(V\tau) = \hat{p}_V(\tau; \nu', \epsilon'),$$

where

$$(4.5) \qquad \hat{p}_V(\tau; \nu', \epsilon') = e^{-2\pi i n\kappa'}\hat{p}_{V^*}(S^n\tau; \nu', \epsilon').$$

Here n and

$$V^* = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} \in \Gamma$$

are as in §III.3. Again $\hat{p}_V(\tau; \nu', \epsilon')$ is a polynomial in τ of degree at most r. Therefore we can state

THEOREM (4.6). *Let $b_1, \cdots, b_\mu$ be any constants and let*

$$(4.7) \qquad \hat{F}(\tau) = \sum_{\nu=1}^{\mu} b_\nu\hat{F}_{\nu'}(\tau).$$

Then in $\mathfrak{I}(\tau)>0$, $\hat{F}(\tau)$ *is regular and satisfies*

$$\hat{F}(\tau) - \epsilon'^{-1}(V)(-i(\gamma\tau+\delta))^r\hat{F}(V\tau) = \hat{p}_V(\tau),$$

where

$$\hat{p}_V(\tau) = \sum_{\nu=1}^{\mu} b_\nu \hat{p}_V(\tau;\nu',\epsilon').$$

2. Using supplementary series we will obtain a necessary and sufficient condition for the function $F(\tau)$, defined by (3.26), to be in $\{\Gamma, r, \epsilon\}$. A simple calculation involving the right-hand side of (3.21) shows that

$$\hat{q}_V(\tau;\nu,\epsilon) = [q_V(\bar{\tau};\nu,\epsilon)]^-$$

for those $V\in\Gamma$ satisfying (2.11). (The bar here denotes complex conjugation, as usual.) We also see from equations (2.1)–(2.3) that in the case $m=\kappa=0$, $a_0(\nu', r, \epsilon') = -\bar{a}_0(\nu, r, \epsilon)$, and therefore

$$\frac{1}{2}\bar{a}_0(\nu, r, \epsilon)[\{1-\epsilon^{-1}(V)(-i(\gamma\bar{\tau}+\delta))^r\}]^-$$
$$= -\frac{1}{2}a_0(\nu', r, \epsilon')\{1-\epsilon'^{-1}(V)(-i(\gamma\tau+\delta))^r\}.$$

It now follows from (3.23) and (4.3) that if $V\in\Gamma$ satisfies (2.11), then

$$\begin{aligned} \hat{p}_V(\tau;\nu',\epsilon') &= [p_V(\bar{\tau};\nu,\epsilon)]^-, && \text{if } \kappa>0,\\ &= a_0(\nu', r, \epsilon')\{1-\epsilon'^{-1}(V)(-i(\gamma\tau+\delta))^r\} \\ &\quad + [p_V(\bar{\tau};\nu,\epsilon)]^-, && \text{if } \kappa=0. \end{aligned} \tag{4.8}$$

On the other hand if $V\in\Gamma$ does not satisfy (2.11), then (3.25) and (4.5) together imply that (4.8) still holds. Hence we have (4.8) for *all* $V\in\Gamma$.

We are now ready to prove

THEOREM (4.9). *Let* $b_1, \cdots, b_\mu$ *be any constants. Let* $F(\tau)=\sum_{\nu=1}^{\mu} b_\nu F_\nu(\tau)$ *and* $\hat{F}(\tau)=\sum_{\nu=1}^{\mu}\bar{b}_\nu\hat{F}_{\nu'}(\tau)$, *where* $F_\nu(\tau)$ *and* $\hat{F}_{\nu'}(\tau)$ *are defined by* (3.5) *and* (4.1), *respectively. We call* $\hat{F}(\tau)$ *the series supplementary to* $F(\tau)$. *Then when* $\kappa>0$ $F(\tau)\in\{\Gamma, r, \epsilon\}$ *if and only if* $\hat{F}(\tau)\equiv 0$. *When* $\kappa=0$, $F(\tau)\in\{\Gamma, r, \epsilon\}$ *if and only if* $\hat{F}(\tau)\equiv\sum_{\nu=1}^{\mu}\bar{b}_\nu a_0(\nu', r, \epsilon')$.

Proof. For $V\in\Gamma$ we have that

$$F(\tau) - \epsilon^{-1}(V)(-i(\gamma\tau+\delta))^r F(V\tau) = p_V(\tau), \tag{4.10}$$

where $p_V(\tau)=\sum_{\nu=1}^{\mu} b_\nu p_V(\tau;\nu,\epsilon)$. We also know that

$$\hat{F}(\tau) - \epsilon'^{-1}(V)(-i(\gamma\tau+\delta))^r\hat{F}(V\tau) = \hat{p}_V(\tau), \tag{4.11}$$

with $\hat{p}_V(\tau)=\sum_{\nu=1}^{\mu}\bar{b}_\nu\hat{p}_V(\tau;\nu',\epsilon')$. By (4.8) we have

$$\hat{p}_V(\tau) = [p_V(\bar{\tau})]^-, \qquad \text{if } \kappa > 0,$$

$$(4.12) \qquad = \left(\sum_{\nu=1}^{\mu} \bar{b}_\nu a_0(\nu', r, \epsilon')\right)\{1 - \epsilon'^{-1}(V)(-i(\gamma\tau + \delta))^r\} + [p_V(\bar{\tau})]^-,$$

$$\text{if } \kappa = 0.$$

By (4.10), $F(\tau) \in \{\Gamma, r, \epsilon\}$ if and only if $p_V(\tau) \equiv 0$, for all $V \in \Gamma$. Suppose $\kappa > 0$. By (4.12) $p_V(\tau) \equiv 0$ if and only if $\hat{p}_V(\tau) \equiv 0$. Thus $F(\tau) \in \{\Gamma, r, \epsilon\}$ if and only if $\hat{p}_V(\tau) \equiv 0$ for all $V \in \Gamma$. Therefore by (4.11), $F(\tau) \in \{\Gamma, r, \epsilon\}$ if and only if $\hat{F}(\tau) \in \{\Gamma, r, \epsilon'\}$. But we can easily see from (4.1) that each $\hat{F}_{\nu'}(\tau)$, and therefore $\hat{F}(\tau)$, is bounded at ∞. Since $r > 0$, by [5, p. 274, Theorem 4], $\hat{F}(\tau) \in \{\Gamma, r, \epsilon'\}$ if and only if $\hat{F}(\tau) \equiv 0$.

Suppose $\kappa = 0$. By (4.12), $p_V(\tau) \equiv 0$ if and only if

$$\hat{p}_V(\tau) \equiv \left(\sum_{\nu=1}^{\mu} \bar{b}_\nu a_0(\nu', r, \epsilon')\right)\{1 - \epsilon'^{-1}(V)(-i(\gamma\tau + \delta))^r\}.$$

Therefore by (4.11), $F(\tau) \in \{\Gamma, r, \epsilon\}$ if and only if $\hat{F}(\tau) - \sum_{\nu=1}^{\mu} \bar{b}_\nu a_0(\nu', r, \epsilon') \in \{\Gamma, r, \epsilon'\}$. By the same reasoning as above, the latter condition is equivalent to $\hat{F}(\tau) \equiv \sum_{\nu=1}^{\mu} \bar{b}_\nu a_0(\nu', r, \epsilon')$.

REMARK. As a special case, this theorem shows that when $\kappa > 0$, $F_\nu(\tau) \in \{\Gamma, r, \epsilon\}$ if and only if $\hat{F}_{\nu'}(\tau) \equiv 0$, and when $\kappa = 0$, $F_\nu(\tau) \in \{\Gamma, r, \epsilon\}$ if and only if $\hat{F}_{\nu'}(\tau) \equiv a_0(\nu', r, \epsilon')$.

If we examine equations (2.1)–(2.3), we see that when $\kappa > 0$,

$$(4.13) \qquad a_m(\nu', r, \epsilon') = e^{\pi i r}[a_{-m-1}(\nu, r, \epsilon)]^-, \qquad \text{for } m \geqq 0,$$

where by $a_{-m-1}(\nu, r, \epsilon)$ is meant the result of replacing m by $-m-1$ in (2.1)–(2.3). In order to derive (4.13) we use

$$A_{c,\nu'}(m, \epsilon') = e^{\pi i r}[A_{c,\nu}(-m - 1, \epsilon)]^-$$

and the fact that $L_c(-m-1, \nu, r, \kappa)$ and $L_c(m, \nu', r, \kappa')$ are real.

When $\kappa = 0$, we find in the same way that

$$(4.14) \qquad \begin{aligned} a_m(\nu', r, \epsilon') &= e^{\pi i r}\bar{a}_{-m}(\nu, r, \epsilon), && \text{for } m \geqq 1, \\ &= -\bar{a}_0(\nu, r, \epsilon), && \text{for } m = 0. \end{aligned}$$

We can now state the following result which is a direct consequence of Theorem (4.9).

COROLLARY (4.15). $F(\tau) \in \{\Gamma, r, \epsilon\}$ *if and only if*

$$(4.16) \qquad \begin{aligned} \sum_{\nu=1}^{\mu} b_\nu a_{-m}(\nu, r, \epsilon) &= b_m e^{\pi i (r+1)}, && \text{for } 1 \leqq m \leqq \mu, \\ &= 0, && \text{for } m \geqq \mu + 1. \end{aligned}$$

REMARKS. Note that this result is the same whether $\kappa > 0$ or $\kappa = 0$. Corollary (4.15) can be applied to obtain information about identically vanishing linear combinations of Poincaré series. This has been carried out in [4].

Proof. Suppose $\kappa>0$. By Theorem (4.9) $F(\tau)\in\{\Gamma, r, \epsilon\}$ if and only if

$$\sum_{\nu=1}^{\mu} \bar{b}_\nu a_{m-1}(\nu', r, \epsilon) = -\bar{b}_m, \qquad \text{for } 1 \leqq m \leqq \mu,$$
$$= 0, \qquad \text{for } m \geqq \mu+1.$$

But by (4.13) this is equivalent to (4.16).

If $\kappa=0$, $F(\tau)\in\{\Gamma, r, \epsilon\}$ if and only if

$$\sum_{\nu=1}^{\mu} \bar{b}_\nu a_m(\nu', r, \epsilon') = -\bar{b}_m, \qquad \text{for } 1 \leqq m \leqq \mu,$$
$$= 0, \qquad \text{for } m \geqq \mu+1.$$

By (4.14) this again is equivalent to (4.16).

3. The function $F(\tau)$ given by (3.1)–(3.2) has been defined only in $\mathscr{I}(\tau)>0$ and we can in fact show that the real axis, $\mathscr{I}(\tau)=0$, is a natural boundary for $F(\tau)$. This follows from Theorem (3.3) and the fact that, by definition of an H-group, Γ is not discontinuous at any point of the real axis. However, following a procedure due to Rademacher [8] we can attach meaning to the *series* (3.1) in $\mathscr{I}(\tau)<0$.

In order to do this we go back to the expression given for $F_\nu(\tau)$ by (3.6)–(3.8). As in [8] we see that $g_c(m)$ is entire function in m of order 1/2, and therefore by a theorem of Wigert the function $\Phi_c(z)$, which is defined by (3.7) only for $|z|<1$, has an analytic continuation to the entire z-plane, with one isolated essential singularity at $z=1$. (This last fact could be used to give another proof that the real axis is a natural boundary for $F_\nu(\tau)$.)

Using the method of [8] we obtain the power series representation, valid in $\mathscr{I}(\tau)<0$,

$$\Phi_c(z) = -\sum_{m=1}^{\infty} g_c(-m)z^{-m+\kappa}, \qquad \text{for } |z| > 1. \tag{4.17}$$

If we insert this into (3.6), with $\mathscr{I}(\tau)<0$, and interchange the order of summation, we obtain a new function $F_\nu^*(\tau)$, regular in $\mathscr{I}(\tau)<0$, given by

$$F_\nu^*(\tau) = e^{-2\pi i(\nu-\kappa)\tau/\lambda} - \sum_{m=1}^{\infty} a_{-m}(\nu, r, \epsilon)e^{2\pi i(-m+\kappa)\tau/\lambda}. \tag{4.18}$$

Note that while $F_\nu(\tau)$ and $F_\nu^*(\tau)$ are both given by the series (3.6), they are of course not analytic continuations of each other.

Putting $F^*(\tau)=\sum_{\nu=1}^{\mu} b_\nu F_\nu^*(\tau)$, we may restate Corollary (4.15) as follows.

COROLLARY (4.19). $F(\tau)\in\{\Gamma, r, \epsilon\}$ *if and only if*

$$F^*(\tau) \equiv (1 - e^{\pi i(r+1)}) \sum_{\nu=1}^{\mu} b_\nu e^{2\pi i(-\nu+\kappa)\tau/\lambda}. \tag{4.20}$$

Note that if r is odd, (4.20) becomes $F^*(\tau) \equiv 0$. Equation (4.20) may then be called an *expansion of zero.*

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