APPROXIMATION IN SYSTEMS OF REAL-VALUED CONTINUOUS FUNCTIONS[1]

BY
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Introduction. If X is a topological space, then we denote by $C(X)$ the set of all real-valued continuous functions defined on X.

One form of Stone's generalization of the classical Weierstrass approximation theorem states[2]: if X is compact Hausdorff and if A is a subalgebra of $C(X)$, then the uniform closure of A in $C(X)$ consists of all $f \in C(X)$ approximated by A on pairs of points of X; in particular, if A separates points of X and contains the identity function 1, then A is uniformly dense in $C(X)$. This result has important applications in obtaining many characterization theorems. For example, let A be a topological algebra with identity. Suppose further that for some compact Hausdorff space X it can be shown that A is isomorphic and homeomorphic to a point separating subalgebra of $C(X)$, where the latter has its uniform topology. Then, if A is complete in its topology, the Stone-Weierstrass theorem implies that A is actually a copy of all of $C(X)$[3].

Hewitt [15] and Henriksen [11] have both stressed the desirability of developing a similar theory of approximation and characterization with no restriction on X other than complete regularity[4]. The purpose of this paper is to develop such a theory in an algebraic setting different from that discussed above for compact X. Before discussing these results, we recall some of the intermediate theories in the literature and we note some of the difficulties inherent in seeking a strict generalization of the known results for the compact case.

Arens [3; 4] has obtained approximation and characterization theorems for $C(X)$ in case X is locally compact and paracompact. Although Arens' approximation theorem [3] not only provides a generalization of the Stone-Weierstrass theorem but also holds for arbitrary X, the characterization he obtains for $C(X)$ appears not to be extendable to the case in which X

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(2) Stone's original generalization appeared in [22]. Since then a considerable number of variations and further generalizations have appeared in the literature; see, for example, Stone [23], Arens [3], Banaschewski [5], Buck [9], Hewitt [13], Hewitt and Zuckerman [16], and Isbell [17]. Stone's second paper [23] contains a particularly full account of several variations.

(3) See Kadison [18] for a detailed exposition of several such applications as well as for further references.

(4) We shall assume Hausdorff separation throughout the paper.

is merely completely regular. One difficulty in obtaining such an extension stems from the mode of approximation (via the k-topology), for in general, although $C(X)$ is a topological algebra (but not necessarily a Q-ring [19]) in the k-topology, it need not be complete(5).

For arbitrary completely regular spaces X, Hewitt [13] has shown that if A is a subalgebra of the algebra $C^*(X)$ of all *bounded* real-valued continuous functions on X, if $1 \in A$, and if A separates every pair of completely separated subsets of X, then A is uniformly dense in $C^*(X)$. This result, again a generalization of the Stone-Weierstrass theorem, is unsuited for many applications in dealing with $C(X)$. For one thing, of course, it fails to provide information about unbounded functions in $C(X)$. A second weakness lies in the choice of topology, for if $C(X)$ contains unbounded functions, then in the uniform topology neither ring multiplication nor scalar multiplication is continuous.

The Hewitt-Henriksen problem then is that of obtaining, for X any completely regular space: first, a topology for $C(X)$ relative to which it is a complete topological algebra; then an approximation theorem for $C(X)$ in this topology; and finally, a characterization of the topological algebra $C(X)$. An unpublished result (cf. [11]) due to H. S. Bear leads one to suspect that there is no topology of the desired kind for $C(X)$, and so, until this matter is settled, the problem, as stated, must be shelved. However, there are two nontrivial complete topologies for $C(X)$: the m-topology [14] and the (weaker) uniform topology. Furthermore, relative to the m-topology $C(X)$ is a topological ring (even a Q-ring with continuous inversion), and relative to the uniform topology, $C(X)$ is a topological l-group. It is these two cases that we shall be concerned with in this paper. In the first case we give several m-approximation theorems in the ring $C(X)$ and then by means of one of these we obtain a characterization of $C(X)$ as a ring. In the second case we obtain an m-approximation theorem (which implies a uniform approximation theorem) in the l-group $C(X)$ and then apply this to obtain a characterization of $C(X)$ as an l-group.

The first main portion of the paper (§§2–3) is devoted to the approximation theorems. To indicate what is involved in an m-approximation theorem, let A be a subring of $C(X)$. We observe first that none of the usual separation conditions alone assures that A is m-dense in $C(X)$. For clearly, even if A is normally separating (§1), its m-closure may lie entirely within $C^*(X)$. Even the additional requirement of inverse-closure (§1) is still inadequate to insure that A is m-dense in $C(X)$. (See Example 3.6.) Thus we are led to seek a still stronger set of conditions, and the ones we impose concern certain countable sums in A. For example, one of our main approximation results states that if A is normally separating and inverse-closed, and if $\sum_n f_n \in A$ whenever

(5) A topological group is *complete* provided that it is complete in the uniformity derived in the usual manner from the neighborhoods of the identity (cf. [7]).

$\{f_n\}$ is a countable subset of A such that the family $\{X-Z(f_n)\}$ of open supports is a star-finite cover of X, then A is m-dense in $C(X)$. Observe that this is not a strict generalization of the Stone-Weierstrass theorem even assuming that A is an algebra, since even if X is compact, point separation in A need not imply normal separation or inverse-closure.

The final portion of the paper (§§4–5) is devoted to obtaining characterizations of the ring and of the l-group $C(X)$. The only previous characterizations of $C(X)$, for X completely regular, were obtained by Blair and the author [2], considering $C(X)$ as an algebra and as a vector lattice. The latter results, although providing complete characterizations of $C(X)$, require certain "external" conditions; that is, to insure that an algebra A is isomorphic to all of $C(X)$, we require, in [2], that A not admit certain types of extensions(6). The characterizations of $C(X)$, as a ring and as an l-group, that we obtain in §5 of this paper do not depend on any conditions of such an external nature. Previously, other "internal" characterizations have been given for $C(X)$ for some special classes of noncompact spaces X. For example, in addition to Arens' characterization [4] for X locally compact and paracompact, Henriksen and Johnson [12] have characterizations for the cases in which X is either Lindelöf, locally compact and σ-compact, extremally disconnected, or discrete; the author [1] and Brainerd [8] have also given characterizations for the case in which X is a P-space.

Finally, I wish to acknowledge my indebtedness to my colleague R. L. Blair for the time he so generously spent in discussing this work during its preparation.

1. Preliminaries. Throughout this paper we shall deal exclusively with completely regular spaces. For such a space X denote by $C(X)$ the set of all real-valued continuous functions on X. We shall consider $C(X)$ variously as a commutative ring with identity and as a commutative l-group. In each of these the operations in $C(X)$ are the usual "pointwise" operations. For a detailed study of the ring $C(X)$ see Gillman and Jerison [25].

The set $C(X)$ admits several natural topologies. In this paper we shall be concerned with two of these, the u-topology and the m-topology [14]. The *u-topology* is simply the familiar topology of uniform convergence on X; that is, as a basis of (not necessarily open) neighborhoods of $f\in C(X)$ we take the collection of all sets

$$\{g \in C(X);\ |f(x) - g(x)| < \epsilon \text{ for all } x \in X\}$$

as ϵ ranges over the positive reals. The *m-topology* is defined by taking as a basis of open neighborhoods of $f\in C(X)$ the collection of all sets

$$\{g \in C(X);\ |f(x) - g(x)| < p(x) \text{ for all } x \in X\}$$

(6) This device, first used by Fan [10] for the case in which X is compact, provides a substitute for the Stone-Weierstrass theorem in obtaining characterizations of $C(X)$.

as p ranges over the set of strictly positive elements of $C(X)$. (An element $p \in C(X)$ is *strictly positive* in case $p(x) > 0$ for all $x \in X$.)

We now review briefly some important facts concerning these two topologies [14]. First, the u-topology is coarser than the m-topology, and they are equivalent if and only if every $f \in C(X)$ is bounded. Next, relative to each of them, $C(X)$ is a topological group under addition and the lattice operations $\vee$ and $\wedge$ are jointly continuous. If $C(X)$ contains any unbounded functions, then multiplication is not continuous in the u-topology; however, $C(X)$ is always a topological ring in the m-topology. In fact, in the m-topology, the set of invertible elements is open (i.e., $C(X)$ is a Q-ring [19]) and inversion is continuous where defined.

If $A \subseteq C(X)$, then we shall denote by A^u and A^m, respectively, the u-closure and the m-closure of A in $C(X)$. We have, of course, that $A^m \subseteq A^u$.

If $f \in C(X)$, then we set

$$Z(f) = \{x \in X; f(x) = 0\}.$$

A subset Z of X is a *zero set* in case $Z = Z(f)$ for some $f \in C(X)$. If $A \subseteq C(X)$, then we set

$$\mathfrak{Z}(A) = \{Z(f); f \in A\}.$$

In particular, we write

$$\mathfrak{Z}(X) = \mathfrak{Z}(C(X)).$$

An important property of $\mathfrak{Z}(X)$ is that it is closed under finite unions and countable(7) intersections [14, Theorem 33].

Two sets A, $B \subseteq X$ are *completely separated* in case there is an $f \in C(X)$ with $f(A) = 0$ and $f(B) = 1$. We shall write

$$A \perp B$$

if and only if A and B are completely separated. A fundamental result in this connection [14, Theorem 19] is that, for A, $B \subseteq X$, we have $A \perp B$ if and only if there exist $Z_1, Z_2 \in \mathfrak{Z}(X)$ such that $A \subseteq Z_1$, $B \subseteq Z_2$, and $Z_1 \cap Z_2 = \emptyset$. In particular, disjoint zero sets are completely separated.

We now turn our attention to properties of subsets of $C(X)$. A subset $A \subseteq C(X)$ is *regular* or *regularly separating* in case for each $x \in X$ and for each open set U of X containing x, there is an $f \in A$ such that $f(x) > 0$ and $f(y) = 0$ for all $y \in X - U$. Similarly, a subset $A \subseteq C(X)$ is *normal* or *normally separating* in case for every pair $Z_1, Z_2 \in \mathfrak{Z}(X)$, if $Z_1 \cap Z_2 = \emptyset$, then there is an $f \in A$ such that $f(Z_1) = 0$ and $f(Z_2) \geqq 1$. It is, of course, clear that $C(X)$ itself is both regular and normal. A normal subset of $C(X)$ is necessarily regular but, obviously, the converse need not hold. An immediate, but important, fact is that if A is an l-subgroup of $C(X)$, if $1 \in A$, and if A is normal, then

(7) We shall interpret "countable" as "at most countable."

$$\{f \in A; 0 \leqq f \leqq 1\}$$

is also normal (but not an l-subgroup).

If $f \in C(X)$ with $Z(f) = \emptyset$, then f has a multiplicative inverse f^{-1} in $C(X)$. We say that a subset $A \subseteq C(X)$ is *inverse-closed* in case whenever $f \in A$ and $f^{-1} \in C(X)$, then $f^{-1} \in A$.

Finally, if X is any topological space and if $S \subseteq X$, then $X - S$ denotes the complement, S^- denotes the closure, and S^0 denotes the interior of S in X. The letter R is reserved for the real number system.

2. **A general approximation theorem.** In this section we obtain certain necessary and sufficient conditions in order that an $f \in C(X)$ be in the m-closure of a subset A of $C(X)$. The significance of these conditions is that they depend only on the behavior of A and f on certain coverings of X.

Let $\mathcal{C}$ be a (not necessarily open) cover of X. If $S \subseteq X$, then we set

$$(S, \mathcal{C})^\star = \bigcup \{T \in \mathcal{C}; S \cap T \neq \emptyset\},$$
$$(S, \mathcal{C})^\wedge = \bigcup \{T \in \mathcal{C}; S \cap T = \emptyset\},$$

and

$$\mathcal{C}^\star = \{(S, \mathcal{C})^\star; S \in \mathcal{C}\}.$$

We now recall some properties of coverings. A cover $\mathcal{D}$ is a *star refinement* of $\mathcal{C}$ in case $\mathcal{D}^\star$ refines $\mathcal{C}$. An open cover $\mathcal{C}$ is *normal* [24] provided that there exists a sequence $\{\mathcal{C}_n\}$ of open covers of X such that $\mathcal{C}_1 = \mathcal{C}$ and $\mathcal{C}_{n+1}$ is a star refinement of $\mathcal{C}_n$ ($n = 1, 2, \cdots$). A cover $\mathcal{C}$ of X is *star finite* in case each $S \in \mathcal{C}$ meets (i.e., has nonvoid intersection with) at most finitely many members of $\mathcal{C}$. A cover $\mathcal{C}$ is *locally finite* in case each $x \in X$ has a neighborhood which meets at most finitely many members of $\mathcal{C}$. In general, neither of these properties implies the other, although a star finite open cover is necessarily locally finite. Finally, a cover $\mathcal{C}$ of X is a *$\mathcal{Z}$-cover* in case each member of $\mathcal{C}$ is a zero set of X.

LEMMA 2.1. *If $\mathcal{C}$ is a locally finite star finite $\mathcal{Z}$-cover of X, then, for each $Z \in \mathcal{C}$, Z is completely separated from $(Z, \mathcal{C})^\wedge$.*

Proof. With no loss of generality we may assume that $Z \neq \emptyset$. Set

$$W = \bigcup \{Z_1 \in \mathcal{C};\ Z_1 \cap (Z, \mathcal{C})^\star \neq \emptyset \text{ and } Z_1 \cap Z = \emptyset\}.$$

Since W is the union of a finite family of zero sets each disjoint from Z, W is a zero set disjoint from Z. Thus, there is an $f \in C(X)$ such that $f(Z) = 1$ and $f(W) = 0$. Define the real-valued function g on X by $g = f$ on $(Z, \mathcal{C})^\star \cup W$ and $g = 0$ otherwise. It follows that $g \in C(X)$ and completely separates Z and $(Z, \mathcal{C})^\wedge$.

Let $\mathcal{S}$ be a family of subsets of X. By an *interval function* on $\mathcal{S}$ we mean a mapping ϕ from $\mathcal{S}$ to the family of all nonempty open intervals of R. An interval function ϕ on $\mathcal{S}$ is *montone* in case for all S, $T\in\mathcal{S}$, if $S\subseteq T$, then $\phi(S)\subseteq\phi(T)$.

LEMMA 2.2. *Let $\mathcal{C}$ be a locally finite star finite $\mathcal{Z}$-cover of X, and, for each $Z\in\mathcal{C}$, let V_Z be an open set such that $Z\perp(X-V_Z)$. Then there exists a star finite normal open cover $\mathcal{C}_0$ of X and a one-one mapping d from $\mathcal{C}$ onto $\mathcal{C}_0$ such that*

(i) *for all $Z\in\mathcal{C}$, $Z\subseteq d(Z)\subseteq V_Z$;*

(ii) *if ϕ is a monotone interval function on $\mathcal{C}\cup\mathcal{C}^\star$, then the interval function ϕ_0 defined on $\mathcal{C}_0\cup\mathcal{C}_0^\star$ by $\phi_0(d(Z))=\phi(Z)$ and $\phi_0((d(Z),\ \mathcal{C}_0)^\star)=\phi((Z,\ \mathcal{C})^\star)$ is monotone.*

Proof. First, let $\mathfrak{F}$ be the set of all pairs $(\mathcal{H}, h)$ where $\mathcal{H}\subseteq\mathcal{C}$ and h is a mapping from $\mathcal{H}$ into the collection of open sets of X such that (a) for all $Z\in\mathcal{H}$, $Z\subseteq h(Z)\subseteq V_Z$ and $h(Z)\perp(Z,\ \mathcal{C})^\wedge$ and such that (b) for all Z, $W\in\mathcal{H}$, if $Z\cap W=\varnothing$, then $h(Z)\cap h(W)=\varnothing$. It follows from Lemma 2.1 that $\mathfrak{F}\neq\varnothing$. Partially order $\mathfrak{F}$ by $(\mathcal{H}_1, h_1)\leqq(\mathcal{H}_2, h_2)$ in case $\mathcal{H}_1\subseteq\mathcal{H}_2$ and h_2 agrees with h_1 on $\mathcal{H}_1$. Since Zorn's Lemma is applicable, there exists a maximal pair $(\mathcal{H}, h)$ in $\mathfrak{F}$. We claim that $\mathcal{H}=\mathcal{C}$. For suppose, on the contrary, that there exists a $Z\in\mathcal{C}-\mathcal{H}$. Let $Z_1, \cdots, Z_n$ be the (necessarily finite) set of elements of $\mathcal{H}$ each of which meets $(Z,\ \mathcal{C})^\star$ and is disjoint from Z. Then since $Z\subseteq(Z_i,\ \mathcal{C})^\wedge$ we have, by Lemma 2.1, that $Z\perp h(Z_i)$ $(i=1, \cdots, n)$. Hence, there is an open set $U_Z\subseteq V_Z$ containing Z and completely separated from

$$(Z,\ \mathcal{C})^\wedge \cup \left(\bigcup_{i=1}^{n} h(Z_i)\right).$$

This is easily seen to contradict the maximality of $(\mathcal{H}, h)$; thus, $\mathcal{H}=\mathcal{C}$, as claimed.

Now, for each pair Z, $W\in\mathcal{C}$, if $W\not\subseteq(Z,\ \mathcal{C})^\star$ and if $W\cap((Z,\ \mathcal{C})^\star,\ \mathcal{C})^\star\neq\varnothing$, choose an element

$$s_{ZW}\in W-(Z,\ \mathcal{C})^\star.$$

Let

$$S=\{s_{ZW};\ Z, W\in\mathcal{C},\ W\not\subseteq(Z,\ \mathcal{C})^\star,\quad\text{and}\quad W\cap((Z,\ \mathcal{C})^\star,\ \mathcal{C})^\star\neq\varnothing\}.$$

Then it is easily seen that S is closed and discrete in X. Next, for each pair Z, $W\in\mathcal{C}$, if $W\not\subseteq Z$, and if $W\cap Z\neq\varnothing$, choose an element

$$t_{ZW}\in W-Z.$$

Let

$$T=\{t_{ZW};\ Z, W\in\mathcal{C},\ W\not\subseteq Z,\ \text{and}\ W\cap Z\neq\varnothing\}.$$

Then T is closed and discrete in X. For each $Z\in\mathcal{C}$, set

$$U_Z=[h(Z)-(S\cup T)]\cup Z.$$

It is clear, then, that

$$\mathcal{C}_0 = \{U_Z; Z \in \mathcal{C}\}$$

is an open cover of X which satisfies, for all Z, $W \in \mathcal{C}$:

(1) $Z \subseteq U_Z \subseteq V_Z$;

(2) $U_W \subseteq U_Z$ implies $W \subseteq Z$;

(3) $U_W \subseteq (U_Z, \mathcal{C}_0)^\star$ implies $W \subseteq (Z, \mathcal{C})^\star$.

In particular, if $\mathcal{C}_0$ is normal, then $\mathcal{C}_0$ and the mapping $d: Z \rightarrow U_Z$ will satisfy the conclusions of the lemma. To see that $\mathcal{C}_0$ is normal, it suffices to show that, for each $Z \in \mathcal{C}$, if $G_Z = \bigcup\{U_W; W \neq Z\}$, then $(X - G_Z) \perp (X - U_Z)$. (Cf. [24, Chapter V, Theorem 5.3 and Theorem 9.3].) But $X - G_Z \subseteq Z$ and $Z \perp (X - U_Z)$. Thus, $\mathcal{C}_0$ is normal and the proof is complete.

Let $\mathcal{C}$ be a cover of X and let $f \in C(X)$. We denote by $\mathcal{C}(f)$ the set of all monotone interval functions ϕ on $\mathcal{C} \cup \mathcal{C}^\star$ having the property that

$$f(S)^- \subseteq \phi((S, \mathcal{C})^\star)$$

for all $S \in \mathcal{C}$. If $\mathfrak{C}$ is a collection of covers of X, if $f \in C(X)$, and if $A \subseteq C(X)$, then we say that A *$\star$-approximates f on $\mathfrak{C}$* in case

$$\mathcal{C}(f) \subseteq \bigcup\{\mathcal{C}(g); g \in A\}$$

for each $\mathcal{C} \in \mathfrak{C}$. We now have the principal result of this section.

THEOREM 2.3. *Let $A \subseteq C(X)$ and let $f \in C(X)$. Then the following statements are equivalent*:

(1) $f \in A^m$;

(2) *A $\star$-approximates f on the collection of all star finite normal open covers of X*;

(3) *A $\star$-approximates f on the collection of all star finite locally finite $\mathfrak{Z}$-covers of X*;

(4) *A $\star$-approximates f on the collection of all countable star finite locally finite $\mathfrak{Z}$-covers of X.*

Proof. (1) *implies* (2): Let $f \in A^m$, let $\mathcal{C}$ be a star finite normal open cover of X, and let $\phi \in \mathcal{C}(f)$. For each $U \in \mathcal{C}$, let (α_U, β_U) be a (possibly unbounded) open interval such that

$$f(U)^- \subseteq (\alpha_U, \beta_U) \subseteq (\alpha_U, \beta_U)^- \subseteq \phi((U, \mathcal{C})^\star).$$

It is clear that if

$$Z = \{x \in X; f(x) \leqq \alpha_U\},$$
$$Y = \{x \in X; f(x) \geqq \beta_U\},$$

then $U \perp (Z \cup Y)$. Also, since $\mathcal{C}$ is normal, $U \perp W$ where $W = X - (U, \mathcal{C})^\star$. Hence, $U \perp (W \cup Z \cup Y)$. Therefore, there exist h_U, $k_U \in C(X)$ having the properties:

(i) $h_U \leqq \alpha_U \vee (f-1)$ and $k_U \geqq \beta_U \wedge (f+1)$ with equality holding in each case on U;

(ii) $h_U = f-1$ and $k_U = f+1$ on $W \cup Z \cup Y$. Now set

$$h = \vee \mathfrak{C}\, h_U \quad \text{and} \quad k = \wedge \mathfrak{C}\, k_U.$$

Since $\mathfrak{C}$ is star finite, it follows that $h, k \in C(X)$. Moreover, it is easily seen that, for each $x \in X$,

$$h(x) < f(x) < k(x).$$

Also, for any $g \in C(X)$, if $h \leqq g \leqq k$, then $\phi \in \mathfrak{C}(g)$. But, since $f \in A^m$, there must exist a $g \in A$, such that $h < g < k$. Hence $\phi \in \mathfrak{C}(g)$ for some $g \in A$, as desired.

(2) *implies* (3): Let $\mathfrak{C}$ be a star finite locally finite $\mathfrak{Z}$-cover of X and let $\phi \in \mathfrak{C}(f)$. For each $Z \in \mathfrak{C}$, let α_Z, β_Z be extended real numbers such that

$$f(Z)^- \subseteq (\alpha_Z, \beta_Z) \subseteq (\alpha_Z, \beta_Z)^- \subseteq \phi((Z, \mathfrak{C})^\star),$$

and set $V_Z = f^{-1}(\alpha_Z, \beta_Z)$. Then $Z \perp (X - V_Z)$ for each $Z \in \mathfrak{C}$. Hence, Lemma 2.2 is applicable. Let $\mathfrak{C}_0$, d, and ϕ_0 be as in the conclusion of that lemma. Then for each $Z \in \mathfrak{C}$,

$$f(d(Z))^- \subseteq f(V_Z)^- \subseteq \phi((Z, \mathfrak{C})^\star) = \phi_0((d(Z), \mathfrak{C}_0)^\star),$$

so that $\phi_0 \in \mathfrak{C}_0(f)$. Therefore, by (2), $\phi_0 \in \mathfrak{C}_0(g)$ for some $g \in A$. But since $Z \subseteq d(Z)$ for each $Z \in \mathfrak{C}$, we clearly have that $\phi \in \mathfrak{C}(g)$, as desired.

(3) *implies* (4): Trivial.

(4) *implies* (1): We must show that if $h \in C(X)$ is a strictly positive function, then there is a $g \in A$ such that

$$|f(x) - g(x)| < h(x)$$

for all $x \in X$. Clearly, we may assume that $h < 1$. Now for each pair m, n of integers with $n > 0$, set

$$H_n = \{x \in X;\ 2^{-n+1} \leqq 1/3h(x) \leqq 2^{-n+2}\}$$

and

$$Z_{mn} = \{x \in H_n;\ (m-1)2^{-n-1} \leqq f(x) \leqq m2^{-n-1}\}.$$

Then $\mathfrak{C} = \{Z_{mn};\ Z_{mn} \neq \emptyset\}$ is obviously a countable $\mathfrak{Z}$-cover of X. Moreover, by routine arguments it follows that if $Z_{mn} \cap Z_{pq} \neq \emptyset$, then exactly one of the following statements holds:

(i) $\quad q = n-1$ and $1/2(m-1) \leqq p \leqq 1/2(m+2)$;

(ii) $\quad q = n$ and $m-1 \leqq p \leqq m+1$;

(iii) $\quad q = n+1$ and $2m-2 \leqq p \leqq 2m+1$.

From this it is evident that $\mathfrak{C}$ is both star finite and locally finite. For each pair m, n with $Z_{mn} \neq \emptyset$, set

$$\alpha_{mn} = \bigwedge \left\{ (p-2)2^{-q-1};\ Z_{pq} \cap Z_{mn} \neq \emptyset \right\},$$
$$\beta_{mn} = \bigvee \left\{ (p+1)2^{-q-1};\ Z_{pq} \cap Z_{mn} \neq \emptyset \right\},$$
$$\overset{\star}{\alpha}_{mn} = \bigwedge \left\{ \alpha_{pq};\ Z_{pq} \cap Z_{mn} \neq \emptyset \right\},$$

and

$$\overset{\star}{\beta}_{mn} = \bigvee \left\{ \beta_{pq};\ Z_{pq} \cap Z_{mn} \neq \emptyset \right\}.$$

It follows that if $Z_{mn} \neq \emptyset$, then

$$(m-5)2^{-n-1} \leqq \alpha_{mn} < \beta_{mn} \leqq (m+4)2^{-n-1}$$

and

$$(m-11)2^{-n-1} \leqq \overset{\star}{\alpha}_{mn} < \overset{\star}{\beta}_{mn} \leqq (m+10)2^{-n-1}.$$

Now define an interval function ϕ on $\mathcal{C} \cup \mathcal{C}^{\star}$ by

$$\phi(Z_{mn}) = (\alpha_{mn}, \beta_{mn}),$$
$$\phi((Z_{mn}, \mathcal{C})^{\star}) = (\overset{\star}{\alpha}_{mn}, \overset{\star}{\beta}_{mn}).$$

Then one easily shows that ϕ is monotone and that $\phi \in \mathcal{C}(f)$. Hence, assuming (4), we have that $\phi \in \mathcal{C}(g)$ for some $g \in A$. Thus, for each $x \in Z_{mn}$,

$$\overset{\star}{\alpha}_{mn} < g(x) < \overset{\star}{\beta}_{mn}.$$

Therefore, if $x \in Z_{mn}$,

$$|f(x) - g(x)| \leqq (2^{-n-1})(11) < h(x).$$

However, since $\mathcal{C}$ is a cover of X, it follows that $|f(x)-g(x)| < h(x)$ for all $x \in X$, and therefore, the proof is complete.

3. **Special ring and l-group approximation theorems.** From our general approximation theorem of the preceding section we now derive some special cases for rings and for l-groups of functions. In essence we show that for such a subsystem of $C(X)$ to be m-dense in $C(X)$ it is sufficient that it contain the rational constants, be normally separating, and be closed under certain countable operations.

We begin with a lemma concerning sequences of positive real numbers.

LEMMA 3.1. *Let (γ_n) and (e_n) be sequences of positive real numbers and assume that $0 < e_n \leqq 1$ for each n. Define*

$$S(1) = \gamma_1,$$
$$S(2) = \gamma_1 e_1 + \gamma_2(1 - e_1),$$

and, for $n>2$,

$$S(n) = \gamma_1 e_1 + \sum_{k=2}^{n-1}\left[\gamma_k e_k \prod_{i=1}^{k-1}(1-e_i)\right] + \gamma_n \prod_{i=1}^{n}(1-e_i).$$

Then, for each n,

$$\bigwedge_{j=1}^{n} \gamma_j \leqq S(n) \leqq \bigvee_{j=1}^{n} \gamma_j.$$

Proof. We obtain the proof by induction. First, it is easily seen that

$$\begin{aligned}
S(n+1) &= S(n) + \gamma_n(e_n - 1)\prod_{i=1}^{n-1}(1-e_i) + \gamma_{n+1}\prod_{i=1}^{n}(1-e_i)\\
&= S(n) + (\gamma_{n+1}-\gamma_n)\prod_{i=1}^{n}(1-e_i)\\
&= S(n-1) + (\gamma_n - \gamma_{n-1})\prod_{i=1}^{n-1}(1-e_i) + (\gamma_{n+1}-\gamma_n)\prod_{i=1}^{n}(1-e_i)\\
&= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&= \gamma_1 + \sum_{k=1}^{n}\left[(\gamma_{k+1}-\gamma_k)\prod_{i=1}^{k}(1-e_i)\right]\\
&= \gamma_1 - \gamma_1(1-e_1)\\
&\quad + (1-e_1)\left\{\gamma_2 + \sum_{k=2}^{n}\left[(\gamma_{k+1}-\gamma_k)\prod_{i=2}^{k}(1-e_i)\right]\right\}.
\end{aligned}$$

Thus, if the desired inequality holds for any set of n terms, we have

$$\bigwedge_{j=2}^{n+1}\gamma_j \leqq \gamma_2 + \sum_{k=2}^{n}\left[(\gamma_{k+1}-\gamma_k)\prod_{i=2}^{k}(1-e_i)\right] \leqq \bigvee_{j=2}^{n+1}\gamma_j.$$

Therefore, since for any $\delta>0$

$$\gamma_1 \wedge \delta \leqq \gamma_1 e_1 + \delta(1-e_1) \leqq \gamma_1 \vee \delta,$$

we have

$$\bigwedge_{j=1}^{n+1}\gamma_j \leqq S(n+1) \leqq \bigvee_{j=1}^{n+1}\gamma_j.$$

Finally, since the lemma is obviously true for $n=1$, the proof is complete.

A countable set $\{f_n\}$ in $C(X)$ is a *$\sigma^\bigstar$-set* in case the family $\{X-Z(f_n);\, n=1, 2, \cdots\}$ is a star finite cover of X. A $\sigma^\bigstar$-set $\{f_n\}$ is positive in case $f_n \geqq 0$ for all n. It is obvious that if $\{f_n\}$ is a $\sigma^\bigstar$-set in $C(X)$, then its pointwise sum $\sum_n f_n$ is in $C(X)$.

THEOREM 3.2. *Let X be completely regular and let A be a divisible*[8] *subring of $C(X)$ which satisfies*:

(1) $\{f\in A; 0\leqq f\leqq 1\}$ *normally separates* X;

(2) *For every positive $\sigma^{\star}$-set $\{f_n\}$ in A, $\sum_n f_n\in A$.*

Then A is m-dense in $C(X)$.

Proof. First, it follows from condition (1) that $1\in A$, and therefore, since A is divisible, A contains all rational constants. Since A is a subgroup of $C(X)$, it will suffice to show that if $f\geqq 0$ in $C(X)$, then $f\in A^m$. So let $f\geqq 0$ in $C(X)$, let $\mathfrak{C}=\{Z_n\}$ be a countable star finite locally finite $\mathfrak{Z}$-cover of X, and let $\phi\in\mathfrak{C}(f)$. For each n, set

$$\alpha_n = 0\vee[\vee\{\wedge\phi((Z_m,\mathfrak{C})^\star); Z_n\subseteq (Z_m,\mathfrak{C})^\star\}],$$
$$\beta_n = \wedge\{\vee\phi((Z_m,\mathfrak{C})^\star); Z_n\subseteq(Z_m,\mathfrak{C})^\star\}.$$

Since $f\geqq 0$ and $\phi\in\mathfrak{C}(f)$, it is clear that $\alpha_n<\beta_n$. Thus, for each n, there is a constant $\gamma_n\in A$ such that $\alpha_n<\gamma_n<\beta_n$. From (1) and Lemma 2.1 it follows that, for each n, there is an $e_n\in A$ such that $0\leqq e_n\leqq 1$, $e_n(Z_n)=1$, and $e_n((Z_n,\mathfrak{C})^\wedge)=0$. Now set

$$f_1=\gamma_1 e_1,$$

and, for all $n>1$, set

$$f_n=\gamma_n e_n\prod_{i=1}^{n-1}(1-e_i).$$

Then, for each n, $f_n\in A$ and $f_n\geqq 0$. Also, the family $\{X-Z(f_n)\}$ is obviously star finite. Moreover, if $x\in X$, then there is a least positive integer k such that $e_k(x)=1$ so that $f_k(x)\neq 0$. Therefore, $\{X-Z(f_n)\}$ is a star finite cover of X. Hence, by (2), $\sum_n f_n\in A$.

To complete the proof it will suffice, in view of Theorem 2.3, to show that $\phi\in\mathfrak{C}(\sum_n f_n)$. So let $Z_n\in\mathfrak{C}$ and $x\in Z_n$. If p is the least positive integer for which $e_p(x)\neq 0$, then $p\leqq n$ and $f_j(x)=0$ whenever $j<p$ or $j>n$. Consequently,

$$\sum_j f_j(x) = \begin{cases}\gamma_p=\gamma_n & \text{if } p=n,\\ \gamma_p e_p(x)+\gamma_{p+1}(1-e_p(x)) & \text{if } p=n-1,\\ \gamma_p e_p(x)+\sum_{k=p+1}^{n}\left[\gamma_k e_k(x)\prod_{i=p}^{k-1}(1-e_i(x))\right] & \text{if } p<n-1.\end{cases}$$

Therefore, it follows from Lemma 3.1, that

$$\wedge\{\gamma_m; e_m(x)\neq 0\}\leqq\sum_j f_j(x)\leqq\vee\{\gamma_m; e_m(x)\neq 0\}.$$

---

[8] A group (ring) A is *divisible* in case for every $f\in A$ and every integer n, there is a $g\in A$ with $f=ng$.

But, if $e_m(x) \neq 0$, then $Z_m \subseteq (Z_n, \mathfrak{C})^\star$. Thus

$$\bigwedge \{\gamma_m; Z_m \subseteq (Z_n, \mathfrak{C})^\star\} \leqq \sum_j f_j(x) \leqq \bigvee \{\gamma_m; Z_m \subseteq (Z_n, \mathfrak{C})^\star\}.$$

However,

$$\bigwedge \phi((Z_n, \mathfrak{C})^\star) < \bigwedge \{\gamma_m; Z_m \subseteq (Z_n, \mathfrak{C})^\star\}$$
$$\leqq \bigvee \{\gamma_m; Z_m \subseteq (Z_n, \mathfrak{C})^\star\} < \bigvee \phi((Z_n, \mathfrak{C})^\star).$$

Thus, $[(\sum_j f_j)(Z_n)]^- \subseteq \phi((Z_n, \mathfrak{C})^\star)$ and the proof is complete.

We say that a subset $A \subseteq C(X)$ is *pseudonormal* in case for every $Z_1, Z_2 \in \mathfrak{Z}(X)$, if $Z_1 \cap Z_2 = \varnothing$, then there is an $f \in A$ such that $Z_1 \subseteq Z(f)$ and $Z_2 \cap Z(f) = \varnothing$. It is clear that a normal subset of $C(X)$ is pseudonormal. The converse, however, is not generally true.

COROLLARY 3.3. *Let X be completely regular and let A be a subring of $C(X)$ which satisfies*:

(1) *A is pseudonormal*;

(2) *A is inverse-closed*;

(3) *for every positive $\sigma^\star$-set $\{f_n\}$ in A, $\sum_n f_n \in A$.*

Then A is m-dense in $C(X)$.

Proof. From (1) it is clear that A contains a nowhere vanishing function. Thus, from (2), we conclude that A contains the rational constants, and so A is divisible. Now let $Z_1, Z_2 \in \mathfrak{Z}(X)$ with $Z_1 \cap Z_2 = \varnothing$. By (1) there exists an $f_1 \in A$ such that $Z_1 \subseteq Z(f_1)$ and $Z_2 \cap Z(f_1) = \varnothing$. In view of the latter fact and (1), there is an $f_2 \in A$ such that $Z_2 \subseteq Z(f_2)$ and $Z(f_1) \cap Z(f_2) = \varnothing$. Thus $f_1^2 + f_2^2$ is a nowhere vanishing function in A, so that

$$f = f_1^2(f_1^2 + f_2^2)^{-1}$$

is in A. Clearly, $0 \leqq f \leqq 1$, $f = 0$ on Z_1, and $f = 1$ on Z_2. That is, $\{f \in A; 0 \leqq f \leqq 1\}$ is normally separating. An application of Theorem 3.2 now completes the proof.

Still another sufficient condition for m-denseness can be based on the concept of "$\sigma^\star$-regularity." Let A be a subring of $C(X)$ and let $\{f_n\}$ be a $\sigma^\star$-set in A. For each n, set(9)

$$f_n^\star = \sum \{f_m; f_m f_n \neq 0\}.$$

We say that A is *$\sigma^\star$-regular* in case for every $\sigma^\star$-set $\{f_n\}$ in A such that $Z(\sum_n f_n) = \varnothing$, there is an $f \in A$ with

$$f f_n^\star f_n = f_n$$

(9) We shall assume that in any group A, $\sum \varnothing = 0$.

for all n. Note that, for a $\sigma^\star$-set $\{f_n\}$, $Z(\sum_n f_n)=\emptyset$ if and only if $\{f_n^\star f_n\}$ is a $\sigma^\star$-set. Also, it is obvious that A is $\sigma^\star$-regular if and only if, for every $\sigma^\star$-set $\{f_n\}$ in A with $Z(\sum_n f_n)=\emptyset$, A contains the inverse $(\sum_n f_n)^{-1}$.

COROLLARY 3.4. *Let X be completely regular and let A be a subring of $C(X)$ which satisfies:*

(1) *A is pseudonormal;*

(2) *A is $\sigma^\star$-regular.*

Then A is m-dense in $C(X)$.

Proof. Since any nowhere vanishing function is a $\sigma^\star$-set, it follows from (2) that A is inverse-closed. If $\{f_n\}$ is a positive $\sigma^\star$-set in A, then clearly $Z(\sum_n f_n)=\emptyset$. Thus, $(\sum_n f_n)^{-1}\in A$, so that, since A is inverse-closed, $\sum_n f_n\in A$. Therefore, in view of the previous corollary, the proof is complete.

A sublattice A of $C(X)$ is said to be $\sigma^\star$*-complete* in case every $\sigma^\star$-set $\{f_n\}$ in A has a least upper bound in A. It is clear that $C(X)$, itself, is $\sigma^\star$-complete. We now prove an m-approximation theorem for l-subgroups of $C(X)$.

THEOREM 3.5. *Let X be completely regular and let A be a divisible l-subgroup of $C(X)$ which satisfies:*

(1) *A is pseudonormal;*

(2) *A is $\sigma^\star$-complete.*

Then A is m-dense (and consequently u-dense) in $C(X)$.

Proof. We obtain the proof by considering three cases of increasing generality.

CASE 1. A is normal and $1\in A$. It will suffice to show that $f\in A^m$ for every $f\geqq 0$ in $C(X)$. So let $f\geqq 0$ in $C(X)$, let $\mathfrak{C}=\{Z_n\}$ be a countable star finite locally finite $\mathfrak{Z}$-cover of X, and let $\phi\in\mathfrak{C}(f)$. Choose α_n, β_n as in the proof of Theorem 3.2. Since $1\in A$ and since A is divisible, A contains all rational constants. Hence, for each n, there is a rational constant $\gamma_n\in A$ with $\alpha_n<\gamma_n<\beta_n$. Since A is a sublattice of $C(X)$, and since $1\in A$, it follows that $\{g\in A;\ 0\leqq g\leqq 1\}$ normally separates X. We may, therefore, choose a sequence $\{e_n\}$ in A as in the proof of Theorem 3.2. Now, for each n, set $f_n=\gamma_n e_n$. Then obviously $\{f_n\}$ is a $\sigma^\star$-set in A, and so, by (2), has a least upper bound $\bigvee_n f_n$ in A. But, by the infinite distributivity in A [6, p. 231], we conclude that $\bigvee_n f_n$ is, in fact, the pointwise supremum of $\{f_n\}$. Finally, from our choice of γ_n, it is clear that, for each n,

$$[(\bigvee_k f_k)(Z_n)]^- \subseteq \phi((Z_n,\ \mathfrak{C})^\star).$$

Thus it follows from Theorem 2.3 that $f\in A^m$.

CASE 2. A is pseudonormal and $1\in A$. Let Z_1, $Z_2\in\mathfrak{Z}(X)$ with $Z_1\cap Z_2=\emptyset$. Then there is an $e\in C(X)$ with $e(Z_1)=1$ and $e(Z_2)=0$. Set

$$Z_3 = \{x \in X;\ e(x) \geqq 1/4\};$$
$$Z_4 = \{x \in X;\ e(x) \leqq 1/2\}.$$

Then Z_3, $Z_4 \in \mathcal{Z}(x)$ with $Z_1 \cap Z_4 = Z_2 \cap Z_3 = \emptyset$ and $Z_3 \cup Z_4 = X$. Since A is pseudonormal and since $1 \in A$, there exist f, $g \in A$ such that $0 \leqq f$, $g \leqq 1/2$, $f(Z_2) = g(Z_1) = 0$, $f(Z_3) > 0$, and $g(Z_4) > 0$. Thus $(f \vee g)(x) > 0$ for all $x \in X$. Now for $n = 1, 2, \cdots$, set

$$U_n = \left\{x \in X; \frac{1}{n+3} < (f \vee g)(x) < \frac{1}{n}\right\},$$

and

$$W_n = \left\{x \in X; \frac{1}{n+2} \leqq (f \vee g)(x) \leqq \frac{1}{n+1}\right\}.$$

Then both $\{U_n; n = 1, 2, \cdots\}$ and $\{W_n; n = 1, 2, \cdots\}$ are star finite covers of X and $W_n \subseteq U_n$ $(n = 1, 2, \cdots)$. Next, for each n, set

$$h_n = [(6nf) \vee g]$$
$$\wedge \left\{(24n^2)\left[\left(\left((f \vee g) - \frac{1}{n+3}\right) \wedge \left(\frac{1}{n} - (f \vee g)\right)\right) \vee 0\right]\right\}.$$

From the readily proved fact that

$$W_n \subseteq X - Z(h_n) \subseteq U_n,$$

it follows that $\{h_n\}$ is a $\sigma^\star$-set in A; thus,

$$h = \vee_n h_n \in A.$$

With a little diligence it can be shown that if $x \in Z_1 \cap W_n$, then (since $g(x) = 0$) $h_n(x) \geqq 2$, and that if $x \in Z_2$, then (since $f(x) = 0$), $h_n(x) \leqq 1/2$ for all n. Therefore, $h(Z_1) \geqq 2$, and $h(Z_2) \leqq 1/2$. Since $(h-1) \vee 0 \in A$, it follows that A normally separates X, and so Case 1 is applicable.

Case 3. A is pseudonormal. The pseudonormality alone implies that A contains some strictly positive element e. Define a mapping $f \rightarrow f^*$ of A into $C(X)$ by

$$f^*(x) = f(x)/e(x)$$

for all $x \in X$. By means of elementary arguments, one verifies that: $f \rightarrow f^*$ is an l-group isomorphism of A onto an l-subgroup A^* of $C(X)$; A^* is pseudonormal and $1 = e^* \in A^*$; a set $\{f_n\}$ in A is a $\sigma^\star$-set if and only if $\{f_n^*\}$ in A^* is a $\sigma^\star$-set; and finally, if $\{f_n\}$ is a $\sigma^\star$-set in A, then

$$(\vee_n f_n)^* = \vee_n f_n^*.$$

Thus, by Case 2, it follows that A^* is m-dense in $C(X)$. Since the mapping ϕ: $g \rightarrow eg$ is obviously an automorphism of $C(X)$ preserving the set of strictly positive functions, we conclude finally that A, the image of A^* under ϕ, is

m-dense in $\phi(C(X)) = C(X)$, and the proof is complete.

We observe that neither Theorem 3.2 nor its corollaries implies the Stone-Weierstrass Theorem, since even for compact spaces, point separation need not imply pseudonormal separation. However, if X is compact, then any divisible l-subgroup of $C(X)$ which separates points and contains a strictly positive function is necessarily pseudonormal. Thus, Theorem 3.5 does imply that such an l-subgroup is uniformly dense in $C(X)$.

Although the sufficient conditions we have given in the above approximation theorems are not always necessary for subrings or l-subgroups of $C(X)$ to be m-dense, the following examples show that none of these conditions can simply be omitted.

EXAMPLE 3.6. Let X be an infinite discrete space and let A be the set of all $f \in C(X)$ such that the range of f is finite. Then, clearly, A is normal, is not u-dense, and is m-closed. Moreover, as a ring, A is inverse-closed, and as an l-group, A is divisible. We observe, however, that by Hewitt's theorem [13], A is u-dense in $C^*(X)$.

EXAMPLE 3.7. Let X be an uncountable discrete space and let A be the set of all $f \in C(X)$ such that for some $\alpha \in R$, $Z(f-\alpha)$ is countable. Then A is regular, is not normal, and is neither u-dense nor m-dense in $C(X)$. As a ring, A is $\sigma^\star$-regular, and as an l-group, A is $\sigma^\star$-complete and σ-complete.

EXAMPLE 3.8. Let X be the rationals in their usual topology. For each pair $\alpha < \beta$ of irrationals, let $\chi_{\alpha\beta}$ be the characteristic function of the open interval (α, β) in X and let $f_{\alpha\beta} \in C(X)$ be defined by

$$f_{\alpha\beta}(x) = \chi_{\alpha\beta}(x)(x-\alpha)^{-2}(x-\beta)^{-2}$$

for all $x \in X$. Let A_0 be the subring of $C(X)$ generated by the collection of all the $f_{\alpha\beta}$, and let A be the smallest subring of $C(X)$ containing the constants R and A_0 such that A is closed under sums of positive $\sigma^\star$-sets. Then A is normal, divisible, and satisfies (2) of Theorem 3.2. However, A is not m-dense in $C(X)$ since, for example, it fails to approximate the characteristic function of $\{x \in X; \pi < x < \infty\}$ in $C(X)$.

4. **Characterizations of normal subsystems.** The main purpose of this section is to give conditions under which a ring (l-group) can be isomorphically represented as a normally separating subring (l-subgroup) of $C(X)$ for some completely regular space X. As corollaries of these and the results of §3 we describe a class of rings (l-groups) isomorphic to m-dense subrings (l-subgroups) of some $C(X)$. En route we state characterizations of regular subsystems which are obvious ring and l-group formulations of those given in [2] for algebras and vector lattices.

Let X be completely regular and let $A \subseteq C(X)$. The weak topology on X determined by A agrees with the original topology on X if and only if there is a sub-basis $\mathcal{B}$ of open sets in X such that, for every $U \in \mathcal{B}$ and $x \in U$, there exists an $\epsilon > 0$ in R and an $f \in A$ such that

$$|f(x) - f(y)| \geqq \epsilon$$

for all $y \in X - U$. When this is the case, we say that A is *weakly pseudoregular* [2]. In general, a weakly pseudoregular subring of $C(X)$ need not be regular. For example, the ring of all polynomial functions on R is such a subring of $C(R)$.

LEMMA 4.1. *Let A be a weakly pseudoregular subring or l-subgroup of $C(X)$ and let $1 \in A$. Then, for every $x \in X$ and every open neighborhood U of x, there is an $f \in A$ such that $0 \leqq f(x) \leqq 1$ and $f(y) \geqq 2$ for all $y \in X - U$. In particular, if A is an l-subgroup, then it is regular.*

Proof. First, let $x \in X$ and let V be a neighborhood of x such that, for some $\epsilon > 0$ and some $g \in A$, $|g(x) - g(y)| \geqq \epsilon$ for all $y \in X - V$. Let $r > 0$ be an integer. Then, for some integer n, $n\epsilon \geqq r+1$, so that

$$|ng(x) - ng(y)| \geqq r + 1$$

for all $y \in X - V$. Let m be an integer such that $|m - ng(x)| < 1$. If A is a ring, set $f = (m - ng)^2$ and if A is an l-group, set $f = |m - ng|$. In either case, $0 \leqq f(x) < 1$ and $f(y) \geqq r$ for all $y \in X - V$.

Now let $x \in X$ and let U be an arbitrary neighborhood of x. Since A is weakly pseudoregular, there exist neighborhoods $V_1, \cdots, V_k$ of x such that

$$\bigcap_{i=1}^{k} V_i \subseteq U$$

and such that, for each i, there is an $\epsilon_i > 0$ in R and a $g_i \in A$ with

$$|g_i(x) - g_i(y)| \geqq \epsilon_i$$

for all $y \in X - V_i$. We have then, from the first paragraph, that there exists, for each $i = 1, \cdots, k$, an $f_i \in A$ such that $f_i \geqq 0$, $0 \leqq f_i(x) < 1$, and $f_i(y) \geqq k+2$ for all $y \in X - V_i$. Set

$$h = \sum_{i=1}^{k} f_i.$$

Let p be the least integer such that $h(x) \leqq p$. Set $f = (h-p)^2$ or $f = |h - p|$ according as A is a ring or l-group. Then, clearly, $0 \leqq f(x) < 1$ and $f(y) \geqq 2$ for all $y \in X - U$. The final statement follows from the fact that if A is an l-group, then $[(f-1) \vee 0] \wedge 1$ is in A.

Let $A \subseteq C(X)$ and $S \subseteq X$. We set

$$I_S = \{f \in A; S \subseteq Z(f)\}.$$

If $S = \{x\}$ is a singleton, we shall let

$$M_x = I_{\{x\}}.$$

LEMMA 4.2. *Let A be a weakly pseudoregular subring of $C(X)$ which contains* 1. *Then A is regular if and only if for every $S \subseteq X$ and every $x \in X$, if $I_S \subseteq M_x$, then*

$$\inf\{f(y); y \in S\} \leqq f(x)$$

for all $f \in A$.

Proof. The necessity follows from the continuity of each $f \in A$, and the sufficiency is an easy consequence of Lemma 4.1.

If A is a point-separating subring (l-subgroup) of $C(X)$, then an ideal (l-ideal) I of A is *fixed* in case

$$\bigcap Z(I) \neq \emptyset,$$

or, equivalently, $I \subseteq I_S$ for some $S \neq \emptyset$ in X. It is clear that $I \subseteq A$ is a maximal fixed ideal (l-ideal) if and only if $I = M_x$ for some $x \in X$. (In general, however, M_x need not be a maximal ideal (l-ideal) of A.) Since for each $x \in X$, the mapping $f \rightarrow f(x)$ is a homomorphism of A into R with kernel M_x, it follows that A/M_x is isomorphic to a subring (l-subgroup) of R. Of course, this property need not characterize the maximal fixed ideals (l-ideals) of A.

It is well known that if $\mathfrak{F}$ is the family of all maximal fixed ideals (l-ideals) of A, then $\mathfrak{F}$ admits the Stone topology and the mapping $x \rightarrow M_x$ is continuous from X onto $\mathfrak{F}$. Moreover, if A is regular, then this mapping is actually a homeomorphism.

Now let A be an arbitrary ring. An ideal M of A is *real* in case A/M is isomorphic to R. For each real ideal M and each $f \in A$ denote by $M(f)$ the image of f under the (necessarily unique) homomorphism of A onto R with kernel M. If $\mathfrak{F}$ is a family of real ideals of A, then each $f \in A$ determines a real-valued function f^* on $\mathfrak{F}$ defined by

$$f^*(M) = M(f)$$

for all $M \in \mathfrak{F}$. In fact, if

$$A^* = \{f^*; f \in A\},$$

then A^* is a ring of real-valued functions on $\mathfrak{F}$ and $f \rightarrow f^*$ is a homomorphism of A onto A^*. If $\bigcap \mathfrak{F} = 0$, then the mapping $f \rightarrow f^*$ is an isomorphism. It is easily shown (cf. [2]) that if $\mathfrak{F}$ is equipped with the weak topology determined by A, then $\mathfrak{F}$ is completely regular, A^* is a weakly pseudoregular subring of $C(\mathfrak{F})$, and $I^* \subseteq A^*$ is a maximal fixed ideal if and only if $I^* = M^* = \{f^*; f \in M\}$ for some $M \in \mathfrak{F}$.

If $\mathfrak{S} \subseteq \mathfrak{F}$ and if $f \in A$, then the *$\mathfrak{S}$-spectrum* of f is the set

$$S(f, \mathfrak{S}) = \{M(f); M \in \mathfrak{S}\}.$$

Thus, $S(f, \mathfrak{S})$ is simply the range of f^* on $\mathfrak{S}$. We say that A is *regular* ($\mathfrak{F}$) in case (i) $\bigcap \mathfrak{F} = 0$, and (ii) for each $\mathfrak{S} \subseteq \mathfrak{F}$ and each $M \in \mathfrak{F}$,

$$\bigcap \mathfrak{S} \subseteq M$$

implies that

$$\bigwedge S(f, \mathfrak{S}) \leqq M(f)$$

for all $f \in A$. Now applying the comments of the above paragraph and Lemma 4.2, we have the following representation theorem (cf. [2, Theorem 2.2]):

THEOREM 4.3. *Let A be a ring with identity and let $\mathfrak{F}$ be a set of real ideals of A. If A is regular ($\mathfrak{F}$), then $\mathfrak{F}$, equipped with its Stone topology, is completely regular and the mapping $f \to f^*$ is an isomorphism of A onto a regular subring A^* of $C(\mathfrak{F})$, the maximal fixed ideals of which are precisely the sets $M^* = \{f^* \in A^*; f \in M\}$ for $M \in \mathfrak{F}$.*

In this last result a (formally stronger) condition equivalent to (ii) in the requirement "regular ($\mathfrak{F}$)" is: for each $\mathfrak{S} \subseteq \mathfrak{F}$ and each $M \in \mathfrak{F}$, if $\bigcap \mathfrak{S} \subseteq M$, then $M(f)$ is in the closure of $S(f, \mathfrak{S})$ for all $f \in A$. The proof of this equivalence is easily obtained via an argument similar to that used in Lemma 4.2.

Suppose now that A is a commutative l-group and let $\mathfrak{F}$ be a family of maximal l-ideals of A. If $e > 0$ in A satisfies $e \notin M$ for all $M \in \mathfrak{F}$, then we say that e is a *unit* ($\mathfrak{F}$). Suppose that e is a unit ($\mathfrak{F}$); then for each $M \in \mathfrak{F}$ there is a unique homomorphism of A onto an l-subgroup of R such that M is the kernel and e is mapped onto 1. Denote the image of $f \in A$ under this homomorphism by $M(f)$. Then, as in the ring case, there is a homomorphism $f \to f^*$ of A onto an l-subgroup A^* of real-valued functions on $\mathfrak{F}$ where, for each $f \in A$, $f^*(M) = M(f)$ for all $M \in \mathfrak{F}$. This homomorphism is an isomorphism provided that $\bigcap \mathfrak{F} = 0$. Finally, if $\mathfrak{F}$ is equipped with its weak topology, then A^* is weakly pseudoregular and thus, by Lemma 4.1, regular in $C(\mathfrak{F})$. We conclude (cf. [20, Theorem 12; Theorem 6.2]):

THEOREM 4.4. *Let A be a commutative l-group and let $\mathfrak{F}$ be a set of maximal l-ideals of A. If e is a unit ($\mathfrak{F}$) and if $\bigcap \mathfrak{F} = 0$, then $\mathfrak{F}$, equipped with its Stone topology, is completely regular and the mapping $f \to f^*$ is an isomorphism of A onto a regular l-subgroup A^* of $C(\mathfrak{F})$. Moreover, $e^* = 1$ and the maximal fixed l-ideals of A^* are precisely the sets $M^* = \{f^*; f \in M\}$ for $M \in \mathfrak{F}$.*

Since any Archimedean l-group is necessarily commutative [6, p. 235], we may, in the above theorem, replace "commutative" by the formally stronger assumption "Archimedean."

We return now to a simultaneous treatment of the ring and l-group cases. In particular, let A be a ring (l-group) satisfying the hypotheses of Theorem 4.3 (Theorem 4.4). For each $S \subseteq A$, set

$$S^{\mathfrak{F}} = \bigcap \{M \in \mathfrak{F}; S \subseteq M\}.$$

Then clearly $S^{\mathfrak{F}}$ is an ideal (l-ideal) of A. We say that an ideal (l-ideal) I of

A is $\mathfrak{F}$-*closed* in case $I = I^{\mathfrak{F}}$. Denote by $\mathfrak{F}(A)$ the distributive lattice of $\mathfrak{F}$-closed ideals (l-ideals) of A. For each $I \in \mathfrak{F}(A)$, set

$$F_I = \{M \in \mathfrak{F}; I \subseteq M\}.$$

Then the mapping $I \to F_I$ is a dual isomorphism of $\mathfrak{F}(A)$ onto the lattice of closed sets of $\mathfrak{F}$ in the Stone topology.

On $\mathfrak{F}(A)$ define a relation $\prec$ by: $I \prec J$ in case $K \wedge I = 0$ and $K \vee J = A$ for some $K \in \mathfrak{F}(A)$. (Note: In $\mathfrak{F}(A)$, $K \wedge I = K \cap I$ and $K \vee J = (K \cup J)^{\mathfrak{F}}$.) Define a second relation $\ll$ on $\mathfrak{F}(A)$ by: $I \ll J$ in case there exists a countable $(\prec)$-dense $(\prec)$-chain $\{K_n\}$ in $\mathfrak{F}(A)$ such that $I \subseteq \bigcap K_n$ and $\bigcup K_n \subseteq J$.

LEMMA 4.5. *Let $I, J \in \mathfrak{F}(A)$. Then*
(1) *$I \prec J$ if and only if $F_J \subseteq F_I^0$,*
(2) *$I \ll J$ if and only if $F_J \perp (\mathfrak{F} - F_I)$.*

Proof. The first statement is a trivial consequence of the fact that $I \to F_I$ is a dual isomorphism of $\mathfrak{F}(A)$ onto the lattice of closed sets of $\mathfrak{F}$. That $F_J \perp (\mathfrak{F} - F_I)$ implies $I \ll J$ follows at once from (1) and the definition of the relation $\ll$. Conversely, suppose that $I \ll J$. Let Q be the set of rationals r such that $0 < r < 1$. Then there exists a mapping $r \to K_r$ from Q into $\mathfrak{F}(A)$ such that $I \subseteq K_r \subseteq J$ for each $r \in Q$, and $r < s$ in Q implies that $K_r \succ K_s$ in $\mathfrak{F}(A)$. For each $r \in Q$ set $F_r = F_{K_r}$. Set $F_0 = \mathfrak{F} - (\bigcap_Q F_r)$ and $F_1 = \bigcup_Q F_r$. Then clearly $\mathfrak{C}_1 = \{F_0, F_1\}$ is an open cover of $\mathfrak{F}$; we claim that it is also normal. For each pair (m, n) of positive integers with $n \geqq 2$ and $2 \leqq m \leqq 4^n - 1$, set

$$U_{m,n} = F^0_{4^{-n}(m+1)} - F_{4^{-n}(m-1)},$$

$$U_{1,n} = F^0_{4^{-n+1}},$$

and

$$U_{4^n,n} = \mathfrak{F} - F_{1-4^{-n}}.$$

Now using routine arguments one easily shows that for each $n \geqq 2$,

$$\mathfrak{C}_n = \{U_{m,n}; m = 1, \cdots, 4^n\}$$

is an open cover of $\mathfrak{F}$ and that the sequence $\{\mathfrak{C}_n\}$ of covers of $\mathfrak{F}$ is normal. We conclude that $\mathfrak{C}_1$ is a normal cover and, therefore, that $\mathfrak{F} - F_0$ and $\mathfrak{F} - F_1$ are completely separated [24, p. 53]. But, since $F_J \subseteq (\mathfrak{F} - F_0)$ and $F_1 \subseteq F_I$, we have $F_J \perp (\mathfrak{F} - F_I)$ as desired.

Let A be a ring and let $\mathfrak{F}$ be a family of real ideals of A. We say that A is *normal* ($\mathfrak{F}$) in case

(i) A is regular ($\mathfrak{F}$);

(ii) for every pair $I, J \in \mathfrak{F}(A)$, $I \ll J$ implies that A contains an identity modulo J which annihilates I.

Similarly, if A is a commutative l-group and if $\mathfrak{F}$ is a set of maximal l-ideals of A, then we say that A is *normal* ($\mathfrak{F}$) in case

(i) $\bigcap\mathfrak{F}=0$ and A contains a unit ($\mathfrak{F}$), say e;

(ii) for every pair I, $J\in\mathfrak{F}(A)$, $I\ll J$ implies that there is an $h\in A$ with $h-e\in J$ and $h\wedge f\leqq 0$ for all $f\in I$.

It is clear that if $C(X)$ is considered as a ring (l-group) and if $\mathfrak{F}$ is the set of all maximal fixed ideals (l-ideals) of $C(X)$, then $C(X)$ is normal ($\mathfrak{F}$). In general, a subring (l-subgroup) of $C(X)$ may be normally separating without being normal relative to any set of real ideals (maximal l-ideals). However, we do have the following:

THEOREM 4.6. *Let A be a ring* (*commutative l-group*) *and let $\mathfrak{F}$ be a set of real ideals* (*maximal l-ideals*) *of A. If A is normal* ($\mathfrak{F}$), *then $\mathfrak{F}$, equipped with its Stone topology, is completely regular and A^* is a normally separating subring* (*l-subgroup*) *of $C(\mathfrak{F})$.*

Proof. An obvious consequence of the definitions of normal ($\mathfrak{F}$) and the preceding results of this section.

Now let A be a ring (l-group) satisfying the hypotheses of Theorem 4.6. A countable set $\{f_n\}$ in A is a *$\sigma^\star$-set* ($\mathfrak{F}$) in case (i) for each n, $f_nf_m \neq 0$ ($|f_n|\wedge|f_m|\neq 0$) for at most finitely many m, and (ii) $\{f_n\}^{\mathfrak{F}}=A$. Clearly, then, $\{f_n\}$ is a $\sigma^\star$-set ($\mathfrak{F}$) in A if and only if $\{f_n^*\}$ is a $\sigma^\star$-set in A^*.

If A is a ring and $\{f_n\}$ is a $\sigma^\star$-set ($\mathfrak{F}$), then for each n, set

$$f_n^\star = \sum \{f_m; f_nf_m \neq 0\}.$$

We say that A is *$\sigma^\star$-regular* ($\mathfrak{F}$) in case for every countable set $\{f_n\}$ in A, if $\{f_n\}$ is a $\sigma^\star$-set ($\mathfrak{F}$) and if $\{f_n^\star f_n\}^{\mathfrak{F}}=A$, then there is an $f\in A$ with

$$ff_n^\star f_n = f_n$$

for all n. It is clear that if A is $\sigma^\star$-regular ($\mathfrak{F}$), then A^* is $\sigma^\star$-regular in $C(\mathfrak{F})$.

If A is an l-group, then we say that A is *$\sigma^\star$-complete* ($\mathfrak{F}$) in case every $\sigma^\star$-set ($\mathfrak{F}$) in A has a least upper bound in A. Again it is obvious that if A is $\sigma^\star$-complete ($\mathfrak{F}$), then A^* is $\sigma^\star$-complete in $C(\mathfrak{F})$.

THEOREM 4.7. *Let $\mathfrak{F}$ be a set of real ideals* (*maximal l-ideals*) *of the ring* (*commutative divisible l-group*) *A. If A is normal* ($\mathfrak{F}$) *and $\sigma^\star$-regular* ($\mathfrak{F}$) (*$\sigma^\star$-complete* ($\mathfrak{F}$)), *then A^* is m-dense and u-dense in $C(\mathfrak{F})$.*

Proof. By Theorem 4.6, Corollary 3.4, and Theorem 3.5.

5. **Characterizations of $C(X)$.** We are now in possession of practically all that is required for internal characterizations of $C(X)$ both as a ring and as an l-group. In this section we develop a bit more machinery and then obtain the desired characterizations.

First let A be a regular inverse-closed subring of $C(X)$ containing 1 and let P be the set of all strictly positive elements of A. Then it is evident that P is an additive semigroup and a multiplicative group, and that P contains the positive rational constants. Moreover, P is directed by $\leqq$. For, if $p, q \in P$, then

$$pq(1 + p)^{-1}(1 + q)^{-1} \leqq p, q.$$

From these observations it follows that if, for each $p \in P$, we set

$$U_p = \{f \in A; -p \leqq f \leqq p\},$$

then A, under addition, is a topological group with $\{U_p; p \in P\}$ as a basis of (not necessarily open) neighborhoods of 0. We denote this topology by $T_m(A)$. Observe that if $Q \subseteq P$ is cofinal in P, then $\{U_q; q \in Q\}$ is also a basis at 0 for $T_m(A)$. In particular, this is the case when Q is taken as the set of all f^2 where $f \in A$ and $Z(f) = \emptyset$.

Of course $T_m(C(X))$ is simply the m-topology of $C(X)$. In general, however, $T_m(A)$ need not coincide with the m-topology of $C(X)$ relativized to A (10). Nevertheless, the topology $T_m(A)$ has many of the desirable features of the m-topology. For example, a special case of a result due to Shirota [21, Lemma 1] is the following:

LEMMA 5.1. *Let A be a normal inverse-closed subring of $C(X)$ which contains 1 and let I be the set of invertible elements in A. Then, relative to the topology $T_m(A)$, A is a topological ring, I is open, and inversion is continuous on I.*

If A is actually m-dense in $C(X)$, then we must have that, on A, the m-topology and $T_m(A)$ are equivalent. For when A is m-dense in $C(X)$, P is clearly cofinal in the directed set of all strictly positive elements of $C(X)$. In particular, we have:

LEMMA 5.2. *If A is a normally separating $\sigma^\star$-regular subring of $C(X)$, then $T_m(A)$ coincides with the m-topology on A.*

Proof. By Corollary 3.4, A is m-dense in $C(X)$.

Now let A be a ring normal $(\mathfrak{F})$ and $\sigma^\star$-regular $(\mathfrak{F})$ where $\mathfrak{F}$ is a set of real ideals of A, and let I be the set of all invertible elements of A. For each $p \in I$, let U_p be the set of all $f \in A$ such that, for each $M \in \mathfrak{F}$, $p^2 - f^2 - g^2 \in M$ for some $g \in A$. That is,

$$U_p = \{f \in A; M(f^2) \leqq M(p^2) \text{ for all } M \in \mathfrak{F}\}.$$

In view of the representation (Theorem 4.7) of A in $C(\mathfrak{F})$, it is evident that A is a topological ring with $\{U_p; p \in I\}$ as a base of neighborhoods at 0. We denote this topology by $T_m(A, \mathfrak{F})$ and note that $f \to f^*$ is actually a homeomorphism of A onto A^* where the latter has the m-topology $T_m(A^*)$. We

(10) In the ring A of Example 3.6 the set of strictly positive elements is closed in the m-topology but is not closed in $T_m(A)$.

now have the following characterization of $C(X)$:

THEOREM 5.3. *A ring A is isomorphic to a ring $C(X)$ for some completely regular space X if and only if relative to some set $\mathfrak{F}$ of real ideals:*
(1) *A is normal* $(\mathfrak{F})$;
(2) *A is $\sigma^\star$-regular* $(\mathfrak{F})$;
(3) *A is complete in the topology $T_m(A, \mathfrak{F})$.*
Moreover, when these conditions hold, the isomorphism from A onto $C(X)$ is actually a homeomorphism of A in the topology $T_m(A, \mathfrak{F})$ onto $C(X)$ is its m-topology.

Proof. The necessity of the conditions is obvious. Conversely, by (1), (2), and Theorem 4.7, we have that A^*, an isomorph of A under $f \rightarrow f^*$, is m-dense in $C(\mathfrak{F})$. Moreover, this isomorphism $f \rightarrow f^*$ is a homeomorphism of A, in the topology $T_m(A, \mathfrak{F})$, onto A^*, in the topology $T_m(A^*)$. By Lemma 5.2, $T_m(A^*)$ is simply the m-topology of $C(\mathfrak{F})$ restricted to A^*. Since, by (3), A^* is complete, and hence closed, in this topology, we have that $A^* = C(\mathfrak{F})$, as desired.

With no less ease we can now characterize $C(X)$ as an l-group. For this let A be a divisible commutative l-group, which relative to some set $\mathfrak{F}$ of maximal l-ideals is normal $(\mathfrak{F})$ and $\sigma^\star$-complete $(\mathfrak{F})$. Let $e \in A$ be a unit $(\mathfrak{F})$. Since A is divisible, it contains all rational multiples of e; for each positive rational α, set

$$U_\alpha = \{f \in A;\ |f| \leqq \alpha e\}.$$

Then, obviously, the family $\{U_\alpha\}$ is a basis of neighborhoods of 0 for a topology, denoted by $T_u(A, e)$, relative to which A is a topological l-group. Since it is equally clear that the isomorphism $f \rightarrow f^*$ is, in fact, a homeomorphism of A onto A^* where the latter has the u-topology of $C(\mathfrak{F})$, we have, applying Theorem 4.7, the following characterization of $C(X)$:

THEOREM 5.4. *A commutative l-group A is isomorphic to an l-group $C(X)$ for some completely regular space X if and only if A is divisible and relative to some set $\mathfrak{F}$ of maximal l-ideals of A:*
(1) *A is normal* $(\mathfrak{F})$;
(2) *A is $\sigma^\star$-complete* $(\mathfrak{F})$;
(3) *A is complete in the topology $T_u(A, e)$.*
Moreover, when these conditions hold, the isomorphism from A onto $C(X)$ may be chosen to be a homeomorphism of A in the topology $T_u(A, e)$ onto $C(X)$ in its u-topology.

We observe that, although each of the above two results calls for completeness in a certain topology, these two characterizations are algebraic, that is, depend solely on the algebraic structure of the ring or l-group A. This is the case since each of the two topologies $T_m(A, \mathfrak{F})$ and $T_u(A, e)$ and the corresponding completeness criteria are intrinsically definable in terms of the algebraic structure of A.

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