PLANE SEMIGROUPS[1]

BY

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Introduction. Let S be the two dimensional Euclidean plane with a continuous, associative multiplication with identity 1. It has been proved [5] that there is an open, connected subgroup G about 1. Since G is of dimension two, and is a subset of the plane, it must be either the cylinder group or one of the two groups on the plane. In the former case, the multiplication in S has been completely determined [6]. When G is one of the latter two, the following is known [5]: Let G^* denote the topological closure of G, and $L = G^* \backslash G$. If L is a (closed) line in S, then

THEOREM A. *If G is abelian, then there are exactly four possibilities for the multiplications in L*:

I. *L is a group.*

II. *L has a zero*, 0, *dividing L into two sets A and B. Then $AB = 0$ and there are the three possibilities*

(a) *A and B are groups*,

(b) $A^2 = B^2 = 0$,

(c) *A is a group and* $B^2 = 0$.

Furthermore, there are at most three distinct orbits xG in L and these are A, B and 0, *respectively.*

THEOREM B. *If G is nonabelian, there can be at most five possibilities for the multiplication in L. These are the four of Theorem* A *and one other as follows*:

III. *If $x, y \in L$, then $xy = x$* (*or $xy = y$*) *and there is precisely one left* (*right*) *orbit Gx* (*yG*).

Examples are known, of course, for all four cases in Theorem A. However, examples in the nonabelian case were given only for IIb and III, and it was conjectured that the other cases are, in fact, impossible [5]. In this paper, we show that, with no restrictions on L (except that it be nonempty), there is but one further possibility in both the abelian and nonabelian cases. Further, it is proved that IIa and IIc are, as conjectured, not possible when G is nonabelian, but an example is given to show I is possible. This, then, completely determines the multiplications in L. Our description, however, does not completely determine the multiplication in G^* since to do this we must show that, if G and G' are isomorphic, and L and L' are isomorphic, then $G \cup L$ is isomorphic to $G' \cup L'$. It seems reasonable to conjecture that this is so.

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The final case is as follows:

IV. *L is a ray, $S=G^*$, the end point of L is a zero, and $L^2=0$.*

Examples are given to show that this multiplication actually does occur in both the abelian and nonabelian cases.

Although no attempt has been made to isolate those results which can be generalized to results on one-parameter transformation groups on the plane, the observant reader will readily see several such.

1. **Preliminaries.** We shall say that G^* is *regular* if L is either a line or ray closed in the topology of S. (Notice that this does not agree, in the latter case, with the usual concept of a manifold with regular boundary.)

Except for the above, standard terminology [9] will be used throughout. A *left* (*right*) *ideal* of a semigroup S is a subset A such that $SA \subset A$ ($AS \subset A$), and an *ideal* is a left ideal which is also a right ideal. A *left* (*right*) *zero* is a right (left) ideal consisting of just one point, and a *zero* is a left zero which is also a right zero. If G is a subgroup of S containing the identity, the *left orbit* through x is the collection $Gx=\{gx: g\in G\}$. Right orbits are defined similarly.

The main tools we shall use are the Jordan curve theorem and the following "folk theorem," a proof, but not a statement, of which can be found in [7]:

If G is a Lie group acting as a transformation group on a completely regular space X, and T is a compact neighborhood of the identity such that for some $p\in X$, $g(p)=p$, $g\in T$ implies $g=1$, then there is a closed neighborhood W of p and a closed set $C\subset W$ such that $C\times T\rightarrow W$ defined by $(x, g)\rightarrow g(x)$ is a homeomorphism onto W. The set C is called a local cross section to the local orbits at p.

Actually, this will only be used in the case where X is the plane and G is a one-parameter transformation group—for which there is an elementary proof.

2. **The main theorem.** Throughout the remainder of this paper, we shall assume that G is homeomorphic to the plane and that L is nonempty. We shall use letters u to z to denote elements of L only, whereas g, h, etc., will be reserved for elements of G. The statements of results will ordinarily be in terms of left (or right) orbits. Clearly a dual result holds for the other orbits.

In what follows, we shall call a set an ideal if it is an ideal relative to G^*. Since L is a closed set which is the boundary of an open set in the plane, it is at most one-dimensional. It is connected [5], and cannot be compact since G is homeomorphic to the plane and cannot be contained in any compact subset of S. Thus, L is precisely of dimension one under our hypotheses. Now G being a planar group implies that it is the product of any two one-parameter subgroups providing one is normal. Thus, since the right isotropy group H of a point $x\in L$ must be at least one-dimensional, if R is a one-parameter group not contained in H, $xR=xG$ if either R or H is normal.

LEMMA 2.1. *If $vG\neq v\in(uG)^*\backslash(uG)$, then L contains a compact right ideal* *(relative to G^*) of the form $(zG)^*$ where z is a limit point of uG. Moreover, there is a simple closed curve A which is the union of two arcs A_1 and A_2, joined at their end points, such that $A_1 \subset uG$, A_2 meets uG only in its end points, and $(zG)^*$ is contained in the interior of the region bounded by Ag for every $g \in G$.*

Proof. Let H be a one-parameter group such that $uH=u$. If H is normal, then also $vH=v$. Choose a different one-parameter subgroup R which is normal if and only if H is not. Then $uG=uR$ and $vG=vR$. Clearly, then, $uR\neq u$ also, and $uR=uG$. There is an element $g\in R$ such that R is represented by $R=\{g^t: t \text{ real}\}$, and $g^tg^s=g^{t+s}$. Let $T=\{g^t: -1\leqq t\leqq 1\}$. There is a local cross section C to the local orbits pT at v, and since a neighborhood of v is then fibred as a product of C and T, we may take C to be an arc [2]. Since v is a limit point of uR, uR must cross C infinitely many times. By choosing $u\in C$ (which we may do by relabelling, since $uG=ugG$ for any $g\in G$), either there are infinitely many $t>1$ such that $ug^t\in C$, or there are infinitely many $t<1$ such that $ug^t\in C$. By a reparameterization of R_1 we can choose the former. Let $s=\min\{t: ug^t\in C,\ t>1\}$, $A_1=\{ug^t: 0\leqq t\leqq s\}$, and A_2 that portion of C between u and ug^s. Then $A=A_1\cup A_2$ is a simple closed curve. Thus it bounds a compact region whose interior we shall denote by Q. Now either $ug^{s+1}\in Q$, or $ug^{-1}\in Q$. The set CT is fibred by the sets Cg^t, $-1\leqq t\leqq 1$. Hence, if $ug^{s+1}\in Q$, as t increases, $t>s+1$, ug^t can enter CT only through Cg^{-1} and leave only through Cg. But the only way it can leave Q is through A_2 since an orbit cannot cross itself. Hence, for $t>s+1$, $ug^t\in Q$. In the case where $ug^{-1}\in Q$, a similar argument shows that $ug^t\in Q$ for all $t<-1$. In either case, there is a limit point $z\in Q\backslash uR$ of that portion of uR contained in Q. Hence, since $z=\lim_n ug^{t(n)}$, where $t(n)\to\infty$ (or $-\infty$ depending on which portion of uR is contained in Q), $zg^t=\lim_n ug^{t(n)+t}\in Q^*$ since $ug^{t(n)+t}\in Q$ if $t(n)$ is sufficiently large for a fixed t. Thus zR has compact closure. Now if H is normal, then $zH=z$ and again $zR=zG$. If H is not normal, then either $zR\neq z$ and we have $zG=zR$, or $zR=z$. Now, if zG is not contained in the interior of Ah for all $h\in G$, there is an element $c\in A_2$ and t such that $cg^t=zh$, where zh is interior to Q for some $h\in G$ which implies zG is not contained in Q and $zR\neq z$ (e.g. $zg^{-t-1}\notin Q^*$ if $ug^{-1}\notin Q$, and otherwise $zg^{-t+1}\notin Q$). Since Ag is interior to A (again assuming the case ug^{s+1} in Q), and zG is interior to Ag, $(zG)^*$ is interior to A, and hence to Ag^t for every t, and thus to Ah for every $h\in G$. The result now follows.

LEMMA 2.2. *If xG is not closed for some x, then there is a compact right orbit.*

Proof. Let $y\in(xG)^*\backslash xG$. If $yG=y$, we are through. If not, the hypotheses of Lemma 2.1 are satisfied and we obtain a simple closed curve A containing a portion of xG such that for some limit point z of xG, zG is interior to Ag for every $g\in G$. We can apply the same reasoning to zG. Consider the set of points (A_u, v) where A_u is a simple closed curve containing a portion of uG and v is a limit point of uG such that vG is contained in the interior of A_ug for every

$g \in G$. Order the pairs as follows:

$$(A_u, v) < (A_y, z)$$

if A_u is interior to $A_y g$ for every $g \in G$. Clearly, this gives us a partial ordering. Let $\{(A_\alpha, x_\alpha)\}$ be a maximal chain and z a limit point of $\{x_\alpha\}$. Now $z \in L$ is closed, and also z is interior to $A_\alpha g$ for every $g \in G$ and every α. If zG is not closed, let y be a limit point of zG. Then uG is interior to A_α for every α. Let R be a one-parameter subgroup of G such that $yR \neq y$. Again, if no such R exists, $yG = y$ is a compact orbit. For some $g \in R$, $R = \{g^t : t \text{ real}\}$ and $g^t g^s = g^{t+s}$. If $yg^t = y$ for any $t \neq 0$, then yG is a compact orbit. Let $T = \{g^t : -1 \leqq t \leqq 1\}$, and C an arc through y which is a local cross section to the local orbits xT. Since y is interior to every $A_\alpha g$, for each α, if we choose β with $(A_\beta, x_\beta) < (A_\alpha, x_\alpha)$, there is a subarc C_α interior to A_β and hence interior to $A_\alpha g$ for every $g \in G$, and such that y is an interior point of C_α relative to its topology as an arc. Since C_α is interior to $A_\alpha g$ for every $g \in G$, $C_\alpha h$ is interior to $A_\alpha g$ for every g, $h \in G$, and hence $C_\alpha G$ is interior to $A_\alpha g$ for every $g \in G$. Now zR meets C_α in infinitely many points. Let B_α be the interval between two consecutive such points zg^t and zg^{t+s}, and $D_\alpha = B_\alpha u \{zg^{t+r} : 0 \leqq r \leqq s\}$. Then $D_\alpha G$ is contained in the interior of A_α. Let D_γ be another such simple closed curve obtained in the same way from (A_γ, x_γ), where we may assume $(A_\gamma, x_\gamma) < (A_\alpha, x_\alpha)$. Clearly $B_\alpha G = D_\alpha G$. Hence, if $A = D_\alpha$, then A is contained in the interior of every A_β in the chain. Let w be a limit point of zG contained in the interior of A, $w \notin zG$. Such a point exists if zG is not compact. Then $(A, w) < (A_\beta, x_\beta)$ for every β in the chain, and hence the chain is not maximal. This contradiction proves the lemma.

LEMMA 2.3. *If L contains a simple closed curve, then it can contain no left or right zero.*

Proof. Let B be a simple closed curve contained in L, and suppose $a \in L$ is a left zero. Let A be a connected set in G^* containing both a and the identity 1. Now G is connected and cannot be contained in a compact part of S. Hence, if p is interior to B, $p \notin G^*$. Thus, $p \notin tB$ and hence the index of p relative to tB is invariant. But the index of p relative to B is nonzero, whereas that of p relative to $aB = a$ is zero.

LEMMA 2.4. *L is not compact.*

Proof. If L is compact, since G cannot be contained in a compact part of S, there is a one cycle in G which bounds in L. But this contradicts the assumption that G is homeomorphic to E_2.

LEMMA 2.5. *If $xG = L$ for $x \in L$, then G^* is regular.*

Proof. Since L is closed and a continuous one-one image of the line in the plane S, the result follows from the fact that the line cannot be wildly embedded as a closed subset of the plane [4].

LEMMA 2.6. *If there is a left zero in G^* which is not a right zero, then $Gx=L$ for every $x\in L$.*

Proof. If there is a nonclosed right orbit, by Lemma 2.2, there is a compact right orbit yG. By Lemma 2.3, yG is a point, contradicting our assumption. Thus, Gx is closed for every $x\in L$, and this implies $Gx=L$.

LEMMA 2.7. *If all left orbits are such that their closures are obtained by adding at most two points, then G^* is regular.*

Proof. If either $xG=L$ or $Gx=x$ for any $x\in L$, then by Lemma 2.5 (and its dual), G^* is regular. Otherwise, by Lemma 2.2, and 2.3, and the assumption that $(Gx)^*\backslash Gx$ consists of at most two points (and is nonempty for some x) there is a left zero and a right zero a and it is in the closure of every orbit. By Lemma 2.3, $Gx\cup\{a\}$ cannot be a simple closed curve, and hence a is an end point for each left orbit. Since G is connected, and S is the plane, there can be at most two orbits other than a, and the union of these is clearly a closed line (or a closed ray if there is only one other orbit; there must be at least one other by our assumption that G is a planar group).

LEMMA 2.8. *A compact orbit is a point.*

Proof. Let Gx be a compact left orbit. If $Gx\neq x$, it is a simple closed curve, and hence $Gx\neq L\neq xG$ by Lemma 2.4. By Lemma 2.2 there is a compact right orbit, and since it is a right ideal, it must meet Gx in a point which, by relabelling, we may take to be x. Now hxG is homeomorphic to xG for every $h\in G$, meets Gx in hx, and is either disjoint from or coincides with $h'xG$, $h'\in G$. Since G is connected and not contained in any compact part of the plane, if xG meets Gx in more than one point, it must meet along an arc. Since $Gx\cup hxG$ is homeomorphic to $Gx\cup xG$, if Gx coincides along an arc with xG, it must coincide with xG, for there can be no point where they can branch. Hence, $Gx\cap xG=\{x\}$, and similarly $Gx\cap hxG=\{hx\}$, or $Gx=xG$. But there cannot be uncountably many disjoint circles in the plane, each meeting a fixed circle in a single point. Hence, $Gx=xG$ and is an ideal. Now Gx has an idempotent e which, by Lemma 2.3, can be neither a left nor a right zero for G^* and hence not for xG since if $e(xG)=e$, $eG=e$, and thus $eG^*=e$. Hence, xG is the circle group [3]. Being the minimal ideal in L, xG is in the closure of every orbit in L. Let $R=\{g^t: t \text{ real}\}$ be a one-parameter subgroup in G such that $xR=xG$, and t_0 the minimum $t>0$ such that $xg^t=x$. Choose $t_1<t_0$, and $T=\{g^t: -t_1\leqq t\leqq t_1\}$, and let C be an arc which is a local cross section to the local right orbit at x. Since CT is homeomorphic to $C\times T$ under the correspondence $(p, t)\to pg^t$, a right orbit yR enters $C\times T$ by first crossing the arc Cg^{-t_1}, and meets C, and then leaves by crossing Cg^{t_1}. Let $y\in L$, $y\notin xG$. Then yR meets C infinitely often, but must remain to one side of x because

the other side contains points not in G^* since xG is a simple closed curve bounding points outside G^*. Thus yR spirals around xG. Let yg^s and yg^{s+v} be two consecutive points on $yR\cap C$, and $z\in L$ a point on the arc between. Now zR must leave CT by crossing Cg^{t_1}, enter through Cg^{-t_1}, and cannot cross yR. In particular, y is not a limit point of zR. This implies that $(yG)^*=yG\cup xG$. Now, since left orbits must behave in a similar manner, Gy must lie along yG, and since it can have no end point by Lemma 2.3, it must coincide with yG. Now $zy\in(zG)^*\cap(Gy)^*=xG$. Since G meets C in a collection of open intervals, there is an arc $A\subset C$ whose interior points (relative to the topology of C) are points of G and whose end points belong to L. If the end points belong to different orbits, and f: $[0, 1]\to A$ is a parameterization of A, then $\{f(t)f(1-t): 0\leqq t\leqq 1\}$ is an arc with end points in xG and the remainder in G which is clearly impossible. Hence, there can be at most one orbit other than xG in L. Similarly, if y, $z\in L\backslash xG$ then $yz\notin xG$, and hence $(yG)^2=yG$. As before, let yg^s and yg^{s+v} be two consecutive points on $yR\cap C$, and F that part of C joining the two. Then $B=\{yg^t: s\leqq t\leqq s+v\}\cup F$ is a simple closed curve containing xG in the compact region it bounds. Let ϕ: $[0, 1]\to G^*$ be defined so that $\phi(1)=y$, $\phi(t)\in G$ otherwise. Now $\phi(t)B$ is a simple closed curve bounding a region containing xG for every $t\neq 1$, $\phi(t)B$ does not meet xG for any t, but $\phi(1)B\subset yG$. Since yG cannot contain a nondegenerate simple closed curve, this is impossible because of invariance of the index under homotopy. Hence, $xG=Gx=L$ contradicting Lemma 2.4, and this then shows that the assumption that $xG\neq x$ is a contradiction.

MAIN THEOREM. *If S is a semigroup with identity* 1 *on the plane, G is the maximal connected subgroup about* 1, *and $L=G^*\backslash G$, then one of the following holds*:

(i) *G is a cylinder group, L is a simple closed curve, and the description of S is known* [6];

(ii) *G is homeomorphic to E_2 and L is either a* (*closed*) *line or a ray in S*;

(iii) *$G=S$.*

Proof. Since (i) is known, we restrict ourselves to (ii). If two left orbits in L are points, then by Lemmas 2.5 and 2.6 $xG=L$ is a closed line in S. Again, if $xG=L$ or $Gx=L$, the result follows from Lemma 2.5, so that we may restrict our attention to the case when there is a nonclosed left orbit, and a nonclosed right orbit, and thus, by Lemmas 2.2, and 2.8, there is a left zero and a right which, of course, must coincide, and which we denote by p. (Clearly, there can be just one such, and it is in the closure of every orbit.) We claim that $(xG)^*\backslash xG=\{p\}$. Suppose, on the contrary, that $y\in(xG^*)\backslash xG$, $y\neq p$. Then $yG\neq y$, and $p\in(yG)^*$. Let A be the arc of Lemma 2.1 relative to x and y, where $A_1\subset xG$. Then p is interior to the region bounded by A. But yG enters the region through A_2 and, because A_2 is a portion of a local cross section to local orbits at y, cannot leave A having once entered (by the same argument as used in the proof of Lemma 2.7). It then follows that points of yG cannot be limit points of xG because yG can cross A_1 just once, and this contradiction proves the assertion. Applying Lemma 2.7 now completes the proof of the theorem.

3. **Multiplications on** L. In this section, we show that only those multiplications listed in the introduction are possible for L.

THEOREM 3.1. *If S is a semigroup with identity on the plane, G is the maximal connected subgroup about the identity, and $L=G^*\backslash G$ is a ray with end point a, then $S=G^*$, a is a zero for S, and $L^2=a$.*

Proof. Clearly, if $L^2\neq a$, since $xG\cup\{a\}=L$ for $x\in L\backslash\{a\}$, there is an element $p\in L$ such that $p^2\neq a$. Moreover, since a must be both a left and a right orbit, $aG^*=a=G^*a$. Let A be a simple closed curve about a, meeting L in precisely the point p. Let B be an arc joining the identity 1 to p such that $B\backslash\{p\}\subset G$. Then $a\notin bA$ for any $b\in B$. Hence, the index of a relative to bA is constant. But this index is nonzero for $b\neq p$, and zero for $b=p$, and this contradiction proves the theorem.

THEOREM 3.2. *Let S be a semigroup with identity on the plane, and suppose the maximal subgroup G about the identity is the nonabelian (affine) group in the plane. If xG is a group for some $x\in L$, then $xG=L$.*

Proof. Let p be the identity of xG. Then $pG=xG$. Let G_p be the isotropy group at p. Since pG is homeomorphic to the line, G_p is a one-parameter subgroup of G. Let $g, h\in G$. Then, since p is the identity for pG

$$(pg)(ph) = (pgp)h = pgh.$$

Hence, the map $g\rightarrow pg$ is a homomorphism whose kernel is G_p, and thus G_p is normal.

Now suppose $L=pG\cup\{a\}\cup yG$, where a is a zero for G^* and the end point of the two disjoint orbits pG and yG. Then $G^*\backslash\{a\}$ is homeomorphic to a strip in the plane. Let H be a one-parameter subgroup of G such that $pH=pG$ and $yH=yG$. (Any one-parameter group other than G_p or G_y will do.) Let K be the fixed point set of H acting on the right in L. Then $K\subset\{a\}\cup yG$ and K is closed. Hence $G^*\backslash K$ is homeomorphic to the plane with at most a countable number of boundary lines. By an argument essentially the same as used in [8], there is a local cross section to the *orbits* of H with end point at y, and another with an end point at p, and since G is a planar group, and these sections must enter G, they can be joined through G to 1 by sections A_1, and A_2. Let $A=A_1\cup A_2$. Then A meets H in precisely the point 1, though it may meet other orbits twice—once in A_1 and once in A_2. Let B be that portion of L between p and y. Then $A\cup B$ is a simple closed curve bounding a compact region Q containing a. Moreover, if $H_+=H\cap Q$, H_+ consists of one "end" of H. Now $A(H\backslash H_+)\cap(Q\cap G)$ is empty, for if $c\in A\cap G$, $h\in H_+$, and $ch^{-1}\in Q$, since $H_+\subset Q$, there is a first place when $ch^{-1}\in Q$, and in this case $ch^{-1}\in A$. But by the construction of A, if say $c\in A_1$, then $ch^{-1}\in A_2$, and the part of A between c and ch^{-1} together with the part of cH between c and ch^{-1} forms a simple closed curve containing H_+, and all this is contained in a compact part of G, which is impossible. By the above, $(Q\cap G)\backslash AH = (Q\cap G)\backslash AH_+$. Now suppose $g_n\to a$, $g_n\in G_p$. Then there exist $v_n\in A$ and $h_n\in H_+$ such that $g_n=v_nh_n$ and we may assume $h_n\to z\in B$. Hence $ph_n\to a$ since $\lim_n ph_n\in B\cap(pH)^*=a$. But $ph_n^{-1}=pv_n$ which is bounded, whereas $pH=pG$ so that either ph_n or ph_n^{-1} is not bounded. Thus, there is a neighborhood of a containing no points of G_p. But this implies that both pG and yG contain no limit points of G_p (for if one point of yG, say, is a limit point of points of G_p, every point of yG is and then a is also). That is, G_p is closed in S. Thus, every element of G in the component of $G^*\backslash G_p$ containing L is expressible as gh where $g\in G_p$ and $h\in H\cap Q$. But clearly this implies that $p=ga=a$ for some $g\in G_p$. Thus the assumption that $xG\neq L$ is false and the theorem is proved.

EXAMPLE A. We now give an example to show that I of the Introduction is possible when G is nonabelian.

Let $G^*=\{(a, b): 0<a<\infty,\ -\infty\leqq b<\infty\}$ with multiplication defined by $(a, b)(c, d)=(ac, bc+d)$. Then $L=\{(a, -\infty): a>0\}$ is clearly a group. Let $\phi: G^*\to P$, where P is a half plane, be a homeomorphism, and $S=(G^*\cup P)/R$ where R is the equivalence relation identifying $x\in L$ with $\phi(x)$. Define multiplication in S by $xy=\phi^{-1}(x)\phi^{-1}(y)$ if $x, y\in P$, $\phi(x\phi^{-1}(y))$ if $x\in G^*$, $y\in P$, $\phi(\phi^{-1}(x)y)$ if $x\in P$, $y\in G^*$, xy if $x, y\in G^*$. One can easily show that S is a topological semigroup on the plane with the properties desired.

EXAMPLE B. By taking the examples for IIb in [5] and identifying the sets A and B in the obvious way, since isotropy groups agree for corresponding elements, we obtain examples for both abelian and nonabelian semigroups of the type described in case IV of the Introduction.

4. Conclusion. Further study of plane semigroups would seem to be nonproductive unless some extra conditions are placed on the semigroup. Horne [1] has characterized quite nicely those abelian semigroups on the plane which have a subsemigroup homeomorphic and isomorphic to the whole real line under multiplication such that its zero and identity are also a zero and identity respectively for the plane. Such a characterization in the nonabelian case would be of interest and within the realm of possibility.

Here, we have described the multiplications on L. However, we have not attempted a complete description of the multiplications in G^*. It does not seem unlikely that there is a relatively small number of such semigroups (up to isomorphism).

The question, raised in [5], of whether a semigroup S with identity on a manifold and no other idempotents is a group is still unsettled except when S is the plane. The related problem of the regularity of $L=G^*\backslash G$, where G is the maximal (open) connected subgroup about the identity is in the same category. It seems quite likely that both of these questions are amenable to similar techniques to those above in the case of certain particular two-dimensional manifolds (e.g., the cylinders).

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