ON THE EXISTENCE OF REAL-CLOSED FIELDS THAT ARE η_α-SETS OF POWER $\aleph_\alpha$(1)

BY

NORMAN L. ALLING

Introduction. In the theory of η_α-sets three main theorems stand out:

I. An η_α-set is universal for totally ordered sets of power not exceeding $\aleph_\alpha$.

II. Two η_α-sets of power $\aleph_\alpha$ are isomorphic.

III. If $\aleph_\alpha$ is regular and if $\sum_{\delta<\alpha} 2^{\aleph_\delta} \leqq \aleph_\alpha$, then an η_α-set of power $\aleph_\alpha$ exists.

These results were proved by Hausdorff [15, pp. 180–185] and Gillman [10]. Clearly they may be viewed as theorems about special objects in a particular category. The author [1] showed that if $\alpha>0$ then these three theorems hold for the category of totally ordered Abelian groups and order preserving (group) isomorphisms; the special group being totally ordered, Abelian, divisible, and an η_α-set.

Erdös, Gillman, and Henriksen [8] (see also Gillman and Jerison [11]) proved that if $\alpha>0$ then I and II hold for the category of totally ordered fields and order preserving (ring) isomorphisms; the special field being real-closed and an η_α-set. It was also shown in [8] that III holds for this category and special object if $\alpha=1$. However in case $\alpha>1$, III was left open both in [8] and in [11].

The initial aim of these researches was to show that, assuming $\alpha>0$, $\aleph_\alpha$ regular, and $\sum_{\delta<\alpha} 2^{\aleph_\delta} \leqq \aleph_\alpha$, a real-closed field exists that is an η_α-set of power $\aleph_\alpha$. The construction is as follows: let G be a totally ordered Abelian divisible group that is an η_α-set of power $\aleph_\alpha$, $\alpha>0$. Let $R\{G\}$ denote the field of formal power series with exponents in G and coefficients in R, the reals. $R\{G\}$ is an η_α-set but its power exceeds $\aleph_\alpha$. Let $R\{G\}_\alpha = \{f \in R\{G\}:$ the support of f is of power less than $\aleph_\alpha\}$. Then $R\{G\}_\alpha$, again a real-closed field, is an η_α-set, and is of power $\aleph_\alpha$.

The only difficult point in these verifications was the proof that $R\{G\}$ and $R\{G\}_\alpha$ are η_α-sets. The proof arrived at by the author did not involve the multiplication in these fields, but depended wholly on their structure as a

Presented to the Society, January 28, 1960; received by the editors December 20, 1960 and, in revised form, July 5, 1961.

(1) This research was in part supported by a grant from the Purdue Research Foundation and in part by a grant from the Office of Naval Research under Contract No. Nonr-1100 (12). Thanks are due Professor Casper Goffman for suggesting in 1958 that the groups considered in the author's dissertation [1] might be represented as Hahn groups, and to Professor Reinhold Baer who suggested in the same year that any connection between the valuation theory of fields and these groups ([1]) be investigated. That both suggestions bore fruit is evidenced by this paper.

totally ordered Abelian group. On analyzing the proof further, the following construction presented itself. Let E be an η_α-set of power $\aleph_\alpha$, $\alpha>0$, and let $R\{E\}$ be the Hahn group on E with real coefficients. Let $R\{E\}_\alpha$ be the subgroup of $R\{E\}$ consisting of all elements with support of power less than $\aleph_\alpha$. Then $R\{E\}$ and $R\{E\}_\alpha$ are totally ordered Abelian divisible groups that are η_α-sets, the latter being of power $\aleph_\alpha$. Further, having proved this, it is immediate that $R\{G\}$ and $R\{G\}_\alpha$ are η_α-sets.

After further study it was found that two of the essential facts used in the proof that $R\{E\}$ and $R\{E\}_\alpha$ are η_α-sets, were that their value sets are η_α-sets and that their factors (i.e., the Archimedean factor groups of their convex subgroups) are conditionally complete. At this point the initial version of the paper was written.

Let α be a nonzero ordinal number. A group (or a field) with valuation is called *α-maximal* if every pseudo-convergent sequence of length less than ω_α has a pseudo-limit. The following was stated in [2].

MAIN THEOREM. *Let G be a totally ordered Abelian group and let α be a nonzero ordinal number. G is an η_α-set if and only if* (1) *its factors are conditionally complete*, (2) *its value set is an η_α-set, and* (3) *it is α-maximal.*

The necessity of these three conditions was shown in [2]. The proof of sufficiency given in [2] rested heavily on some lemmas proved in the initial version of this paper. Having been requested by the referee to revise the initial version of the paper extensively, it seemed appropriate to write this revised version around the Main Theorem. The sufficiency of these conditions will be proved in §1. In §2 it will be shown that various examples, given in [2], satisfy conditions (1), (2), and (3) and thus are η_α-sets. In §3 it will be shown that $R\{G\}_\alpha$ is a real-closed field.

The following is an immediate consequence of the Main Theorem.

MAIN COROLLARY. *Let K be a totally ordered field and let α be a nonzero ordinal number. K is an η_α-set if and only if* (1) *its residue class field is isomorphic to the reals*, (2) *its value group is an η_α-set, and* (3) *it is α-maximal.*

Background. Let T be a totally ordered set and let α be an ordinal number. T is said to be an *η_α-set* if given H, $K\subset T$ such that $H<K$ and $|H|+|K|<\aleph_\alpha$ there is a $t\in T$ such that $H<\{t\}<K$. ($H<K$ ($H\leqq K$) if given $h\in H$ and $k\in K$ then $h<k$ ($h\leqq k$).) The initial results on η_α-sets can be found in Hausdorff [15, pp. 180–185]. For more recent results see Sierpiński [21] and Gillman [10].

Let ρ be an ordinal number. By $W(\rho)$ is meant the set of all ordinal numbers δ less than ρ. For an ordinal α, ω_α is defined to be the least ordinal such that $W(\omega_\alpha)$ is of power $\aleph_\alpha$. A nonempty totally ordered set T without a greatest element is said to have *upper character* ω_α if the smallest cardinal number of a cofinal subset of T is $\aleph_\alpha$; in this case T has a cofinal subset isomorphic to

$W(\omega_\alpha)$ and ω_α is the smallest such ordinal. The lower character of T is the dual concept. Let $t\in T$. If $\{x\in T: x<t\}$ has no greatest element and if its upper character is ω_α, then the *left character of t* is defined to be ω_α. The dual definition defines the *right character of t.*

By a *gap* in T is meant a pair (H, K) of nonempty subsets of T such that H has no greatest element, K has no least element, $H<K$, and $H\cup K=T$. Let (H, K) be a gap in T. By the *left character of* (H, K) is meant the upper character of H. The dual definition defines the *right character of* (H, K). Hausdorff [15, pp. 142–147] developed the idea of characters as here presented, although his notation has here been modified somewhat.

The following is readily verified: a nonempty totally ordered set that is dense in itself and that has neither a least nor a greatest element is an η_α-set if and only if its upper and lower characters are not less than ω_α; given a point, its right and left characters are not less than ω_α; and given a gap, at least one of its characters is not less than ω_α.

To conclude, Hausdorff [15] and Gillman [10] have shown that an η_α-set of power $\aleph_\alpha$ exists if and only if $\aleph_\alpha$ is regular and $\sum_{\delta<\alpha} 2^{\aleph_\delta}\leqq\aleph_\alpha$.

Let G be a totally ordered Abelian group. For $g\in G$ let $|g|=\max(g, -g)$. A subgroup G' of G will be called *convex* if given $g\in G$ and $g'\in G'$ such that $|g|\leqq|g'|$, then $g\in G'$. The set of all convex subgroups of G is totally ordered under inclusion and, under this order, is a complete lattice. For $a\in G$ let $X(a)$ be the smallest convex subgroup of G that contains a. Such subgroups will be called *principal convex subgroups.* Clearly X has the following properties, given $a, b\in G$:

(0.1) $|a|\leqq|b|$ implies $X(a)\subset X(b)$.

(0.2) $X(a)=X(b)$ if and only if there exists a positive integer n such that $|a|\leqq n|b|$ and $|b|\leqq n|a|$.

(0.3) $X(a\pm b)\subset X(a)\cup X(b)$, and if $X(a)\neq X(b)$ then $X(a\pm b) = \max(X(a), X(b))$.

Let G^* be the nonzero elements of G and let $P=X(G^*)$. The totally ordered set P will be called the *value set of G under X.* (Note: by (0.1), $X(a)=\{0\}$ if and only if $a=0$.) Clearly $X(G)=\{0\}\cup P$. A mapping of G onto a totally ordered set, with a least element, that satisfies conditions (0.1)–(0.3) will be called a *natural valuation of G.* Clearly two natural valuations of G are essentially identical.

The valuation on G induces a uniform structure on G. It generates the interval topology (under which G is a topological group) if P has no least element and the discrete topology if P has a least element.

Let $a\in G$ and let N denote the set of positive integers. It is easily seen that $X(a)=\{g\in G: |g|\leqq n|a| \text{ for some } n\in N\}$. Let $a\neq 0$ and let $Y(a) =\{g\in G: n|g|<|a| \text{ for all } n\in N\}$. Clearly $Y(a)$ is the largest proper convex subgroup of $X(a)$; thus $X(a)/Y(a)$, which will be denoted by $G(a)$ and will be called a *factor* of G, is an Archimedean totally ordered group under the natural order on the cosets of $Y(a)$ in $X(a)$. Let K be a totally ordered field and let X be the natural valuation on K, considered as a totally ordered group. Let a, $b \in K^*$ and let $X(a)+X(b)=X(ab)$. Under this addition, which is independent of the choice of representative, P is a totally ordered Abelian group. Further, K is a field with valuation X and value group P whose valuation ring is $X(1)$, whose valuation ideal is $Y(1)$, and whose residue class field $k=X(1)/Y(1)$ is totally ordered and Archimedean. Although anticipated by Hahn [14], these ideas are due to Baer [4] and Krull [17]. More recent researches have been done by Conrad [6; 7], Gravett [12; 13], and Fleischer [9].

Let ρ be a nonzero limit ordinal number and let G be a totally ordered Abelian group. A sequence $(g_\delta)_{\delta<\rho}$ of elements of G is said to be *pseudo-convergent* if given $\delta<\gamma<\mu<\rho$ then $X(g_\gamma-g_\delta)>X(g_\mu-g_\gamma)$. By the *length* of the sequence is meant the ordinal number ρ. Let $(g_\delta)_{\delta<\rho}$ be a pseudo-convergent sequence and let $p_\delta=X(g_{\delta+1}-g_\delta)$ for all $\delta<\rho$. Clearly $(p_\delta)_{\delta<\rho}$ is a strictly decreasing sequence in P, the value set of G. It is well known [16] that if $\delta<\gamma<\rho$ then $X(g_\gamma-g_\delta)=p_\delta$. An element g of G is called a *pseudo-limit* of $(g_\delta)_{\delta<\rho}$ if $X(g-g_\delta)=p_\delta$ for all $\delta<\rho$. Let $B=\{h\in G: X(h)<p_\delta \text{ for all } \delta<\rho\}$. B is known as the *breadth* of $(g_\delta)_{\delta<\rho}$. It is well known [16] that if the sequence has a pseudo-limit in G it is uniquely determined modulo B. These ideas, introduced by Ostrowski [20], were extended by Kaplansky [16] and have been used on valued linear spaces and ordered Abelian groups by Gravett [12; 13], and on normed modules by Fleischer [9].

Clearly B is a convex subgroup of G. Let $G'=G/B$ and let $P'=\{p\in P: p_\delta<p$ for some $\delta<\rho\}$. X induces a natural valuation X' on G' whose value set is P'. Let g_δ' be the image of g_δ in G'. Then $(g_\delta')_{\delta<\rho}$ is a Cauchy sequence in the valuation uniformity on G' whose topology, since P' has no least element, is the same as the interval topology on G'. Let g be a pseudo-limit of $(g_\delta)_{\delta<\rho}$ in G and let g' be the image of g in G'. It is easily seen that g' is the limit of $(g_\delta')_{\delta<\rho}$. Conversely, let g' be the limit of $(g_\delta')_{\delta<\rho}$ in G' and let g be a pre-image of g' in G. Then g is a pseudo-limit of $(g_\delta)_{\delta<\rho}$. This may be shown by the following argument: let $\delta<\rho$. Since $(g_\delta')_{\delta<\rho}$ converges to g' in G', there exists γ, $\delta<\gamma<\rho$, such that $X'(g'-g_\gamma')<p_\delta$. Thus, $X(g-g_\gamma)<p_\delta$ and $X(g-g_\delta)=X(g-g_\gamma+g_\gamma-g_\delta)=p_\delta$. The equivalence of the pseudo-convergence of $(g_\delta)_{\delta<\rho}$ in G and of the convergence of $(g_\delta')_{\delta<\rho}$ in G', a result which is stated implicitly in [9], will be used several times in this paper.

Let α be a nonzero ordinal number. G is called *α-maximal* if every pseudo-convergent sequence of length less than ω_α has a pseudo-limit in G. This definition, introduced in [2], is closely related to the idea of a maximal field with valuation ([17; 20] and [16]), and a maximal group with valuation ([6; 12] and [9]). For the sake of completeness it can be said that a totally ordered group G is *maximal* if every pseudo-convergent sequence in G has a pseudo-limit in G.

1. Let G be a totally ordered Abelian group, let X be its natural valuation, and let P be its value set. In this section it will be shown that the three conditions of the Main Theorem are sufficient.

THEOREM 1.1. *If the upper character of P is ω_α then both the upper and the lower characters of G are equal to ω_α. If the lower character of P is ω_α, then the point character of G is ω_α.*

Proof. Let A be a subset of positive elements of G. If A is cofinal in G then $X(A)$ is cofinal in P. If A is coinitial in $\{g\in G: g>0\}$ then $X(A)$ is coinitial in P, proving the theorem.

Assuming the conditions of the Main Theorem we see that it remains to be proved that given a gap in G, one of its characters is at least ω_α. A gap in G is associated with a unique point in the Dedekind completion of G. It is sufficient therefore to consider the Dedekind completion of G and to prove that no point in it exists both of whose characters are less than ω_α.

Assume now that G has no least positive element. (This will certainly be assured if P is nonempty and without a least element.) Thus G is nondiscrete in the interval topology, under which G is a topological group. Let Σ be the Dedekind completion of G: i.e., let Σ be a totally ordered set, conditionally complete as a lattice, without a least or a greatest element, in which G is dense.

Let $a, b\in\Sigma$ and let $aLb=\text{l.u.b. }\{g+h: g, h\in G, g\leqq a, \text{ and } h\leqq b\}$ and let $aUb=\text{g.l.b. }\{g+h: g, h\in G, g\geqq a, \text{ and } h\geqq b\}$. Clearly $aLb\leqq aUb$. Clifford [5] has shown that Σ is a semigroup under L and U, that L is lower semi-continuous and U is upper semi-continuous, and that L and U extend the addition on G. Let $g\in G$. For $x\in G$ the mapping $x\to x+g$ is a one-to-one order preserving mapping of G onto G. Clearly this mapping extends uniquely to a one-to-one order preserving mapping of Σ onto Σ; thus $aLg=aUg$ for any $a\in\Sigma$. In case a or $b\in G$, we will write $a+b$ instead of aLb or aUb.

The mapping $x\to -x$ is a one-to-one order reversing mapping of G onto G. Clearly it extends, uniquely, to a one-to-one order reversing mapping of Σ onto Σ. For $a\in\Sigma$ let a^* be the image of a under this mapping. Clearly $a^{**}=a$ and $(aLb)^*=a^*Ub^*$ for all $a, b\in\Sigma$. A subset Σ' of Σ will be called *symmetric* if it is closed under the mapping $a\to a^*$. Let Σ' be a symmetric subset of Σ. Then Σ' is closed under L if and only if it is closed under U, since $(aLb)^*=a^*Ub^*$. A subset Σ' of Σ will be called *convex* if given $a, c\in\Sigma'$ and $b\in\Sigma$ such that $a<b<c$, then $b\in\Sigma'$. The set of all convex, symmetric subsemigroups (under L or equivalently under U) of Σ is totally ordered under inclusion and as a lattice is complete. For $a\in\Sigma$ let $S(a)$ be the smallest convex, symmetric subsemigroup of Σ that contains a. It is easy to verify that S has the following properties: $S(a)=\{0\}$ if and only if $a=0$; if $|a|\leqq|b|$, where $|a|=\max(a, a^*)$, then $S(a)\subset S(b)$; and $S(aLb)\cup S(aUb)\subset S(a)\cup S(b)$. Let $V(a)=S(a)\cap G$. Clearly V satisfies these properties and in addition $V(a)$ is a convex subgroup of G. Further, V extends X: i.e., given $g\in G$,

$V(g)=X(g)$. However, $V(a)$ need not be in P. Hence the natural valuation X on G has been extended to V, the *natural valuation of* Σ.

It should be noted, in passing, that if $2G=G$ and if G' is a proper, convex subgroup of G, then $V(\text{l.u.b. } G')=G'$. In the following example this does not hold, however. Let $G=\{f\in R^{W(\omega_0+1)}: \text{the support of } f \text{ is finite and } f(\omega_0)\in Z\}$ and let it be ordered anti-lexicographically. Let $G'=\{f\in G: f(\omega_0)=0\}$. Then G' is a proper, convex subgroup of G and yet $V(\text{l.u.b. } G')=G$. Further, $G'\notin V(\Sigma)$ (2).

Our task is to show that if G satisfies the conditions of the Main Theorem, then there are no elements $a\in\Sigma$ both of whose characters are less than ω_α. We will proceed by contradiction, after proving three lemmas.

LEMMA 1.2. *Let α be a nonzero ordinal number and let G be such that P is an η_α-set. Let $a\in\Sigma$, $a>0$, such that both of its characters are less than ω_α. There exist h, $k\in G$ such that $0<h<a<k$ and $X(h)=X(k)$; thus $V(a)=X(h)\in P$.*

Proof. Since the characters of a are both less than ω_α, there exist nonempty subsets of positive elements of G, H, and K, such that $H<K$, $|H|+|K|<\aleph_\alpha$, and l.u.b. $H=a=$g.l.b. K. Clearly $X(H)\leqq X(K)$. Were $X(H)<X(K)$ then, since P is an η_α-set, there would exist $p\in P$ such that $X(H)<\{p\}<X(K)$. Let g be a positive element of $X^{-1}(p)$. Then $H<\{g\}<K$; thus $g=a$. Since P is an η_α-set, its lower character is at least ω_α; thus, by Theorem 1.1, the point character of G is at least ω_α. Hence the characters of $g(=a)$ are at least ω_α, which is a contradiction. We conclude that there exist $h\in H$ and $k\in K$ such that $X(h)=X(k)$, proving the lemma.

On assuming the second condition of the Main Theorem, that P is an η_α-set, Lemma 1.2 may be invoked. Some machinery will now be introduced that will allow us to exploit the first condition of the Main Theorem: that the factors of G be conditionally complete.

LEMMA 1.3. *Let $h\in G$, $h>0$, and let f be the canonical homomorphism of $X(h)$ onto $X(h)/Y(h)=G(h)$. Assume that $Y(h)\neq\{0\}$ and that $G(h)$ is conditionally complete. For $b\in S(h)$, the Dedekind completion of $X(h)$, let $F_0(b)=\text{l.u.b. }\{f(g): g\in X(h) \text{ and } g<b\}$ and let $F_1(b)=\text{g.l.b. }\{f(g): g\in X(h) \text{ and } g>b\}$. Then F_0 and F_1 are order preserving mappings of $S(h)$ onto $G(h)$ that extend f, $F_0(b)\leqq F_1(b)$, $F_0(bLc)=F_0(b)+F_0(c)$, $F_1(bUc)=F_1(b)+F_1(c)$, $F_0(b^*)=-F_1(b)$, and $F_1(b^*)=-F_0(b)$ for all b, $c\in S(h)$. Finally, if $G(h)$ is dense, $F_0=F_1$.*

Proof. $S(h)=\{b\in\Sigma: -nh<b<nh \text{ for some } n\in N\}$ and thus is conditionally complete; hence it is the Dedekind completion of $X(h)$. That F_0 and

(2) If $f\in G$ such that $\{f\}>G'$ then $f(\omega_0)\geqq 1$; thus if $a=\text{l.u.b. } G'$, the sequence a, aUa, $(aUa)Ua, \cdots$ is cofinal in Σ. Hence $S(a)=\Sigma$ and $V(a)=G$. Let $b\in\Sigma$, $0\leqq b<a$. There exists $g\in G'$ such that $b<g<a$. Since the value set of G' is isomorphic to $W(\omega_0)$, $V(g)$ is a proper subgroup of G'; thus $V(b)$ is a proper subgroup of G', showing that $G'\notin V(\Sigma)$.

F_1 are order preserving is obvious, as is the fact that $F_0(b) \leqq F_1(b)$ for all $b \in S(h)$. Since $Y(h) \neq \{0\}$, F_0 and F_1 extend f, but of course since $G(h)$ is Dedekind complete, the ranges of f, F_0 and F_1 coincide. Let $b, c \in S(h)$. Given $g, g' \in X(h)$ such that $g \leqq b$ and $g' \leqq c$ then $g+g' \leqq bLc$; thus $F_0(b)+F_0(c) \leqq F_0(bLc)$. Since L is lower semi-continuous, a fact proved by Clifford [5], given $g'' \in X(h)$ such that $g'' < bLc$, there exist $g, g' \in X(h)$ such that $g < b$, $g' < c$, and $g'' < g+g'$. Thus $F_0(bLc) \leqq F_0(b)+F_0(c)$. By a dual argument $F_1(bUc) = F_1(b)+F_1(c)$. Further, $F_0(b^*) = \text{l.u.b.}\{f(g)\colon g \in X(h),\ g < b^*\} = -\text{g.l.b.}\ \{f(-g)\colon -g \in X(h),\ -g > b\} = -F_1(b)$. Using this, $F_1(b^*) = -F_0(b^{**}) = -F_0(b)$. Assume that $G(h)$ is dense and, for a moment, that $F_0(b) < F_1(b)$ for some $b \in S(h)$; then there exists $g \in X(h)$ such that $F_0(b) < f(g) < F_1(b)$. If $g = b$ then $F_0(b) = f(g)$. If $g < b$ then $f(g) \leqq F_0(b)$, and if $g > b$ then $f(g) \geqq F_1(b)$, which is absurd. Hence $F_0(b) = F_1(b)$ for all $b \in S(h)$ if $G(h)$ is dense, proving the lemma.

The following example shows that if $G(h)$ is not dense, F_0 need not equal F_1. Let $G = Z \times R$, lexicographically ordered. Let $h = (1, 0)$; then $G(h)$ is isomorphic to Z. Let $b = \text{l.u.b.}\ \{(0, r)\colon r \in R\}$. Then $F_0(b) = 0$ and $F_1(b) = 1$.

In Lemma 1.3 condition (1) was used. In the next lemma conditions (1) and (2) are used in the hypothesis; in its conclusion, a result appears which will be used to exploit condition (3).

LEMMA 1.4. *Let G be such that P is an η_α-set, $\alpha > 0$, and such that the factors of G are conditionally complete. Let $a \in \Sigma$ such that both of its characters are less than ω_α. Then there exists $g \in G$ such that $V(a-g) < V(a)$.*

Proof. Since P is an η_α-set its lower character is at least ω_α. By Theorem 1.1, the point character of G is at least ω_α. Hence $a \neq 0$. Without loss of generality we may assume that $a > 0$ merely by replacing a with a^* if $a < 0$. By Lemma 1.2 there exist $h, k \in G$ such that $0 < h < a < k$ and $X(h) = X(k) = V(a)$. Since P does not have a least element, $Y(h) \neq \{0\}$. Let T be a cofinal subset of positive elements of $Y(h)$. Clearly $X(T)$ is a cofinal subset of $\{p \in P\colon p < X(h)\}$. Since P is an η_α-set, $|T| \geqq \aleph_\alpha$; thus the upper (and lower) character of $Y(h)$ is at least ω_α. Clearly $a \in S(h)$. Let f be the canonical homomorphism of $X(h)$ onto $G(h) (\equiv X(h)/Y(h))$. Let F_0 and F_1 be the mappings of $S(h)$ onto $G(h)$ defined in Lemma 1.3. If $G(h)$ is dense, $F_0 = F_1$. If $G(h)$ is not dense, it is isomorphic to Z, the additive group of integers; thus $X(h)$ is isomorphic to $Z \times Y(h)$, ordered lexicographically. Since the upper (and lower) character of $Y(h)$ is at least ω_α and since both characters of a are less than ω_α, a lies over a coset of $Y(h)$ in $X(h)$ rather than between two such cosets; thus $F_0(a) = F_1(a)$.

Let $r = F_0(a)$. Let $g \in X(h)$ such that $f(g) = r$. Using Lemma 1.3 we see that $F_0(a-g) = 0 = F_1(a-g)$. Let $K = F_0^{-1}(0) \cap F_1^{-1}(0)$; thus $a - g \in K$. Since F_0 and F_1 are order preserving, K is convex. Let $b, c \in K$. Then $F_0(bLc) = F_0(b)+F_0(c) = 0 = F_1(b)+F_1(c) = F_1(bUc)$. Since $F_0 \leqq F_1$ and $bLc \leqq bUc$,

$0=F_0(bLc)\leqq F_1(bLc)\leqq F_1(bUc)=0$; thus $bLc\in K$, showing that K is closed under L. Since $F_0(b^*)=-F_1(b)=0$ and $F_1(b^*)=-F_0(b)=0$, K is symmetric. Thus K is a symmetric, convex subsemigroup of Σ. Hence $S(a-g)\cap G\subset K\cap G$ $=f^{-1}(0)=Y(h)<X(h)=V(a)$, proving the lemma.

The reader may wonder at the complexity of the last paragraph, and in particular at the use of both F_0 and F_1 to define K. In case $G(h)$ is discrete, $F_0^{-1}(0)$ and $F_1^{-1}(0)$ will not be symmetric, for if $d=\text{l.u.b. } Y(h)$ then $F_0(d)=0$, $F_0(d^*)=-1$, $F_1(d)=1$, and $F_1(d^*)=0$.

With this machinery we can now prove the sufficiency of the conditions in the Main Theorem.

THEOREM 1.5. *Let α be a nonzero ordinal number and let G be such that* (1) *its factors are conditionally complete,* (2) *its value set P is an η_α-set, and* (3) *it is α-maximal. Then G is an η_α-set.*

Proof. Since P is an η_α-set its upper and lower characters are at least ω_α. Applying Theorem 1.1 we see that the upper, lower, and point characters of G are all at least ω_α. Let $a\in\Sigma$, the Dedekind completion of G. It suffices to show that at least one of the characters of a is at least ω_α. Assume for a moment that both of the characters of a are less than ω_α. Without loss of generality we may assume that $a>0$ merely by replacing a with a^* if $a<0$. There exist nonempty subsets of G such that $\{0\}<H<\{a\}<K$, $|H|+|K|<\aleph_\alpha$, and l.u.b. $H=a=$ g.l.b. K. Let $P_0=\{V(a-x):x\in G\}$. By Lemma 1.2, $P_0\subset P$. By Lemma 1.4, given $x\in G$ there exists $g\in G$ such that $V(a-x-g)<V(a-x)$; thus P_0 has no least element. Let $P_0'=\{V(a-x):x\in H\cup K\}$. Clearly P_0' is coinitial in P_0. Hence the lower character of P_0 is less than ω_α. Let $B=\{g\in G: \{X(g)\}<P_0\}$. Since G is α-maximal, $G'=G/B$ is Cauchy complete. For $p\in P_0$ let $U_p=\{g\in G: V(a-g)<p\}$. Clearly $(U_p)_{p\in P_0}$ is a filter base of open sets in G. Let U_p' be the image of U_p in G'. Clearly $(U_p')_{p\in P_0}$ is a Cauchy filter base of open sets in G'. Since G' is Cauchy complete, this filter base has a limit g' in G'. Let g be a pre-image of g' in G. Thus $g\in U_p$ for all $p\in P_0$ and therefore $V(a-g)<p$ for all $p\in P_0$. But $V(a-g)\in P_0$, which is absurd, proving that at least one of the characters of a is not less than ω_α[3].

2. Having proved the Main Theorem, we can now give a number of examples of totally ordered Abelian groups that are η_α-sets, $\alpha>0$.

Let T be a totally ordered set, let k be a field, and let $f\in k^T$. Let $s(f)$ $\equiv\{t\in T: f(t)\neq 0\}$. Then $s(f)$ will be called the *support* of f. The set $s(f)$ is

[3] Given $a, b\in\Sigma$ such that $V(a)\neq V(b)$ then $V(aLb)=V(a)\cup V(b)=V(aUb)$. Using this result (which is not as easily obtained as the corresponding result for $a, b\in G$), the last part of this proof can be simplified as follows: choose a sequence $(g_\delta)_{\delta<\rho}$ in $H\cup K$ such that $(V(a-g_\delta))_{\delta<\rho}$ is strictly decreasing and is coinitial in P_0'. Clearly ρ is a limit ordinal less than ω_α. With the aid of the result which begins this footnote it is easily seen that $(g_\delta)_{\delta<\rho}$ is pseudo-convergent by observing that if $\delta<\epsilon<\rho$ and if $V(a-g_\delta)\neq V(g_\delta-g_\epsilon)$ then $V(a-g_\epsilon)=V(a-g_\delta+g_\delta-g_\epsilon)=V(a-g_\delta)$ $\cup V(g_\delta-g_\epsilon)\geqq V(a-g_\delta)$, which is absurd. Since G is α-maximal there exists $g\in G$ that is a pseudo-limit of this sequence. Then $V(a-g)<p$ for all $p\in P_0$, which is absurd, completing the proof.

called *anti-wellordered* if every nonempty subset of it contains a greatest element. Let $k\{T\}$ be the set of all $f\in k^T$ such that $s(f)$ is anti-wellordered. $k\{T\}$ is an Abelian group under pointwise addition. Let f be a nonzero element in $k\{T\}$ and let $d(f)$ be the greatest element in $s(f)$. Let k be a totally ordered field and define $f>0$ if $f(d(f))>0$. Under this ordering $k\{T\}$ is a totally ordered Abelian group. Further, if k is Archimedean d is a natural valuation of $k\{T\}$ with value set T. The group $k\{T\}$ is called the *Hahn group on T with coefficients in k*. See Hahn [14], Zelinsky [23], and B. H. Neumann [19] for details.

Let E be an η_α-set, $\alpha>0$, and $G=R\{E\}$, R being the field of real numbers. Let $G_\alpha=\{f\in G: |s(f)|<\aleph_\alpha\}$. Clearly G_α is a subgroup of G. Let E_0 be a nonempty subset of E and let $M=\{f\in G$: given $e\in E_0, f(e)\in Z\}$. Clearly M is a subgroup of G. Let $M_\alpha=M\cap G_\alpha$. It will be shown in this section that G and M are η_α-sets and if $\aleph_\alpha$ is a regular cardinal number, then G_α and M_α are η_α-sets. Clearly G and G_α are divisible, whereas M and M_α are not. Finally it will be shown that if E is of power $\aleph_\alpha$, then G_α and M_α are of power $\aleph_\alpha$.

Clearly G, G_α, M and M_α have as their value set E, an η_α-set. Further, in each case the factors are either the reals or the integers; thus conditions (1) and (2) of the Main Theorem hold. Clearly G and M are maximal, thus α-maximal, and by the Main Theorem are η_α-sets.

THEOREM 2.1. *Assume that $\aleph_\alpha$ is regular. Then G_α and M_α are α-maximal.*

Proof. Let ρ be a nonzero limit ordinal less than ω_α and let $(e_\delta)_{\delta<\rho}$ be a strictly decreasing sequence in E. Let $B=\{g\in G: d(g)<e_\delta$ for all $\delta<\rho\}$ and let $B_\alpha=B\cap G_\alpha$. Clearly $B(B_\alpha)$ is a convex subgroup of $G(G_\alpha)$. To show that G_α is α-maximal it suffices to show that G_α/B_α is Cauchy complete. Clearly $E'=\{e\in E: e\geqq e_\delta$ for some $\delta<\rho\}$ is the value set of G_α/B_α and G/B under natural valuations. Further G_α/B_α may be identified, in a natural way, with a subgroup of G/B. Let a' be in G/B and be the limit of a Cauchy sequence in G_α/B_α. Let a be a pre-image of a' in G. Let $s(a)=(x_\mu)_{\mu<\gamma}$, the sequence being strictly decreasing. Let $a(x_\mu)=r_\mu$ for all $\mu<\gamma$. Let $c(x_\mu)$ be the characteristic function of x_μ. Clearly $a=\sum_{\mu<\gamma} r_\mu c(x_\mu)$. If $(x_\mu)_{\mu<\gamma}\cap E'=\varnothing$ then $a'=0$ and is in G_α/B_α. Assume that the intersection is not empty. Since given $p'\in E'$ and $p\in P$ such that $p'<p$ then $p\in E'$, and since $(x_\mu)_{\mu<\gamma}$ is strictly decreasing, there exists an ordinal γ' such that $x_\mu\in E'$ if and only if $\mu<\gamma'$. In the next paragraph it will be shown that $\gamma'<\omega_\alpha$. Let $b=\sum_{\mu<\gamma'} r_\mu c(x_\mu)$. Thus $b\in G_\alpha$ and a' is the image of b in G_α/B_α, proving that G_α/B_α is Cauchy complete.

Since a' is the limit of a Cauchy sequence in G_α/B_α, given $\delta<\rho$ there exists $g_\delta\in G_\alpha$ such that $d(a-g_\delta)<e_\delta$. Thus $\{x_\mu: \mu<\gamma$ and $x_\mu\geqq e_\delta\}\subset s(g_\delta)$. Hence $\{x_\mu: \mu<\gamma'\}=\bigcup_{\delta<\rho}\{x_\mu: \mu<\gamma$ and $x_\mu\geqq e_\delta\}\subset\bigcup_{\delta<\rho} s(g_\delta)$. Therefore $|\{\mu:\mu<\gamma'\}|=|\{x_\mu:\mu<\gamma'\}|\leqq\sum_{\delta<\rho}|s(g_\delta)|$. Since $g_\delta\in G_\alpha$, $|s(g_\delta)|<\aleph_\alpha$. Since

$\rho<\omega_\alpha$ and since $\aleph_\alpha$ is regular, $\sum_{\delta<\rho}|s(g_\delta)|<\aleph_\alpha$. This proves that $\gamma'<\omega_\alpha$. A similar argument shows that M_α is α-maximal, proving the theorem.

A more general version of this theorem can be proved by dropping the assumption that $\aleph_\alpha$ is regular and by replacing "α-maximal" by "$cf(\alpha)$-maximal" in the conclusion, $\omega_{cf(\alpha)}$ being the upper character of $W(\omega_\alpha)$ $(=\{\delta: \delta<\omega_\alpha\})$. (See Tarski [22] for details on $cf(\alpha)$.) Of course, $\aleph_\alpha$ is regular if and only if $\alpha=cf(\alpha)$. If $\aleph_\alpha$ is singular (i.e., not regular), then $cf(\alpha)<\alpha$. That $cf(\alpha)$ is the largest ordinal number for which G_α (or similarly M_α) is $cf(\alpha)$-maximal may be seen as follows. Since E is an η_α-set, its lower character is at least ω_α; thus there exists a strictly decreasing sequence $(e_\delta)_{\delta<\omega_\alpha}$ in E. By definition $\omega_{cf(\alpha)}(\equiv\rho)$ is the upper character of $W(\omega_\alpha)$; thus there exists a strictly increasing sequence $(\pi(\delta))_{\delta<\rho}$ that is cofinal in $W(\omega_\alpha)$. Let $\pi(\rho)=\omega_\alpha$. For $\epsilon\leqq\rho$ let $g_\epsilon=\sum_{\delta<\pi(\epsilon)}c(e_\delta)$. Clearly $g_\epsilon\in G_\alpha$ for all $\epsilon<\rho$, however $g_\rho\notin G_\alpha$. Clearly g_ρ is a pseudo-limit of $(g_\epsilon)_{\epsilon<\rho}$. Given any pseudo-limit h of $(g_\epsilon)_{\epsilon<\rho}$, then $(e_\delta)_{\delta<\pi(\rho)}\subset s(h)$; thus $h\notin G_\alpha$, proving that G_α is not $(cf(\alpha)+1)$-maximal.

COROLLARY 2.2. *G and M are η_α-sets. If $\aleph_\alpha$ is regular then G_α and M_α are η_α-sets.*

THEOREM 2.3. *Let E be of power $\aleph_\alpha$. Then G_α and M_α are of power $\aleph_\alpha$.*

Proof. Clearly $|M_\alpha|\geqq\aleph_\alpha$; thus it suffices to show that $|G_\alpha|\leqq\aleph_\alpha$. Hausdorff [15] and Gillman [10] have shown that the existence of an η_α-set of power $\aleph_\alpha$ implies that $\aleph_\alpha$ is regular and that $\sum_{\delta<\alpha}2^{\aleph_\delta}\leqq\aleph_\alpha$. Let the ordering on E, under which it is an η_α-set, be suppressed and let E be identified with $W(\omega_\alpha)$. For $\pi\in W(\omega_\alpha)$ let $A_\pi=\{f\in G_\alpha: s(f)\subset W(\pi)\}$. Let $f\in G_\alpha$. Since $|s(f)|<\aleph_\alpha$ and since $\aleph_\alpha$ is regular, there exists $\pi\in W(\omega_\alpha)$ such that $s(f)\subset W(\pi)$; thus $G_\alpha\subset\bigcup_{\pi<\omega_\alpha}A_\pi$. Hence $|G_\alpha|\leqq\sum_{\pi<\omega_\alpha}|A_\pi|\leqq\sum_{\pi<\omega_\alpha}(2^{\aleph_0})^{|W(\pi)|}$ $\leqq\sum_{\delta<\alpha}\aleph_{\delta+1}2^{\aleph_0\aleph_\delta}\leqq\aleph_\alpha\sum_{\delta<\alpha}2^{\aleph_\delta}\leqq\aleph_\alpha^2=\aleph_\alpha$, proving the theorem.

COROLLARY 2.4. *If E is of power $\aleph_\alpha$ then G_α and M_α are η_α-sets of power $\aleph_\alpha$.*

3. Having these results on totally ordered Abelian groups, we can apply them to totally ordered fields, which of course are totally ordered Abelian groups under addition.

Let α be a nonzero ordinal number and let G be a totally ordered Abelian group that is an η_α-set. Let $K=R\{G\}$; thus K is a totally ordered group. For $a, b\in K$ let $ab(g)=\sum_{x\in G}a(x)b(-x+g)$. Hahn [14] showed that under this definition, K is a field, often referred to as the *field of formal power series with coefficients in R and exponents in G.* (For a more modern presentation see [19].) Clearly G is isomorphic to the value group of K, and R is isomorphic to the residue class field of K. Since K is maximal [17], the Main Corollary applies and asures us that K is an η_α-set. Let $K_\alpha=R\{G\}_\alpha$. By Corollary 2.2, if $\aleph_\alpha$ is regular then K_α is an η_α-set. By Corollary 2.4, if G is of power $\aleph_\alpha$ then K_α is of power $\aleph_\alpha$.

It is easily verified that two necessary conditions for a totally ordered field to be real-closed are that its value group be divisible and that its residue class field be real-closed. As shown in §2, if an η_α-set exists, then nondivisible Abelian groups G that are η_α-sets exist; thus for G nondivisible, K and K_α are not real-closed.

Krull [17] observed that if G is divisible then $R\{G\}$, which is maximal, is real-closed. MacLane [18] has shown that the field of formal power series with coefficients in an algebraically closed field and exponents in a totally ordered Abelian divisible group, is algebraically closed. Knowing [3] that a totally ordered field F is real-closed if and only if $F(i)$ is algebraically closed, i being a root of x^2+1, it is clear that "algebraically" can be replaced by "real-" in the last sentence(4). After stating an obious lemma, this result will be somewhat extended.

LEMMA 3.1. *Let F be a field, let H be a totally ordered Abelian group, and let H' be a subgroup of H. For $a' \in F\{H'\}$ let $t(a')$ be the extension of a' to H, letting its values outside of H' be zero. Then t is an isomorphism of $F\{H'\}$ into $F\{H\}$ and if F is a totally ordered field then t is order preserving.*

COROLLARY 3.2. *Let F be a real- (algebraically) closed field, let H be a totally ordered Abelian divisible group, and let α be a nonzero ordinal number. Then $F\{H\}_\alpha$ is a real- (algebraically) closed field.*

Proof. Let $f(x)=\sum_{j=0}^n a_j x^j$ be a nonzero polynomial with coefficients in $F\{H\}_\alpha$. Let H' be the smallest divisible subgroup of H containing $\bigcup_{j=0}^n s(a_j)$. Since $a_j \in F\{H\}_\alpha$, $|s(a_j)| < \aleph_\alpha$; thus $|H'| < \aleph_\alpha$. Let a_j' be the restriction of a_j to H' and let $f'(x)=\sum_{j=0}^n a_j' x^j$. By results of Krull [17] and MacLane [18] stated above, $F\{H'\}$ is a real- (algebraically) closed field. Clearly the isomorphism t, defined in Lemma 3.1, extends to an isomorphism t^* of $F\{H'\}[x]$ into $F\{H\}[x]$, which sends $f'(x)$ to $f(x)$.

Let $f(x)=x^2-b$, $b>0$. Since $F\{H'\}$ is real-closed, $f(x)$ has a root in $F\{H\}_\alpha$. If $f(x)$ is any polynomial of odd degree then, since $F\{H'\}$ is real-closed, $f(x)$ has a root in $F\{H\}_\alpha$, proving that $F\{H\}_\alpha$ is a real-closed field.

A similar argument holds in case F is algebraically closed, proving the corollary.

The following example shows that the two necessary conditions for a totally ordered field to be real-closed, given earlier in this section, are not sufficient. Let $U=R\{R\{W(\omega_1)\}\}$ and let $U_1=R\{R\{W(\omega_1)\}\}_1$. By Corollary 3.2, U_1 is a real-closed field, as is U. Further, U_1 is a proper subfield of U,

(4) This rests on the fact that $(F(i))\{G\}$ is isomorphic to $(F\{G\})(i)$, as can be seen by the following: clearly $F\{G\}$ is a subfield of $(F(i))\{G\}$, which is algebraically-closed. Hence $(F\{G\})(i)$ may be identified with $\{a+bi\colon a, b \in F\{G\}\}$, a subfield of $(F(i))\{G\}$. For $z \in (F(i))\{G\}$, $s(z)$ (the support of z) is an anti-wellordered subset of G. For $g \in G$ let $z(g) = a(g)+b(g)i$, $a(g)$, $b(g) \in F$. Clearly $s(a)$, $s(b) \subset s(z)$; thus $a, b \in F\{G\}$ and $z \in (F\{G\})(i)$.

since there are uncountable anti-well ordered subsets in $R\{W(\omega_1)\}$. Let a be a positive element in U that is not in U_1. Then $U_1(a)$ is a transcendental extension of U_1; hence the square root of a, which is in U, is not in $U_1(a)$, showing that $U_1(a)$ is not real-closed. However, its value group is divisible and its residue class field is real-closed.

To conclude, a construction of the fields referred to in the title of the paper can be achieved as follows. Let E be an η_α-set of power $\aleph_\alpha$, $\alpha>0$. Then $R\{E\}_\alpha$ is a totally ordered Abelian divisible group that is an η_α-set of power $\aleph_\alpha$. Finally, $R\{R\{E\}_\alpha\}_\alpha$ is a real-closed field that is an η_α-set of power $\aleph_\alpha$.

BIBLIOGRAPHY

1. N. L. Alling, *On ordered divisible groups*, Trans. Amer. Math. Soc. **94** (1960), 498–514.

2. ———, *A characterization of Abelian η_α groups in terms of their natural valuation*, Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 711–713.

3. E. Artin and O. Schreier, *Algebraische Konstruktion reeler Körper*, Abh. Math. Sem. Univ. Hamburg **5** (1926), 85–99.

4. R. Baer, *Über nicht-Archimedisch geordnete Körper*, S.-B. Heidelberger Akad. **8** (1927), 3–13.

5. A. H. Clifford, *Completion of semi-continuous ordered commutative semigroups*, Duke Math. J. **26** (1958), 41–59.

6. P. Conrad, *Embedding theorems for Abelian groups with valuations*, Amer. J. Math. **75** (1953), 1–29.

7. ———, *On ordered division rings*, Proc. Amer. Math. Soc. **5** (1954), 323–328.

8. P. Erdös, L. Gillman and M. Henriksen, *An isomorphism theorem for real-closed fields*, Ann. of Math. **61** (1955), 542–554.

9. I. Fleischer, *Maximality and ultracompleteness in normed modules*, Proc. Amer. Math. Soc. **9** (1958), 151–157.

10. L. Gillman, *Some remarks on η_α-sets*, Fund. Math. **43** (1956), 77–82.

11. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960.

12. K. A. H. Gravett, *Valued linear spaces*, Quart. J. Math. Oxford Ser. (2) **6** (1955), 309–315.

13. ———, *Ordered Abelian groups*, Quart. J. Math. Oxford Ser. (2) **7** (1956), 57–63.

14. H. Hahn, *Über die nichtarchimedischen Grössensysteme*, S.-B. K. Akad. Wiss. Vienna **116** (1907), 601–653.

15. F. Hausdorff, *Grundzüge der Mengenlehre*, Verlag von Veit, Leipzig, 1914.

16. I. Kaplansky, *Maximal fields with valuation*, Duke Math. J. **9** (1942), 303–321.

17. W. Krull, *Allgemeine Bewertungstheorie*, J. Reine Angew. Math. **167** (1932), 160–196.

18. S. MacLane, *The universality of formal power series fields*, Bull. Amer. Math. Soc. **45** (1939), 888–890.

19. B. H. Neumann, *On ordered division rings*, Trans. Amer. Math. Soc. **66** (1949), 202–252.

20. A. Ostrowski, *Untersuchungen zur arithmetischen Theorie der Körper*, Math. Z. **39** (1935), 269–404.

21. W. Sierpiński, *Sur une propriété des ensembles ordonnés*, Fund. Math. **36** (1949), 56–67.

22. A. Tarski, *Sur les classes d'ensembles closes par rapport à certaines opérations élémentaires*, Fund. Math. **16** (1930), 181–304.

23. D. Zelinsky, *Nonassociative valuations*, Bull. Amer. Math. Soc. **54** (1948), 175–183.

PURDUE UNIVERSITY,
LAFAYETTE, INDIANA