ON GLOBAL ASYMPTOTIC STABILITY OF SOLUTIONS OF DIFFERENTIAL EQUATIONS(1)

BY

PHILIP HARTMAN AND CZESŁAW OLECH

1. **Introduction.** Consider a system of real differential equations for $x=(x^1, \cdots, x^n)$,

$$x' = f(x) \tag{1.1}$$

in which $f(x)$ is of class C^1 on E^n. Let $J(x)=(\partial f/\partial x)$ denote the Jacobian matrix of f and let $H(x)=(J+J^*)/2$ be the symmetric part of $J(x)$. One of the results of [2] is to the effect that if

$$f(0) = 0 \tag{1.2}$$

and

$$H(x) \text{ is negative definite (for fixed } x \neq 0), \tag{1.3}$$

then $x=0$ is a globally asymptotically stable solution of (1.1); i.e., every solution $x=x(t)$ or (1.1) exists for large t and $x(t)\to 0$ as $t\to\infty$. Among the results of [4], which deals with the case $n=2$, is the following: if

$$x=0 \text{ is a locally asymptotically stable solution of (1.1)} \tag{1.4}$$

and (1.3) is replaced by the conditions

$$\operatorname{tr} J(x) = \operatorname{tr} H(x) \leqq 0, \tag{1.5}$$

$$|f(x)| \geqq \text{Const.} > 0 \quad \text{for} \quad |x| \geqq \text{const.} > 0, \tag{1.6}$$

then again $x=0$ is a globally asymptotically stable solution of (1.1).

One of the main results of the first part of this paper will be a generalization of the latter theorem to the case of arbitrary $n\geqq 2$. In this situation, the trace of $H(x)$ will be replaced by the function

$$\alpha(x) = \max(\lambda_i(x) + \lambda_j(x)) \qquad \text{for } 1 \leqq i < j \leqq n, \tag{1.7}$$

where $\lambda_1(x), \cdots, \lambda_n(x)$ are the eigenvalues of $H(x)$, the condition (1.5) by

$$\alpha(x) \leqq 0, \tag{1.8}$$

Received by the editors August 22, 1961.

(1) This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under contracts No. AF 18(603)-41 and No. AF 49(638)-382 and in part by the U. S. Army, Army Ballistic Missile Agency under contract DA 36-034-ORD-3514 RD. Reproduction in whole or in part is permitted for any purpose of the United States Government.

and the condition (1.6) relaxed to

$$|x|\,|f(x)| \geqq \text{Const.} > 0 \qquad \text{for } |x| \geqq \text{const.} > 0,$$

or even to

$$\int^{\infty}\left[\min_{|x|=\rho} |f(x)|\right] d\rho = \infty. \tag{1.9}$$

Instead of dealing with (1.7), it turns out to be more convenient to treat the quantity introduced by Borg [1]

$$\gamma(x) = \max J(x)w\cdot w \qquad \text{for } |w| = 1,\ w\cdot f(x) = 0, \tag{1.10}$$

where $w=(w^1, \cdots, w^n)$. In particular, it will be seen that (1.4) and

$$\gamma(x) \leqq 0 \tag{1.11}$$

are sufficient for the global asymptotic stability of $x=0$. The relationship between the conditions (1.8)–(1.9) and (1.11) is indicated below; cf. (2.6).

The proof of these facts depends on a modification of arguments of [4] in which the plane E^2 of [4] is replaced by a piece of a 2-dimensional surface of E^n covered by a 1-parameter family of solutions of (1.1).

The second part of the paper concerns the orbital stability of bounded, nontrivial, solutions of (1.1) and is related to a result of Borg [1] dealing with a bounded solution $x=x_0(t)$ of (1.1) in a portion D of E^n where $|f(x)| \geqq \text{Const.}>0$ and $\gamma(x)<0$. A modification and simplification of Borg's arguments will be used to obtain stronger results. One theorem to be obtained implies, as in [1], that $x=x_0(t)$ has a periodic limit cycle $x=x^*(t)$ which has asymptotic orbital stability. Borg's condition $\gamma(x)<0$ will be relaxed, however, to one concerning the indefinite integral of $\gamma(x_0(t))$. Borg has also pointed out that his conditions enforce a "global" stability for $x=x^*(t)$ under suitable conditions on $f(x)$ on the boundary ∂D of D.

Finally, in the third part of the paper, some of the results will be extended to nonautonomous systems using the principles of Ważewski [5]. This method can also be used to obtain some of the results of the second part of the paper.

The authors wish to thank Professor J. Moser for calling their attention to the paper [1] of Borg.

2. **Trivial and unbounded solutions.** The notations J, H, α, γ, w, etc. introduced above will be employed.

THEOREM 2.1. *Let $f(x)$ be an n-dimensional vector function of class C^1 on E^n such that*

$$f(0) = 0; \qquad f(x) \neq 0 \qquad \text{if } x \neq 0; \tag{2.1}$$

and (1.4) *holds. Assume either that* (1.8)–(1.9) *holds or that* (1.11) *holds. Then $x=0$ is a globally asymptotically stable solution of* (1.1).

It is possible to formulate a condition which is more general than either (1.8)–(1.9) or (1.11). This condition will involve the existence of a suitable function $p(x)>0$.

THEOREM 2.2. *Let $f(x)$ be of class C^1 on E^n and such that* (2.1) *and* (1.4) *hold. Outside of a sufficiently small sphere $|x|<\epsilon$, let there exist a positive function $p(x)$ of class C^1 satisfying the inequalities*

$$p'(x) + \gamma(x)p(x) \leqq 0, \tag{2.2}$$

where $p'(x) = \operatorname{grad} p(x)\cdot f(x)$, *and* $|x|p(x) \geqq c > 0$ *or, more generally,*

$$\int^{\infty}\left[\min_{|x|=\rho} p(x)\right] d\rho = \infty. \tag{2.3}$$

Then $x=0$ is a globally asymptotically stable solution of (1.1).

By the set of attraction of the asymptotically stable solution $x=0$ is meant the set of points x_1 such that the solution $x=x(t)$ determined by the initial condition $x(0)=x_1$ exists for $t\geqq 0$ and satisfies $x(t)\to 0$ as $t\to\infty$. The only condition on $\epsilon>0$ in the last theorem is that the sphere $|x|\leqq\epsilon$ be in the set of attraction of $x=0$.

Theorem 2.1 is contained in Theorem 2.2. In order to see this, first note that if (1.11) holds, then $p(x)$ can be chosen to be $p(x)=1$. Also, note that if (1.9) holds and

$$\gamma(x)\,|f(x)|^2 + J(x)f(x)\cdot f(x) \leqq 0, \tag{2.4}$$

then $p(x)$ can be chosen to be $p(x)=|f(x)|$. It will now be verified that (1.8) implies (2.4).

For fixed x and a pair of constant vectors v and w,

$$\begin{aligned}(Jw\cdot w)\,|v|^2 + |w|^2(Jv\cdot v) - (v\cdot w)[(Jv\cdot w) + (v\cdot Jw)]&\\ \leqq \alpha[|v|^2|w|^2 - (v\cdot w)^2]&.\end{aligned} \tag{2.5}$$

In fact, the left side of (2.5) is unchanged if J is replaced by its symmetric part H and v, w are subjected to an orthogonal transformation; so that, without loss of generality, it can be supposed that $H=\operatorname{diag}(\lambda_1,\cdots,\lambda_n)$ at the given point x. The left side of (2.5) is then seen to be

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i(w^iw^iv^jv^j + w^jw^jv^iv^i - 2v^iw^iv^jw^j) = \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i(w^iv^j - w^jv^i)^2,$$

which is

$$\frac{1}{2}\sum_{i\neq j}\sum(\lambda_i+\lambda_j)(w^iv^j - w^jv^i)^2 \leqq \frac{1}{2}\,\alpha\sum_{i=1}^{n}\sum_{j=1}^{n}(w^iv^j - w^jv^i)^2.$$

Hence (2.5). If, in (2.5), $v=f(x)$ and w are subject to the conditions in the definition (1.10) of γ, it follows that

$$\gamma(x)\,|f(x)|^2 + J(x)f(x)\cdot f(x) \leqq \alpha(x)\,|f(x)|^2.$$

Hence (1.8) implies (2.4).

For later reference, note that the last inequality can be written as

$$\gamma(x) \leqq \alpha(x) - (\log |f(x)|)', \tag{2.6}$$

where the prime has the same significance as in (2.2).

Theorem 2.2 can be further generalized as follows:

THEOREM 2.3. *Let x denote a point on a complete (noncompact) Riemannian manifold M^n having a positive definite metric $ds^2 = g_{jk}(x)dx^j dx^k$ of class C^1. Let $x=0$ denote a fixed point of M^n. Let $f(x)$ be a contravariant vector field on M^n of class C^1 such that* (2.1) *and* (1.4) *hold. Outside of a sufficiently small neighborhood of $x=0$, let the tensor $e_{ij}=g_{ik}f^k{}_{,j}$ satisfy*

$$e_{ij}(x)w^i w^j \leqq 0 \qquad \text{if} \qquad g_{ij}(x)f^i(x)w^j = 0. \tag{2.7}$$

Then $x=0$ is a globally asymptotically stable solution of (1.1).

In the definition of e_{ij}, the tensor $f^k{}_{,j}$ is the covariant derivative of f; so that in local coordinates,

$$e_{ij} = g_{ik}\partial f^k/\partial x^j + \frac{1}{2}(\partial g_{ij}/\partial x^k)f^k + \frac{1}{2}[\partial g_{ki}/\partial x^j - \partial g_{kj}/\partial x^i]f^k. \tag{2.8}$$

In order to see that Theorem 2.2 is contained in Theorem 2.3, let $p(x)$ be a positive function of class C^1 on all of E^n satisfying (2.2) for $|x| \geqq \epsilon$. Let M^n be the manifold of points x of E^n with the metric $ds^2 = p^2(x)|dx|^2$, where $|dx|$ is the Euclidean element of arc-length. It is readily verified that (1.10) and (2.2) imply (2.7). In fact, (e_{ij}) is the matrix $p^2J(x) + pp'I + p(f^i\partial p/\partial x^j - f^j\partial p/\partial x^i)$, where I is the unit matrix and the last term is a skew-symmetric matrix; thus $e_{ij}w^iw^j = p(pJw\cdot w + p'|w|^2)$. Condition (2.3) assures that the distance from 0 to x on M^n tends to ∞ as $|x|\to\infty$, so that M^n is complete.

The condition (1.11) of Theorem 2.1 or (2.7) of Theorem 2.3 contrasts with the conditions in [2] involving negativity of $J(x)f(x)\cdot f(x)$ or $e_{ij}(x)f^i(x)f^j(x)$.

Conditions of the type (1.8) or (1.11) or those involving $p(x)$ imply some type of orbital stability, not only for trivial solutions ($x(t)\equiv$const.) but, for all solutions of (1.1). The following theorem is applicable to both bounded and unbounded solutions (and, for bounded solutions, it is not contained in those of §5 unless "$\leqq$" is replaced by "$<$" in (2.2)). The case of bounded (nontrivial) solutions will be considered in detail in §5 under conditions somewhat different from those of Theorem 2.4.

THEOREM 2.4. *Let D be a domain in E^n. Let $f(x)\neq 0$ be of class C^1 on D and let there exist a function $p(x)$ of class C^1 on D satisfying*

$$p(x) \geqq c > 0 \quad . \tag{2.9}$$

and (2.2). *Let* $x=x_0(t)$ *be a solution of* (1.1) *defined on the maximal interval* $0 \leqq t < \omega (\leqq \infty)$ *of* $t \geqq 0$ *with the property that there exists a number* $d>0$ *such that* $\operatorname{dist}(x_0(t), \partial D) > d > 0$ *for* $0 \leqq t < \omega$. *Then there are positive constants* δ, K *such that, for any solution* $x=x(t)$ *of* (1.1) *with* $|x(0)-x_0(0)| < \delta$, *there exists an increasing function* $s=s(t)$, $0 \leqq t < \omega$, *such that* $s(0)=0$, $0 \leqq t < s(\omega) \leqq \infty$ *is the maximal interval on* $t \geqq 0$ *on which* $x=x(t)$ *exists and, finally,* $|x(s(t))-x_0(t)| \leqq K|x(0)-x_0(0)|$ *for* $0 \leqq t < \omega$.

The conclusion of this assertion is very crude in the sense that it does not supply any estimates for $|s(t)-t|$. On the other hand, the assumptions are very light in the sense that, for example, there is no assumption concerning the uniform continuity of $J(x)$ to permit the replacement of (1.1) by a system of differential equations which is a "small" perturbation of a nonautonomous linear system.

The remarks above dealing with the relationship of Theorem 2.1 to Theorem 2.2 imply

COROLLARY. *Theorem* 2.4 *remains valid if the assumptions concerning* $p(x)$ *are replaced by either* (1.8) *and* $|f(x)| \geqq c > 0$ *or by* (1.11).

Theorem 2.4 has a generalization analogous to that of Theorem 2.2 (cf. Theorem 2.3).

THEOREM 2.5. *Let* x *denote a point on a Riemannian manifold* M^n *having a positive definite metric* $ds^2 = g_{jk}(x)dx^j dx^k$ *of class* C^1. *Let* $f(x) \neq 0$ *be a contravariant vector field on* M^n *of class* C^1 *such that the tensor* $e_{ij} = g_{ik}f^k{}_{,j}$ *satisfies* (2.7). *Let* $x=x_0(t)$ *be a solution of* (1.1) *on the maximal interval* $0 \leqq t < \omega$ $(\leqq \infty)$ *of* $t \geqq 0$ *with the property that there exists a number* $d>0$ *such that* $\operatorname{dist}(x_0(t), \partial M^n) > d > 0$. *Then there are positive constants* δ, K *such that, for any solution* $x=x(t)$ *with* $\operatorname{dist}(x_0(0), x(0)) < \delta$, *there exists an increasing function* $s=s(t)$, $0 \leqq t < \omega$, *such that* $s(0)=0$, $0 \leqq t < s(\omega) \leqq \infty$ *is the maximal interval of* $t \geqq 0$ *on which* $x=x(t)$ *exists and, finally,* $\operatorname{dist}(x(s(t)), x_0(t)) \leqq K \operatorname{dist}(x(0), x_0(0))$ *for* $0 \leqq t < \omega$.

If x, y are points of M^n, $\operatorname{dist}(x, y)$ is defined as usual as the infimum of the Riemannian lengths of the rectifiable arcs on M^n joining x, y; $\operatorname{dist}(x_0(t), \partial M^n) > d$ is understood to mean, for example, that all geodesics issuing from the point $x_0(t)$ can be extended to a length greater than d.

This theorem has the following corollary (which reduces to Theorem 2.4 for the choice $(g_{jk}(x)) = p(x)I$, where I is the unit matrix).

COROLLARY. *Let* D *be a domain in* E^n *on which there is defined a positive definite, symmetric,* n *by* n *matrix* $(g_{jk}(x))$ *of class* C^1 *satisfying* $(g_{jk}(x)) \geqq c^2 I$ *for some constant* $c>0$, *that is,*

$$g_{jk}(x)w^j w^k \geqq c^2 |w|^2 \textit{ for all } w = (w^1, \cdots, w^n). \tag{2.10}$$

Consider D to be a Riemannian manifold M^n with the metric $ds^2 = g_{jk}(x)dx^j dx^k$. Let $f(x) \neq 0$ be a (contravariant) vector field on $D = M^n$ of class C^1 such that $e_{ij}(x)$ defined by (2.8) *satisfies* (2.7). *Then Theorem* 2.5 *remains valid if* "dist" *is interpreted to be "Euclidean distance" in both assumption and conclusion.*

Theorem 2.5 and its Corollary will be proved in the next section. It will be shown in §4 that Theorem 2.3 is a consequence of Theorem 2.5.

3. **Proof of Theorem 2.5.** Let π be a piece of hypersurface of class C^1 through $x_0(0)$ orthogonal to $f(x_0(0))$ at $x_0(0)$. It can be supposed that π has a parametrization of the form $x = z(r, u)$, where u is a unit vector at $x_0(0)$ orthogonal to $f(x_0(0))$ and $0 \leqq r \leqq r_1$; r is arc-length along the arc $x = z(r, u)$, for fixed u, which starts at $r = 0$ in the direction u at $x_0(0)$; furthermore, π with $x_0(0)$ deleted is covered in a one-to-one manner by this family of arcs. [The existence of π and the parametrization $x = z(r, u)$ is clear if one considers local coordinates for which $(g_{jk}(x))$ reduces to the identity matrix at $x = x_0(0)$.] It is also clear that all solutions of (1.1) with initial point near $x_0(0)$ cross π.

For a fixed u and r, let $x = x(t, r)$ be the solution of (1.1) determined by the initial condition $x(0, r) = z(r, u)$. Let $0 \leqq t < \omega(r) \leqq \infty$ be the largest interval on $t \geqq 0$ on which $x = x(t, r)$ exists. Thus $x(t, 0) = x_0(t)$ and $\omega(0) = \omega$.

For fixed u, consider the 2-dimensional surface $S: x = x(t, r)$ defined on some (t, r)-set containing $0 \leqq t < \omega(r)$, $0 \leqq r \leqq r_1$. On S, consider the differential equation for the orthogonal trajectories to the parameter arcs $r = \text{const.}$ (i.e., to the solution paths of (1.1) on S) determined by the relation $g_{jk}(x)f^j(x)(dx^k/dr) = 0$, where $x = x(t, r)$ and $t = t(r)$. Let $t = T(r, s)$ be the solution of this differential equation,

$$dt/dr = -g_{jk}(x)f^j(x)x^k{}_r / g_{jk}(x)f^j(x)f^k(x), \qquad x = x(t, r), \tag{3.1}$$

with initial condition

$$T(0, s) = s \tag{3.2}$$

(so that the corresponding orthogonal trajectory starts at the point $x = x_0(s)$). In (3.1) and below, subscripts r, s denote partial differentiation.

Since the right side of (3.1) has a continuous partial derivative with respect to the dependent variable t, the solution $t = T(r, s)$ of (3.1), (3.2) has a continuous second mixed derivative $T_{sr} = T_{rs}$ on its domain of existence. Furthermore as a function of r, $T_s(r, s)$ satisfies a homogeneous linear differential equation, so that (3.2) implies that $T_s(0, s) = 1 > 0$ and

$$T_s(r, s) > 0. \tag{3.3}$$

The reparametrization of S given by

$$S: x = y(s, r) \equiv x(T(r, s), r) \tag{3.4}$$

will be used below.

Let D be the open set on M^n which is the union of the "spheres" $\operatorname{dist}(x, x_0(t)) < d/2$ for $0 \leqq t < \omega$. Thus $\operatorname{dist}(x, \partial M^n) \geqq d/2$ if $x \in D$. There is a constant $b > 0$, independent of u, such that $T(r, 0)$ exists for $0 \leqq r \leqq b$ for every u (so that the orthogonal trajectory starting at $x_0(0)$ reaches, for every fixed u, the solution path of (1.1) through $x(0, b)$) and that

$$\int_0^b [g_{jk}(y(0, r))y^j_r(0, r)y^k_r(0, r)]^{1/2} dr < d/2. \tag{3.5}$$

Since the integral in (3.5) is not less than $\operatorname{dist}(x_0(0), y(0, r))$ for $0 \leqq r \leqq b$, where $y(0, 0) = x_0(0)$, it follows that $x = y(0, r) \in D$ for $0 \leqq r \leqq b$.

The set of positive s-values for which $t = T(r, s)$ exists for $0 \leqq r \leqq b$ is open. Let s_0, $0 < s_0 \leqq \omega$, be the least upper bound for this set. Define

$$L(s, \sigma, \tau) = \int_\sigma^\tau [g_{jk}(y(s, r))y^j_r(s, r)y^k_r(s, r)]^{1/2} dr \tag{3.6}$$

for $0 \leqq \sigma < \tau \leqq b$ and $0 \leqq s < s_0$. It will be shown that $L(s, \sigma, \tau)$ is nonincreasing with respect to s (for fixed σ, τ). In fact, if $I(s, r)$ is the square of the integrand of (3.6), then $I_s \equiv \partial I/\partial s \leqq 0$. In order to see this, note that $y_s \equiv \partial y/\partial s = T_s f(y)$. Hence, for $y \equiv y(s, r)$, $y_{rs} = T_s J(y) y_r + T_{rs} f(y)$ and, since $g_{jk}(y) f^j(y) y^k_r = 0$ by (3.1),

$$I_s = 2T_s\left[g_{jk}(\partial f^k/\partial x^m) y^m_r y^j_r + \frac{1}{2} (\partial g_{jk}/\partial x^m) f^m y^j_r y^k_r\right].$$

Hence, by (2.8), $I_s = 2e_{jk}(y) y^j_r y^k_r T_s$ since the last term in (2.8) is skew-symmetric in i, j. Thus $I_s \leqq 0$ follows from (2.7) with $w = y_r$ and (3.3), and so $L_s(s, \sigma, \tau) \leqq 0$.

Thus $L(s, 0, b) \leqq L(0, 0, b)$ if $0 \leqq s < s_0$. Since the integral in (3.5) is $L(0, 0, b)$, it follows that $L(s, 0, b) < d/2$ and so, $x = y(s, r) \in D$ for $0 \leqq s < s_0$, $0 \leqq r \leqq b$.

It will now be shown that

$$T(s, r) \to \omega(r), \qquad s \to s_0, \tag{3.7}$$

for $0 \leqq r \leqq b$, where $0 \leqq t < \omega(r)$ is the maximal interval of existence of $x = x(t, r)$ on $t \geqq 0$. Suppose, if possible, that (3.7) fails to hold for some $r = r^0$, $0 \leqq r^0 \leqq b$. Since the arguments to follow do not depend on the position of $r = r^0$ in $[0, b]$, let $r^0 = b$. Thus (3.7) fails for $r = b$. In particular, $y_0 = \lim y(s, b)$ exists as $s \to s_0$ and y_0 is in the closure $\bar{D}$ of D. There exists an orthogonal trajectory $x = y(r)$ on S such that $y(b) = y_0$ and $y(r)$ is defined on some interval $(0 \leqq)\sigma < r \leqq b$. In particular, the solutions $x = x(t, r)$ of (1.1) for $\sigma < r \leqq b$ cross $x = y(r)$ with increasing t near $t = T(s_0, b)$.

From the continuous dependence of solutions on initial conditions, it follows that $y(r)$ [and hence $y_r(r)$] is the uniform limit of $y(s, r)$ [and $y_r(s, r)$, respectively] as $s \to s_0$ on every closed interval $(\sigma <)\tau \leqq r \leqq b$. Consequently $L(s, \tau, b)$ is continuous at $s = s_0$ if $L(s_0, \tau, b)$ is defined by

$$L(s_0, \tau, b) = \int_\tau^b [g_{jk}(y(r)) y^j_r(r) y^k_r(r)]^{1/2} dr$$

and $\sigma < \tau < b$. By the monotone property of L, $L(s_0, \tau, b) \leqq L(0, 0, b) < d/2$. Thus the arc $x = y(r)$, $\sigma < r \leqq b$, has a finite arc-length. Since $y(r) \in D$ and $\operatorname{dist}(x, \partial M^n) \geqq d/2 > 0$ for $x \in D$, it follows that $y(\sigma) = \lim y(r)$ as $r \to \sigma$ and $y(\sigma) \in \bar{D}$.

The limit relation $y(s, r) \to y(r)$, $s \to s_0$, holds uniformly on the closed interval $\sigma \leqq r \leqq b$ since $y(s, r)$ is equicontinuous with respect to r on $\sigma \leqq r \leqq b$ for $0 \leqq s < s_0$. In fact,

$$\operatorname{dist}(y(s, r_1), y(s, r_2)) \leqq L(s, r_1, r_2) \leqq L(0, r_1, r_2) \tag{3.8}$$

and $L(0, r_1, r_2) \to 0$ as $r_1 - r_2 \to 0$.

Now it is easy to see that $y = y(r)$ can be continued over the interval $0 \leqq r \leqq b$. For if $\sigma > 0$, the arguments above can be applied to $r = \sigma$, instead of $r = b$, to obtain an extension to an interval $\sigma_1 \leqq r \leqq b$, where $0 \leqq \sigma_1 < \sigma$. Furthermore, the set of $r = \sigma_1$ which can be so reached is both open and closed relative to $0 \leqq r < b$, so that $r = 0$ can be reached.

This implies that $y = y(s, r)$ can be defined for $0 \leqq s \leqq s_0$, $0 \leqq r \leqq b$, and hence for $0 \leqq s \leqq s_0 + \epsilon$, $0 \leqq r \leqq b$ for some $\epsilon > 0$. But this contradicts the definition of $s = s_0$. Thus the assumption that (3.7) fails to hold for some r is untenable. In particular, $s_0 = \omega$.

By (3.8) with $r_1 = 0$, $r_2 = r$, and the definition of $y(s, r)$ in (3.4), it follows that

$$\operatorname{dist}(x(T(s, r), r), x_0(s)) \leqq L(0, 0, r) \qquad \text{for } 0 \leqq s < \omega. \tag{3.9}$$

The continuity of $(g_{jk}(x))$ at $x = x_0(0)$ implies that if $b > 0$ is sufficiently small, then there is a constant K_1 such that $L(0, 0, r) \leqq K_1 r$ if $0 \leqq r \leqq b$. Furthermore, $b > 0$ and K_1 can be chosen independent of the unit vector u determining $S: x = x(t, r)$. Also, since r is arc-length on the arc $x = z(r, u)$ on π, $\operatorname{dist}(z(r, u), x_0(0)) \geqq K_2 r$ for $0 \leqq r \leqq b$ if $b > 0$ and $K_2 > 0$ are suitably chosen (independent of u). Thus if $K = K_1/K_2$, then, for $0 \leqq s < \omega$ and $0 \leqq r \leqq b$,

$$\operatorname{dist}(x(T(s, r), r), x_0(s)) \leqq K \operatorname{dist}(x(0, r), x_0(0)). \tag{3.10}$$

Hence if $x = x(t)$ is a solution with an initial point $x(0) = x(0, r)$ for some r, $0 \leqq r \leqq b$, and some u, the assertion of Theorem 2.5, except for $s(0) = 0$, follows with $s(t) = T(t, r)$. On the other hand, if $x(0)$ is sufficiently near to $x_0(0)$, then there exists a small $|t_1|$ such that $x = x(t)$ crosses π at $t = t_1$ near $x_0(0)$, i.e., $x(t_1) = x(0, r)$ for some small $r \geqq 0$ and u. Also, it is clear that $\operatorname{dist}(x(0, r), x_0(0))$ is majorized by a constant times $\operatorname{dist}(x(0), x_0(0))$. Thus, if K is suitably altered,

$$\text{dist}\, x(t_1 + T(s, r), x_0(s)) \leqq K \,\text{dist}(x(0), x_0(0))$$

for $0 \leqq s < \omega$. Thus, the assertion of Theorem 2.5 for $x = x(t)$, except for $s(0) = 0$, follows with $s(t) = t_1 + T(t, r)$. In either of the two cases just considered, the modification of $s(t)$ so as to satisfy $s(0) = 0$ is trivial. This proves Theorem 2.5.

On the Corollary of Theorem 2.5. This corollary is an immediate consequence of Theorem 2.5. This is clear if it is first noted that E-dist$(x_0(t), \partial D) > d$ implies D-dist$(x_0(t), \partial D) > cd$, where E-dist is Euclidean distance and D-dist is distance on $D = M^n$ relative to $ds^2 = g_{ik}dx^i dx^k$. Also, if $\delta > 0$, there are constants $\delta_1 > 0$, $K_1 > 0$ such that the Euclidean sphere $|x_0(0) - x| < \delta_1$ is contained in D-dist$(x_0(0), x) < \delta$ and D-dist$(x(0), x) \leqq K_1 |x(0) - x|$ if $|x(0) - x| < \delta_1$. Finally, $c|x(s(t)) - x_0(t)| \leqq D$-dist$(x(s(t)), x_0(t))$.

4. **Proof of Theorem 2.3.** Let "dist" refer to distance on M^n. Let $\epsilon > 0$ be so small that the closed sphere $\sum(\epsilon)$: dist$(0, x) \leqq \epsilon$ is in the domain of attraction of $x = 0$ and that (2.7) holds if $x \notin \sum(\epsilon)$. Since the domain of attraction of $x = 0$ is an open set, there exists a $d > 0$ such that $\sum(\epsilon + d)$ is also in the domain of attraction of $x = 0$. Let M denote the manifold obtained by deleting $\sum(\epsilon)$ from M^n. Then M, $f(x) \neq 0$ on M, and the metric $ds^2 = g_{jk}(x)dx^j dx^k$ on M satisfy the assumptions of Theorem 2.5.

Suppose, if possible, that Theorem 2.3 is false. Then there exists a point x_0 on the boundary of the domain of attraction of $x = 0$. Let $x = x_0(t)$ be the solution of (1.1) with initial condition $x_0(0) = x_0$. Suppose that $0 \leqq t < \omega (\leqq \infty)$ is the maximal interval of existence of $x_0(t)$ on $t \geqq 0$. It is clear that $x_0(t) \notin \sum(\epsilon + d)$ for $0 \leqq t < \omega$. Hence dist$(x_0(t), \partial M) > d$ for $0 \leqq t < \omega$.

Thus, Theorem 2.5 is applicable and all solutions $x = x(t)$ starting at initial points $x(0)$ sufficiently near to $x_0(0)$ remain close to the path $x = x_0(t)$ in the sense of Theorem 2.5. In particular, $x = x(t) \in M$ on its entire interval of existence on $t \geqq 0$. But this contradicts the assumption that $x_0(0) = x_0$ is on the boundary of the domain of attraction of $x = 0$.

5. **Bounded solutions.** Although the main subject of this section is the stability of a bounded solution $x = x_0(t)$ of (1.1), the first two theorems will be stated so as to be applicable to unbounded solutions in some situations.

The following notations will be used below:

$$(5.1) \qquad f_0(t) = f(x_0(t)), \qquad J(t) = J(x_0(t)), \qquad \gamma(t) = \gamma(x_0(t));$$

$$(5.2) \qquad \Gamma_c(t) = \int_0^t \gamma(r)dr + ct \quad \text{and} \quad \Gamma(t) = \Gamma_0(t);$$

$h(r)$ will denote a continuous, nondecreasing function for $r \geqq 0$ satisfying $h(0) = 0$ and

$$(5.3) \qquad |J(x) - J(y)| \leqq h(|x - y|),$$

where the expression on the left denotes the norm of the matrix $J(x) - J(y)$ as an operator from E^n to E^n.

In the theorems in this section, the following hypothesis will be used:

(H) Let $f(x)$ be of class C^1 on a (not necessarily bounded) domain D in E^n and have a Jacobian matrix $J(x)$ which is bounded and uniformly continuous on D. Let $x=x_0(t)$ be a solution of (1.1) for $t\geqq 0$ with the properties that there exist a number $d>0$ satisfying

$$\text{dist}(x_0(t), \partial D) \geqq d > 0 \qquad \text{for } t \geqq 0 \tag{5.4}$$

and a number $m>0$ such that

$$|f_0(t)| \geqq m > 0 \qquad \text{for } t \geqq 0. \tag{5.5}$$

When D is bounded, it can be replaced by a somewhat smaller domain so that the assumption that $J(x)$ is bounded and uniformly continuous becomes redundant. Similarly, in this case, (5.5) can be replaced by $f(x)\neq 0$.

THEOREM 5.1. *Assume* (H) *and the existence of constants* $c>0$, C *such that*

$$\Gamma_c(t) \leqq C \qquad \textit{for } t \geqq 0. \tag{5.6}$$

Then $x=x_0(t)$ *exhibits the following type of asymptotic stability: for* $0<b<c$, *there exist a* $\delta=\delta(b)>0$ *and a* $K=K(b)$ *with the properties that if* $x=x(t)$ *is a solution of* (1.1) *with* $|x(0)-x_0(0)|<\delta$, *then* $x=x(t)$ *exists for* $t\geqq 0$ *and there exist an increasing function* $s(t)$ *for* $t\geqq 0$ *and a number* t_0 *satisfying*

$$|x(s(t)) - x_0(t)| \leqq K|x(0) - x_0(0)| \exp \Gamma_b(t) \qquad \textit{for } t \geqq 0, \tag{5.7$_b$}$$

$$|s(t) - (t + t_0)| \leqq K|x(0) - x_0(0)| \int_t^\infty \exp \Gamma_b(r)dr \qquad \textit{for } t \geqq 0, \tag{5.8$_b$}$$

$$|t_0| \leqq K|x(0) - x_0(0)|. \tag{5.9}$$

If, in addition, $f(x)$ *is bounded on* D, *then, for* $t\geqq 0$,

$$|x(t + t_0) - x_0(t)| \leqq K|x(0) - x_0(0)| \int_t^\infty \exp \Gamma_b(r)dr. \tag{5.10$_b$}$$

In analogy to (5.1), put

$$\alpha(t) = \alpha(x_0(t)); \tag{5.11}$$

cf. (1.7). It is clear from (2.6) and (5.5) that condition (5.6) in this theorem can be replaced by

$$\mathrm{A}_c(t) \equiv \int_0^t \alpha(r)dr + ct \leqq C \text{ for some } c > 0 \text{ and } t \geqq 0. \tag{5.12}$$

The condition (5.6) on $\gamma(t)$ can be lightened if some assumptions are placed on a degree of continuity $h(r)$ of $J(x)$.

THEOREM 5.2. *Assume* (H) *and that*

$$\Gamma(t) \to -\infty \quad \textit{as} \quad t \to \infty, \tag{5.13}$$

$$\int_0^\infty h(c \exp \Gamma(t))dt < \infty \qquad \textit{for some } c > 0, \tag{5.14}$$

where $h(r)$ *is a nondecreasing function satisfying* $h(0)=0$ *and* (5.3). *Then* $x=x_0(t)$ *exhibits the following type of asymptotic stability: there exist* $\delta>0$, $K>0$ *such that if* $x=x(t)$ *is a solution of* (1.1) *with* $|x(0)-x_0(0)|<\delta$, *then* $x(t)$ *exists for* $t\geqq 0$ *and there exist an increasing* $s=s(t)$, $t\geqq 0$, *and a number* t_0 *satisfying* (5.7_0), (5.8_0), (5.9). *In addition, if* $f(x)$ *is bounded on* D, *then* (5.10_0) *holds.*

(Incidentally, (5.13)–(5.14) imply $\int^\infty \exp \Gamma(t)dt<\infty$; cf. (6.2).)

If, for example, $J(x)$ is uniformly Hölder continuous of order τ, $0<\tau\leqq 1$, then $h(r)$ can be chosen to be Const. r^τ so that (5.14) reduces to the condition $\int^\infty \exp \tau\Gamma(t)dt<\infty$.

Corollary. *In Theorem* 5.1 [*or Theorem* 5.2] *replace condition* (5.5) *by the conditions* $f_0(t)\neq 0$,

$$\int_t^\infty |f_0(r)|^{-1} \exp \Gamma_b(r)dr < \infty \quad \textit{and} \quad |f_0(t)|^{-1} \exp \Gamma_b(t) \leqq C \tag{5.15_b}$$

for some b, $0<b<c$ [*or* $b=0$]. *Then assertions* (5.7_b) [*or* (5.7_0)] *and* (5.9) *remain valid, but* (5.8_b) [*or* (5.8_0)] *must be altered by replacing the integral there by the integral in* (5.15_b) [*or* (5.15_0)].

When $x=x_0(t)$ in Theorems 5.1 or 5.2 is bounded and the main assumptions are made "uniform" so as to imply a "uniform" type of asymptotic stability, there results an analogue of Borg's theorem [1] asserting the existence of a periodic solution having asymptotic orbital stability.

Theorem 5.3. *In Theorem* 5.1, *let D be bounded and* (5.6) *replaced by*

$$\Gamma_c(t) - \Gamma_c(s) \leqq C \quad \textit{for } 0 \leqq s < t < \infty \textit{ and some } c > 0. \tag{5.16}$$

Then (1.1) *possesses a periodic solution* $x=x^*(t)$ *which is a limit cycle of* $x=x_0(t)$ *and has* $n-1$ *characteristic exponents with negative real parts; in particular,* $x=x^*(t)$ *has asymptotic orbital stability and the solution paths near* $x=x^*(t)$ *possess asymptotic phases.*

This theorem is due to Borg [1] under the stronger assumption $\gamma(x)<0$ on D. Since, without loss of generality, it can be assumed that $\log |f(x)|$ is bounded on D, (2.6) shows that Theorem 5.3 remains correct if the function Γ_c is replaced by the function Λ_c of (5.12).

Theorem 5.4. *In Theorem* 5.2, *let D be bounded and, in addition, to* (5.13)–(5.14), *assume that*

$$\Gamma(t) - \Gamma(s) \leqq C \qquad \textit{for } 0 \leqq s < t < \infty, \tag{5.17}$$

$$\int_t^\infty h(c \exp(\Gamma(r) - \Gamma(t)))dr \leqq C \qquad \textit{for } t \geqq 0. \tag{5.18}$$

Then the conclusion of Theorem 5.3 *are valid.*

The proofs of Theorem 5.1–5.4 will be modifications of that of Borg in which his use of the (local) implicit function theorem for $s=s(t)$ is replaced by a differential equation for $s(t)$ applicable for all t.

6. **Proof of Theorem 5.1.** For aid in proving Theorem 5.3, the proof of Theorem 5.1 will be carried out with particular attention to the quantities on which δ and K depend. Let K_0 satisfy

$$|J(x)| \leqq K_0 \qquad \text{for } x \text{ on } D. \tag{6.1}$$

The case that $J(x)$ is constant is simple and will not be considered. Hence, by (5.3), there is a constant $c_0>0$ such that, for small $r \geqq 0$,

$$h(r) \geqq c_0 r. \tag{6.2}$$

Let δ_1, K_1 denote constants (not always the same) depending only on K_0, c_0, the degree of continuity of $J(x)$, and the numbers d, m in (5.4), (5.5).

Consider a solution $x=x(t)$ of (1.1) on some t-interval and the differential equation for a function $s=s(t)$, if it exists, such that

$$w(t) = x(s(t)) - x_0(t) \tag{6.3}$$

satisfies

$$w(t)\cdot f_0(t) = 0. \tag{6.4}$$

Inserting (6.4) into (6.3) and differentiating with respect to t gives

$$s' = (|f_0|^2 - w\cdot f_0')/f(w + x_0(t))\cdot f_0 \equiv S(t, w) \tag{6.5}$$

when the denominator is different from 0. Correspondingly, a differentiation of (6.3) gives

$$w' = f(w + x_0(t))S(t, w) - f_0. \tag{6.6}$$

Equations (6.5), (6.6) can be considered a system of differential equations for the $(n+1)$-vector (s, w). This system splits in the sense that s does not occur in (6.6) and $s=s(t)$ is obtained by a quadrature if $w(t)$ is known.

Conversely, let $s=s(t)$, $w=w(t)$ be a solution of (6.5)–(6.6) and suppose that $s'(t)>0$, then

$$x(s) = w(t(s)) + x_0(t(s)), \tag{6.7}$$

where $t=t(s)$ is the inverse of $s=s(t)$, satisfies $dx/ds=f(x(s))$, i.e., is a solution of (1.1). For $dx/ds=(w'+x_0')/S$ is $f(x(s))$ by (6.6). Furthermore, if (6.4) holds at some t-value, then it holds for all points t for which $w(t)$ is defined.

It will be shown that if $|w_0|$ is sufficiently small and

$$w_0 \cdot f_0(0) = 0, \tag{6.8}$$

then the solution of (6.5), (6.6) satisfying

$$s(0) = 0, \qquad w(0) = w_0 \tag{6.9}$$

exists and $s'(t) > 0$ for $t \geqq 0$.

To this end, let

$$\Delta \equiv \Delta(t, w) = f(w + x_0(t)) - f_0(t). \tag{6.10}$$

Then

$$\Delta - J(t)w = \int_0^1 [J(x_0(t) + \tau w) - J(t)]w d\tau,$$

so that, by (5.3) and the monotony of $h(r)$,

$$|\Delta - J(t)w| \leqq h(|w|)|w| \tag{6.11}$$

and, by (6.1),

$$|\Delta| \leqq K_0 |w|. \tag{6.12}$$

By (6.10),

$$f(w + x_0(t)) \cdot f_0 = |f_0|^2 + \Delta \cdot f_0 = |f_0|^2(1 + \Delta \cdot f_0/|f_0|^2);$$

so that

$$1/f(w + x_0(t)) \cdot f_0 = |f_0|^{-2}(1 + \Delta \cdot f_0/|f_0|^2)^{-1} \leqq |f_0|^{-2}(1 + K_1|w|), \tag{6.13}$$

by virtue of (5.5) and (6.12). Since $x = x_0(t)$ is a solution of (1.1), $f_0' = Jf_0$ and so, $|w \cdot f_0'|/|f_0|^2 \leqq K_0|w|/|f_0| \leqq K_1|w|$. Hence (6.5) gives

$$|s' - 1| \equiv |S - 1| \leqq K_1|w| < 1 \quad \text{for } |w| \leqq \delta_1. \tag{6.14}$$

From (6.6), $w' = S\Delta + (S-1)f_0$ and so, by (6.4), $w \cdot w' = S\Delta \cdot w$. Hence (6.2), (6.11), (6.12) and (6.14) show that

$$w \cdot w' \leqq J(t)w \cdot w + K_1 h(|w|)|w|^2 \quad \text{for } |w| \leqq \delta_1. \tag{6.15}$$

Let $\delta_1(b) > 0$ be chosen so that

$$K_1 h(|w|) \leqq b \quad \text{for } |w| \leqq \delta_1(b) \leqq \delta_1. \tag{6.16}$$

Then $|w|' \leqq (\gamma(t) + b)|w|$ for $|w| \leqq \delta_1(b)$ and so, from (5.2),

$$|w(t)| \leqq |w(0)| \exp \Gamma_b(t). \tag{6.17}$$

Thus, if

$$d_0 = \sup \exp \Gamma_b(t) \quad \text{for } t \geqq 0, \tag{6.18}$$

then the inequality (6.17) makes it clear that the solution of (6.6) with initial condition $w(0)=w_0$, subject to $|w_0|\leqq\delta_1(b)/d_0$ and (6.8), exists and satisfies (6.17) for $t\geqq 0$. Also, in this case, $s(t)$ exists for $t\geqq 0$ and $s'>0$ by (6.14). From (6.14) and (6.18), $t_0=\lim\,(s(t)-t)$, $t\to\infty$, exists and

$$(6.19)\qquad |s(t)-(t+t_0)| \leqq K_1|w(0)|\int_t^\infty \exp\Gamma_b(r)dr;$$

furthermore $s(0)=0$ shows that

$$(6.20)\qquad |t_0| \leqq K_1|w(0)|\int_0^\infty \exp\Gamma_b(t)dt.$$

Consequently, it follows that if $\delta=\delta(b)$ is chosen to be $\delta_1(b)/d_0$, where $\delta_1(b)$ is subject only to (6.16), and $K=K(b)$ is chosen to be

$$(6.21)\qquad K = \max\left(1, K_1, K_1\int_0^\infty \exp\Gamma_b(t)dt\right),$$

then the assertions (5.7)–(5.9) of Theorem 5.1 hold for solutions $x(t)$ with initial conditions $x(0)$ satisfying $|x(0)-x_0(0)|\leqq\delta$, provided that $x(0)$ lies on the hyperplane $(x-x_0(0))\cdot f_0(0)=0$.

It is clear that this last proviso can be removed by decreasing δ in a manner which depends only on $1/|f_0(0)|$ and on the degree of continuity of $f(x)$ at $x=x_0(0)$, i.e., only on m and K_0. For if $|x(0)-x_0(0)|\leqq\delta_1$, there is a t-value $t=t_1$ such that the solution $x(t)$ with initial point $x(0)$ satisfies $(x(t_1)-x_0(0))\cdot f_0(0)=0$ and $|t_1|\leqq K_1|x(0)-x_0(0)|$. Thus it suffices to apply the above result to the solution $x=x(t+t_1)$ of (1.1), replacing $s(t)$ by $s(t)+t_1$ and t_0 by t_0+t_1.

As to (5.10), suppose that $|f(x)|\leqq M$ on D, then (5.7), (5.8) and the mean value theorem of differential calculus give

$$(6.22)\qquad |x(s(t))-x(t+t_0)| \leqq KM|x(0)-x_0(0)|\int_t^\infty \exp\Gamma_b(r)dr.$$

This together with (5.7) implies (6.10), for a suitable K, if it is noted that

$$(6.23)\qquad \exp\Gamma_b(t)\Big/\int_t^\infty \exp\Gamma_b(r)dr \leqq K_1.$$

For the derivative of the numerator is $(\gamma(t)+b)\exp\Gamma_b(t)$ and of the denominator $-\exp\Gamma_b(t)$. Thus (6.23) follows from $|\gamma(t)+b|\leqq K_1$.

7. **A lemma.** The following standard type of lemma (cf. [3]) will be useful in the proof of Theorem 5.2.

LEMMA 7.1. *In the differential equation*

$$(7.1)\qquad w' = J(t)w + g(t, w),$$

let $J(t)$ be an n by n matrix, continuous for $t \geqq 0$ and $g(t, w)$ a continuous vector function for $t \geqq 0$, $|w| \leqq c$. Let

$$\gamma(t) = \max_{|w|=1} J(t)w \cdot w \tag{7.2}$$

satisfy

$$\Gamma(t) \equiv \int_0^t \gamma(r)dr \to -\infty \quad as \quad t \to \infty. \tag{7.3}$$

Let $h(t, r)$ be a continuous scalar function for $t \geqq 0$, $r \geqq 0$ which is nondecreasing in r (for fixed t) and satisfies $h(t, 0) = 0$,

$$|g(t, w)| \leqq h(t, |w|)|w| \tag{7.4}$$

and

$$d_1 \equiv \int_0^\infty h(t, c \exp \Gamma(t))dt < \infty. \tag{7.5}$$

Let $\delta = c \min (1, 1/\sup \exp (d_1 + \Gamma(t)))$. Then any solution $w(t)$ such that $|w(0)| < \delta$ exists for $t \geqq 0$ and

$$|w(t)| \leqq |w(0)| \exp (d_1 + \Gamma(t)) \qquad for\ t \geqq 0. \tag{7.6}$$

Proof. If $w = w(t)$ is a solution of (7.1) on some t-interval $[0, T]$, then

$$|w|' \leqq \gamma(t)|w| + h(t, |w|)|w|. \tag{7.7}$$

Let

$$v = |w(t)|/\exp \Gamma(t), \tag{7.8}$$

so that

$$v' \leqq h(t, v \exp \Gamma(t))v. \tag{7.9}$$

If $v = v(t)$ is a solution of this differential inequality on some interval $[0, T]$ and $0 \leqq v(0) \exp d_1 < c$, then by the monotony of h,

$$v(t) \leqq v(0) \exp \int_0^t h(r, c \exp \Gamma(r))dr \leqq v(0) \exp d_1$$

on any interval $[0, T_0]$ on which $v(t) \leqq c$. But then $v(t) \leqq v(0) \exp d_1 < c$ on its entire interval of existence. Consequently, (7.8) implies (7.6) if $|w(0)| \exp d_1 < c$. Since $w(t)$ can be continued for increasing t as long as $|w(t)| < c$, the lemma follows.

8. **Proof of Theorem 5.2.** The arguments at the beginning of the proof of Theorem 5.1 lead to (6.15) or, equivalently, to

$$|w|' \leqq \gamma(t)|w| + K_1 h(|w|)|w| \qquad \text{if } |w| \leqq \delta_1 \tag{8.1}$$

and $w(0) = w_0$ satisfies (6.8). The proof of Lemma 7.1 shows that the solution $w = w(t)$ of (6.6) then exists for $t \geqq 0$ and satisfies

$$|w(t)| \leqq |w(0)| \exp(d_1 + \Gamma(t)) \qquad \text{for } t \geqq 0 \tag{8.2}$$

if

$$|w(0)| < \delta \equiv \delta_1 \min(1, 1/\sup \exp(d_1 + \Gamma(t))), \tag{8.3}$$

where

$$d_1 = K_1 \int_0^\infty h(\delta_1 \exp \Gamma(t)) dt \tag{8.4}$$

is finite if $\delta_1 \leqq c$. Also, $s' > 0$ by (6.14).

Note that (6.2) and (5.14) imply that

$$\int_0^\infty \exp \Gamma(t) dt < \infty, \tag{8.5}$$

hence $\int^\infty |w(t)| dt < \infty$. Thus the proof of Theorem 5.2 can be completed as was the proof of Theorem 5.1.

(For the proof of Theorem 5.4, note that δ and K in Theorem 5.2 depend only on K_0, $h(r)$, m, d and the quantities d_0, d_1 in (6.18), (8.4).)

9. **Proof of the Corollary.** If (5.5) is replaced by $f_0(t) \neq 0$, the derivation of (6.14), (6.15) leads instead to

$$|s' - 1| \equiv |S - 1| \leqq K_1 |w| / |f_0| \quad \text{if } |w| \leqq \delta_1 \min(1, |f_0|), \tag{9.1}$$

$$|w|' \leqq \gamma(t) |w| + h(|w|) |w| + K_1 |w|^2 / |f_0| \qquad \text{if } |w| \leqq \delta_1 \min(1, |f_0|). \tag{9.2}$$

In order to obtain an analogue of Theorem 5.1, the last inequality can be written

$$|w|' \leqq (\gamma + b) |w| + K_1 |w|^2 / |f_0| \quad \text{if } |w| \leqq \delta_1(b) \min(1, |f_0|). \tag{9.3}$$

It follows from the proof of Lemma 7.1, with $h(t, r) = K_1 r / |f_0(t)|$, that

$$|w(t)| \leqq |w(0)| \exp(d_2 + \Gamma_b(t)) \tag{9.4}$$

if $|w(0)|$ is sufficiently small and

$$d_2 = K_1 \int_0^\infty |f_0(t)|^{-1} \exp \Gamma_b(t) dt < \infty. \tag{9.5}$$

In view of (5.15_b), it is clear that one obtains the desired assertions as in the proof of Theorem 5.1.

In order to obtain the analogue of Theorem 5.2, note that the proof of Lemma 7.1, with $h(t, r) = h(r) + K_1 r / |f_0(t)|$, show that if $|w(0)|$ is sufficiently small, then (9.2) implies (9.4), with $b=0$, where d_2 must be replaced by the sum of the integrals in (8.4) and (9.5). The proof is completed as before.

10. **Proof of Theorem 5.3.** It can be supposed that D is replaced by a slightly smaller domain, so that $f(x)\neq 0$ is of class C^1 on the closure of D. Thus $J(x)$ is bounded and uniformly continuous on D and $|f(x)|$, $1/|f(x)|$ are bounded.

Let $\Sigma_\delta(T)$ denote the intersection of the sphere $|x-x_0(T)|<\delta$ and the hyperplane $(x-x_0(T))\cdot f_0(T)=0$. The proof of Theorem 5.1 shows that there exist a $\delta>0$ and a K [depending only on bounds for $|f(x)|$, $1/|f_0(t)|$, $|J(x)|$, the degree of continuity of $J(x)$, the number d in (5.4), a number b in the range $0<b<c$, a bound for $\Gamma_b(t)-\Gamma_b(T)$ for $t\geqq T$ and the number $\int_T^\infty \exp(\Gamma_b(r)-\Gamma_b(T))dr$] with the property that if $x=x(t)$ is a solution of (1.1) with $x(T)\in\Sigma_\delta(T)$, then there exists an increasing function $s=s(t)$ and a number $t=t_0$ such that $s(T)=T$,

$$(10.1)\quad (x(s(t))-x_0(t))\cdot f_0(t)=0,$$

$$(10.2)\quad |x(s(t))-x_0(t)|\leqq K|x(T)-x_0(T)|\exp(\Gamma_b(t)-\Gamma_b(T))\ \text{for}\ t\geqq T,$$

$$(10.3)\quad |s(t)-(t+t_0)|\leqq K|x(T)-x_0(T)|\int_t^\infty \exp(\Gamma_b(r)-\Gamma_b(T))dr \quad \text{for}\ t\geqq T,$$

$$(10.4)\quad |t_0|\leqq K|x(T)-x_0(T)|.$$

Note that $\Gamma_b(r)-\Gamma_b(T)=\Gamma_c(r)-\Gamma_c(T)-(c-b)(t-T)$, so that (5.16) implies

$$\int_T^\infty \exp(\Gamma_b(r)-\Gamma_b(T))dr\leqq \exp C/(c-b).$$

This makes it clear that δ and K can be chosen independent of T and that (10.2), (10.3) can be replaced by

$$(10.5)\quad |x(s(t))-x_0(t)|\leqq K|x(T)-x_0(T)|e^{-(c-b)(t-T)}\quad \text{for}\ t\geqq T,$$

$$(10.6)\quad |s(t)-(t+t_0)|\leqq K|x(T)-x_0(T)|e^{-(c-b)(t-T)}\quad \text{for}\ t\geqq T.$$

Let $x=x^*$ be a limit point of $x_0(t)$ as $t\to\infty$. Let Σ_δ^* be the intersection of the sphere $|x^*-x|<\delta$ and the hyperplane $(x-x^*)\cdot f(x^*)=0$. Choose $t=T$ with the property that $|x^*-x_0(T)|$ is so small that all solutions of (1.1) starting at a point of $\Sigma_{\delta/2}^*$ for $t=0$ cross $\Sigma_\delta(T)$ for some small $|t|$. Correspondingly, solutions starting, for $t=T$, at a point of $\Sigma_\delta(T)$ remain close to $x_0(t)$ for $t\geqq T$ in the sense of (10.5).

If τ is large and suitably chosen, then $|x_0(T+\tau)-x^*|$ and $\exp(b-c)\tau$ are arbitrarily small. This makes it clear that solutions starting on $\Sigma_{\delta/2}^*$ for $t=0$ cross $\Sigma_{\delta/2}^*$ again for a unique t-value near $t=\tau$. This defines a continuous map of $\Sigma_{\delta/2}^*$ into itself which has a fixed point by Brouwer's theorem. Since the image of $\Sigma^*_{\delta/2}$ under this map can be made an arbitrarily small set (arbitrarily near x^*) by choosing τ sufficiently large, it follows that $x=x^*$ is a fixed point.

Thus the solution $x=x^*(t)$ of (1.1) with $x^*(0)=x^*$ is periodic with a period which is near τ. It also follows that there exists a number t_0 such that $|x_0(t)-x^*(t-T+t_0)|\leqq K|x_0(T)-x^*|$ for $t\geqq T$. Thus, if $|x_0(T)-x^*|$ is sufficiently small, then $|\gamma(t)-\gamma(x)|$, where $x=x^*(t-T+t_0)$, is arbitrarily small (say, $<c-b$) for $t\geqq T$. It follows that if π is a period of $x=x^*(t)$, then

$$\int_0^\pi \gamma(x^*(t))dt < 0.$$

Thus Theorem 5.1 can be applied to $x=x^*(t)$, instead of $x=x_0(t)$. Correspondingly, the analogues of (10.1), (10.5) and (10.6) imply the statements concerning the asymptotic orbital stability of $x=x^*(t)$.

11. **Proof of Theorem 5.4.** The last proof shows that in order to prove the existence of a periodic solution $x=x^*(t)$, it is sufficient to verify that if $t=0$ is replaced by $t=T$ in Theorem 5.2, then δ and K in the assertion of that theorem can be chosen independent of T. From the remark at the end of §8, this will be the case when (5.17), (5.18) hold.

In order to examine the stability properties of $x=x^*(t)$, consider first the degree of continuity of $\gamma(x)$. Let $|w|=1$ and $w\cdot f(x)=0$. For y near x, write $w=w_0+w_1$, where $w_0=(w\cdot f(y))f(y)/|f(y)|^2$ is the component of w_0 in the direction of $f(y)$ and $w_1\cdot f(y)=0$. Since $w\cdot f(x)=0$ and $|f|$, $1/|f|$ are bounded, it is clear that $|w_0|\leqq K_1|f(x)-f(y)|$, so that $|w_0|\leqq K_1|x-y|$. Thus, $J(x)w\cdot w \leqq J(x)w_1\cdot w_1+K_1|x-y|$ which, in turn, is at most $J(y)w_1\cdot w_1+K_1|x-y| +h(|x-y|)$. It follows that $\gamma(x)\leqq\gamma(y)+K_1|x-y|+h(|x-y|)$. Since x and y can be interchanged in this argument, (6.2) shows that

$$|\gamma(x)-\gamma(y)| \leqq K_1 h(|x-y|). \tag{11.1}$$

Arguing as in the last section, it is seen that there exists T and t_0 such that

$$|x^*(t-T+t_0)-x_0(t)| \leqq K|x_0(T)-x^*|\int_t^\infty \exp\Gamma(r)dr \qquad \text{for } t\geqq T,$$

where $|x_0(T)-x^*|$ can be made arbitrarily small, say, $<\epsilon/K$. Thus by (11.1),

$$\gamma(x^*(t-T+t_0)) \leqq \gamma(t) + K_1 h\left(\epsilon\int_t^\infty \exp\Gamma(r)dr\right) \qquad \text{for } t\geqq T. \tag{11.2}$$

By (6.2) and (5.18),

$$c_0c\int_t^\infty \exp(\Gamma(r)-\Gamma(t))dt \leqq \int_t^\infty h(c\exp(\Gamma(r)-\Gamma(t)))dr < C \qquad \text{for } t\geqq 0;$$

in particular, $\epsilon\int_t^\infty \exp\Gamma(r)dr\leqq(C\epsilon/c_0c)\exp\Gamma(t)$. Thus, if $C\epsilon/c_0<c^2$, (11.2) implies

$$\gamma(x^*(t - T + t_0)) \leqq \gamma(t) + K_1 h(c \exp \Gamma(t)) \qquad \text{for } t \geqq T.$$

Consequently, (5.13) and (5.14) show that

$$\int_0^t \gamma(x^*(r))dr \to -\infty \quad \text{as} \quad t \to \infty.$$

Since $x^*(t)$ is periodic, the last integral is of the form $-c_1 t + O(1)$, as $t \to \infty$, for some $c_1 > 0$. Thus, again Theorem 5.1 can be applied to $x = x^*(t)$, instead of $x = x_0(t)$, and the desired stability properties follow.

12. **Nonautonomous systems.** In order to illustrate the methods to be used and the type of results to be obtained in this part of the paper, the simple case of a linear system of equations

$$x' = J(t)x \tag{12.1}$$

will be considered first.

THEOREM 12.1. *Let $J(t)$ be a continuous, bounded, n by n matrix for $t \geqq 0$. Let* (12.1) *possess a solution $x = x_0(t) \not\equiv 0$ such that*

$$\gamma(t) = \max J(t)w \cdot w \qquad \text{for } |w| = 1,\ w \cdot x_0(t) = 0, \tag{12.2}$$

satisfies

$$\int^\infty |x_0(t)|^{-1} \exp \Gamma(t)dt < \infty, \quad \text{where} \quad \Gamma(t) = \int_0^t \gamma(r)dr. \tag{12.3}$$

Then (12.1) *possesses an $n-1$ parameter family of solutions $x = x(t)$ satisfying*

$$|x(t)| \leqq K|x(0)|\,|x_0(t)| \int_t^\infty |x_0(r)|^{-1} \exp \Gamma(r)dr \qquad \text{for } t \geqq 0, \tag{12.4}$$

where K is a constant (independent of $x(t)$).

This theorem can be generalized in case that one has knowledge of a k-dimensional integral manifold of (12.1).

THEOREM 12.2. *Let $J(t)$ be a continuous, bounded, n by n matrix for $t \geqq 0$. Let* (12.1) *possess k, $1 \leqq k < n$, linearly independent solutions $x_1(t), \cdots, x_k(t)$ with the property that if $L(t)$ is the linear manifold spanned by $x_1(t), \cdots, x_k(t)$, then the integrals $\mathrm{B}(t)$, $\Gamma(t)$ over $[0, t]$ of the functions*

$$\beta(t) = \min J(t)v \cdot v \qquad \text{for } |v| = 1,\ v \in L(t), \tag{12.5}$$

$$\gamma(t) = \max J(t)w \cdot w \qquad \text{for } |w| = 1,\ w \perp L(t), \tag{12.6}$$

satisfy

$$\int^\infty \exp(\Gamma(t) - \mathrm{B}(t))dt < \infty. \tag{12.7}$$

Then (2.1) *possesses an* $n-k$ *parameter family of solutions* $x=x(t)$ *such that*

$$(12.8) \qquad |x(t)| \leqq K|x(0)| (\exp \mathrm{B}(t)) \int_t^\infty \exp(\Gamma(r) - \mathrm{B}(r))dr \qquad \textit{for } t \geqq 0,$$

where K *is a constant* (*independent of* $x(t)$).

The symbol $w \perp L(t)$ in (12.6) means that w is orthogonal to $L(t)$, i.e., $w \cdot x_1(t) = \cdots = w \cdot x_k(t) = 0$. For $k=1$, this theorem reduces to Theorem 12.1 since $\exp \mathrm{B}(t)$ is $|x_1(t)|$, up to a constant factor.

In order to state an analogue of Theorem 12.1 for nonlinear, nonautonomous systems

$$(12.9) \qquad x' = f(t, x)$$

introduce the following notations: $J(t, x)$ is the Jacobian matrix $(\partial f(t, x)/\partial x)$; $f_t(t, x) = \partial f(t, x)/\partial t$; for a given solution $x = x_0(t)$, let $f_0(t) = f(t, x_0(t))$, $J(t) = J(t, x_0(t))$, $f_{t0}(t) = f_t(t, x_0(t))$, and finally the symbol $\Gamma_c(t)$ as in (5.2), where

$$(12.10) \qquad \gamma(t) = \max J(t)w \cdot w \qquad \text{for } |w| = 1,\ w \cdot f_0(t) = 0.$$

Note that $f_0' = f_{t0} + J(t) f_0$.

Theorem 12.3. *Let* D *be a domain in* E^n *and* $I: 0 \leqq t < \infty$. *Let* $f(t, x)$ *be of class* C^1 *on* $I \times D$ *and have a bounded and uniformly continuous Jacobian matrix* $J(t, x) = (\partial f/\partial x)$ *on* $I \times D$. *Let* (12.9) *possess a solution* $x = x_0(t)$, $t \geqq 0$, *with the properties* (i) (5.4) *holds for some constant* d; (ii) *there exist constants* $m > 0$, M *such that* $0 < m \leqq |f_0(t)| \leqq M$ *for* $t \geqq 0$; (iii) (5.16) *holds for some constants* $c > 0$, C; *finally*, (iv) f_{t0} *satisfies*

$$(12.11) \qquad f_{t0}(t) \to 0 \quad \textit{as} \quad t \to \infty.$$

Then (12.9) *possesses an* $n-1$ *parameter family of solutions* $x = x(t)$ *satisfying*

$$(12.12) \qquad |x(t) - x_0(t)| \leqq K \exp \Gamma_b(t),$$

where $K = K(b)$ *is a constant* (*independent of* $x(t)$) *and* b *is any number on the range* $0 < b < c$.

The proof will show that if b is suitably restricted, then (12.11) can be relaxed to the assumption that $\limsup |f_{t0}|/|f_0|$ is sufficiently small, where the "smallness" depends on the constants in (5.16).

It is also possible to refine this theorem by introducing the type of conditions occurring in Theorem 5.2 and the Corollary following it. This type of refinement will be stated only for the case that (12.9) is replaced by an autonomous system (1.1). The notations of §5 will be used.

Theorem 12.4. *Let* $f(x)$ *be of class* C^1 *on a domain* D *and possess a bounded and uniformly continuous Jacobian matrix. Let* (1.1) *have a solution* $x = x_0(t)$, $t \geqq 0$, *satisfying* (5.4) *for some constant* d *and let the equation of variation* (12.1) *along* $x=x_0(t)$ *possess* $k(<n)$ *linearly independent solutions* $x_1=f_0(t)$, $x_2(t)$, $\cdots$, $x_k(t)$ *such that if* $L(t)$, $\beta(t)$, $\gamma(t)$, $\mathrm{B}(t)$, $\Gamma(t)$ *are defined as in Theorem* 12.2, *then* (5.13), (5.14) *and*

$$\int_t^\infty \exp\left((\Gamma(r)-\mathrm{B}(r))-(\Gamma(t)-\mathrm{B}(t))\right)dr \leqq C \qquad \text{for } t \geqq 0 \tag{12.13}$$

hold. Then (1.1) *has an* $n-k$ *parameter family of solutions* $x=x(t)$ *satisfying*

$$|x(t)-x_0(t)| \leqq K\exp\Gamma(t) \qquad \text{for } t\geqq 0. \tag{12.14}$$

The proofs of the theorems of this section will depend on the method of Ważewski; cf. [5] also for further references. For the convenience of the reader, the definitions and the main result in the form to be used below will be recalled. Let $f(t, x)$ in (12.9) be of class C^1 on $I\times D$ (as in Theorem 12.3) and let Ω be a (t, x)-domain in $I\times D$. A point $P=(t_0, x_0)$ of $\partial\Omega$ is called an egress point with respect to Ω and (12.9) if the solution $x=x(t)$ through P satisfies $(t, x(t))\in\Omega$ for $t_0-\epsilon<t<t_0$ and some $\epsilon>0$. If, in addition, $(t, x(t))\notin\Omega$ for $t_0\leqq t<t_0+\epsilon$, then P is called a strict egress point. Let S denote the set of egress points on $\partial\Omega$, S^* the set of strict egress points. A set A of a topological space B is called a retract of B if there exists a continuous map $u: B\rightarrow A$ defined on all of B such that $u(P)=P$ if $P\in A$. Ważewski's main theorem is as follows: If $S=S^*$ and Z is a subset of $\Omega\cup S$ such that $Z\cap S$ is a retract of S but is not a retract of Z, then there is at least one point $P=(t_0, x_0)$ in Z such that the solution $x=x(t)$ of (12.9) determined by $x(t_0)=x_0$ exists and satisfies $(t, x(t))\in\Omega$ for $t\geqq t_0$.

13. **Proof of Theorem 12.2.** Let $E(t)$ denote the matrix which gives the orthogonal projection of E^n onto $L(t)$. For a given x, let v and w be the components of x in $L(t)$ and orthogonal to $L(t)$,

$$v = E(t)x, \qquad w = x - v = x - E(t)x. \tag{13.1}$$

Then along a solution $x=x(t)$ of (12.1),

$$w\cdot w' = J(t)w\cdot w \leqq \gamma(t)\,|w|^2. \tag{13.2}$$

For $E(t)x$ is a linear combination of $x_1(t), \cdots, x_k(t)$, say $E(t)x=\sum c_j(t)x_j(t)$, where $c_j(t)$ are scalars. Differentiating this, it is seen that $(E(t)x)'-J(t)E(t)x \in L(t)$, so that $w'-J(t)w\in L(t)$ and (13.2) follows from $w\perp L(t)$. The relations (13.2) give

$$|w|' \leqq \gamma(t)\,|w|. \tag{13.3}$$

In order to obtain analogous inequalities for $|v|'$, note that $|v|^2=|x|^2-|w|^2$. Hence $|v|'=|v|^{-1}(J(t)x\cdot x-J(t)w\cdot w)$ or, since $x=v+w$, $|v|'=|v|^{-1}(J(t)v\cdot v +2H(t)v\cdot w)$, where $H=(J+J^*)/2$. Thus, if K_0 is a constant satisfying $|J(t)|\leqq K_0$, then, by (12.5),

$$|v'| \geqq \beta(t)|v| - 2K_0|w|. \tag{13.4}$$

In terms of two positive continuously differentiable functions $\phi(t)$, $\psi(t)$ to be specified in a moment, put

$$W = |w| - \phi(t), \qquad V = |v| - \psi(t), \tag{13.5}$$

where v, w are given by (13.1). Then

$$W' \leqq \gamma\phi - \phi' \qquad \text{if } W = 0, \tag{13.6}$$

where W' is the derivative of $W(x, t)$ along a solution of (12.1). Similarly

$$V' \geqq \beta\psi - 2K_0\phi - \psi' \qquad \text{if } V = 0,\ W \leqq 0. \tag{13.7}$$

Let $q(r) > 0$ be continuous, bounded and integrable for $r \geqq 0$ and let $Q(t)$ be the integral of $q(r)$ over $0 \leqq r \leqq t$. Put

$$\phi(t) = \exp(\Gamma(t) + Q(t)), \tag{13.8}$$

$$\psi(t) = 3K_0 \exp[\mathrm{B}(t)] \int_t^\infty \phi(r) \exp[-\mathrm{B}(r)]dr. \tag{13.9}$$

Then

$$\gamma\phi - \phi' = -q\phi < 0, \qquad \beta\psi - 2K_0\phi - \psi' = K_0\phi > 0. \tag{13.10}$$

In $I \times E^n$, consider the set of points $\Omega = \{(t, x): t > 0,\ W < 0,\ V < 0\}$. In view of (13.6), (13.7) and (13.10) the set S of egress points with respect to Ω and system (12.1) is $S = \{(t, x): t > 0,\ V = 0,\ W < 0\}$. Also the points of S are strict egress points, so that $S = S^*$. Let t_0 and w_0 satisfy the inequalities $t_0 > 0$, $|w_0| < \phi(t_0)$. Put $Z = \{(t, x): t = t_0,\ w = w_0,\ V \leqq 0\}$. Note that S is the cartesian product of the solid cylinder $C = \{(t;\ w): t > 0,\ W < 0\}$ and the surface of the sphere $B = \{v:\ V \leqq 0\}$, that Z is the product of the point $P = (t_0,\ w_0) \in C$ and the sphere B, and that $Z \cap S$ is the product of P and the surface of B. Thus $Z \cap S$ is not a retract of Z but is a retract of S. Therefore from Ważewski's topological principle, it follows that there exists a v_0, $|v_0| < \psi(t_0)$, such that the solution $x(t)$ of (12.1) with initial condition $x(t_0) = v_0 + w_0$ is contained in Ω for $t \geqq t_0$; that is, $(t, x(t)) \in \Omega$ for $t \geqq t_0$ or, by the definition of Ω, $|w(t)| < \phi(t)$, $|v(t)| < \psi(t)$, where $x(t) = v(t) + w(t)$.

Actually, $\phi(t)/\psi(t)$ is bounded for $t \geqq 0$. In order to see this, note that

$$3K_0\phi/\psi = \phi \exp(-\mathrm{B}) \Big/ \int_t^\infty \phi \exp(-\mathrm{B})dr,$$

where the derivative of the numerator is $\phi \exp(-\mathrm{B})$ times $\gamma + q - \beta$ and the derivative of the denominator is $-\phi \exp(-\mathrm{B})$. Since β, γ, q are bounded for $t \geqq 0$, the ratio ϕ/ψ is also bounded. Hence $(t, x(t)) \in \Omega$ implies $|x(t)| \leqq |v(t)| + |w(t)| \leqq \text{Const. } \psi(t)$, so that (12.8) holds for a suitable constant K. Finally, since w_0, in the definition of Z, can be chosen arbitrarily from an $n-k$ dimensional sphere $|w_0| < \phi(t_0)$ the existence of a family of solutions of (12.1) satisfying (12.8) and depending on exactly $n-k$ parameters follows from the linearity of (12.1). Also, since (12.1) is linear, K can be chosen the same for all solutions of this family.

14. **Proof of Theorem 12.3.** Introduce the new dependent variable

$$y = x - x_0(t), \tag{14.1}$$

so that (12.9) becomes

$$y' = J(t)y + g(t, y), \tag{14.2}$$

where the uniform continuity of $J(t, x)$ implies that

$$|g(t, y)|/|y| \to 0, \quad |y| \to 0, \quad \text{uniformly for } t \geqq 0. \tag{14.3}$$

For a given y and t, introduce the component of y in the direction $f_0(t)$ and orthogonal to it,

$$v = (y\cdot f_0)f_0/|f_0|^2, \qquad w = y - v = y - (y\cdot f_0)f_0/|f_0|^2. \tag{14.4}$$

By (14.2),

$$w' = J(t)w - (y\cdot f_0)f_{t0}/|f_0|^2 + g(t, y) + (\cdot\cdot\cdot)f_0.$$

Hence $w\cdot f_0 = 0$ implies

$$|w|' \leqq \gamma(t)|w| + k(t)|v| + |g(t, y)|, \tag{14.5}$$

where

$$k(t) = |f_{t0}(t)|/|f_0(t)|. \tag{14.6}$$

By an argument similar to that leading to (13.4) in the linear case,

$$|v|' = |v|^{-1}(Jv\cdot v + 2Hv\cdot w - (y\cdot f_0)(f_{t0}\cdot w)/|f_0|^2 + g\cdot v).$$

From the definition of v and from $f_0' = Jf_0 + f_{t0}$, it follows that

$$|v|' \geqq |v|(\log|f_0|)' - k(t)|v| - (2K_0 + k(t)|w|) - |g(t, y)|. \tag{14.7}$$

Introduce the functions $W(y, t)$, $V(y, t)$ of (13.5), where

$$\phi(t) = \exp \Gamma_b(t), \tag{14.8}$$

$$\psi(t) = C_1|f_0(t)|\int_t^\infty |f_0(r)|^{-1}\exp \Gamma_b(r)dr, \tag{14.9}$$

$0 < b < c$ and C_1 is a constant to be specified below. Note that $\psi(t)/\phi(t)$ is at most $C_1(M/m)$ times $\int_t^\infty \exp(\Gamma_b(r) - \Gamma_b(t))dr$ and this integral is bounded for $t \geqq 0$ by virtue of (5.16) and $b < c$. Thus, for some constant $C_2 = C_2(C, c, M, m)$,

$$\psi(t) \leqq C_1C_2\phi(t). \tag{14.10}$$

If $W \leqq 0$ and $V \leqq 0$, then $|y| \leqq |v| + |w| \leqq \phi + \psi \leqq (1 + C_1C_2)\phi$. Hence

$W' = |w|' - \phi'$, (14.3), (14.5), (14.10) and (12.11) imply that

$$(14.11) \qquad W' < (\gamma(t) + b)\phi - \phi' = 0 \qquad \text{if } W = 0,\ V \leqq 0$$

and t is sufficiently large. Similarly, from (14.7),

$$(14.12) \qquad V' > \psi(\log |f_0|)' - \psi' - C_3\phi \qquad \text{if } V = 0,\ W \leqq 0$$

for large t, where C_3 is a constant (which is arbitrarily near to $2K_0$). Thus, (14.9) shows that $V' > 0$ if $C_1 > C_3$.

This makes it clear that the proof of Theorem 12.3 can be completed in the same way that Theorem 12.2 was, using a set Ω: $\{(t, x) : t > T,\ W < 0,\ V < 0\}$, where T is a fixed large number.

15. **Proof of Theorem 12.4.** Rewrite (1.1) as (14.2) by introducing the dependent variable (14.1). Then (14.3) holds and, in fact,

$$(15.1) \qquad |g(t, y)| \leqq h(|y|)|y|.$$

Let $E(t)$ denote the matrix of the orthogonal projection on $L(t)$, the manifold spanned by $x_1(t), \cdots, x_k(t)$. For a given y and t, put

$$(15.2) \qquad v = E(t)y, \qquad w = y - v = y - E(t)y.$$

Then a modification of the derivation of (13.3), (13.4) give

$$(15.3) \qquad |w|' \leqq \gamma(t)|w| + h(|v| + |w|)(|v| + |w|),$$

$$(15.4) \qquad |v|' \geqq \beta(t)|v| - 2K_0|w| - h(|v| + |w|)(|v| + |w|).$$

Introduce the functions W, V of (13.5), where

$$(15.5) \qquad 0 < \psi(t) \leqq (c_1 - 1)\phi(t)$$

for a constant $c_1 > 1$ to be specified. Then

$$(15.6) \qquad W' \leqq \gamma(t)\phi + c_1 h(c_1\phi)\phi - \phi' \qquad \text{if } W = 0,\ V \leqq 0,$$

$$(15.7) \qquad V' \geqq \beta(t)\psi - (2K_0 + c_1 h(c_1\phi))\phi - \psi' \qquad \text{if } V = 0,\ W \leqq 0.$$

Choose ϕ, ψ to be

$$(15.8) \qquad \phi(t) = \epsilon \exp\left(\Gamma(t) + c_1 \int_t^\infty h(c \exp \Gamma(r))dr\right),$$

$$(15.9) \qquad \psi(t) = C_1(\exp \mathrm{B}(t)) \int_t^\infty \phi(r) \exp(-\mathrm{B}(r))dr,$$

where $\epsilon > 0$, C_1 are to be specified.

If $\epsilon > 0$ is sufficiently small, then $c_1\phi(t) < c \exp \Gamma(t)$. Hence $\phi' = (\gamma + c_1 h(c \exp \Gamma))\phi > (\gamma + c_1 h(c_1\phi))\phi$; so that $W' < 0$ if $W = 0$, $V \leqq 0$. Similarly, $V' > 0$ when $V = 0$, $W \leqq 0$, provided that $C_1 > 2K_0 + c_1 h(c_1\phi)$. This proviso is satisfied if $C_1 > 2K_0$ and t is sufficiently large (since $h(c_1\phi) \to 0$ as $t \to \infty$).

The existence of a constant c_1 satisfying (15.5) is clear from (12.13) and (15.8), (15.9).

The proof of Theorem 12.4 can be completed by using Ważewski's principle as above.

REFERENCES

1. G. Borg, *A condition for the existence of orbitally stable solutions of dynamical systems*, Kungl. Tekn. Högsk. Handl. **153** (1960), 12 pp.

2. P. Hartman, *On stability in the large for systems of ordinary differential equations*, Canad. J. Math. **13** (1961), 480–492.

3. P. Hartman and A. Wintner, *Asymptotic integrations of ordinary non-linear differential equations*, Amer. J. Math. **77** (1955), 692–724.

4. C. Olech *On the global stability of autonomous systems in the plane*, to appear.

5. T. Ważewski, *Sur une méthode topologique de l'examen de l'allure asymptotique des intégrales des équations differentielles*, Proc. Internat. Congress of Mathematicians (Amsterdam, 1954), Vol. 3, pp. 132–139.

THE JOHNS HOPKINS UNIVERSITY,
BALTIMORE, MARYLAND
RIAS,
BALTIMORE, MARYLAND
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES,
KRAKÓW, POLAND