ON UNITARY EQUIVALENCE OF ARBITRARY MATRICES(1)

BY

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1. **Introduction.** The problem we wish to study is that of deciding whether two given square matrices A and B over the field of complex numbers are unitarily equivalent, i.e., whether there exists a unitary matrix U such that $B = U^{-1}AU$. This decision can be made easily if a computable set of canonical forms for all matrices is obtained, that is, if there exists an algorithm which associates with any given matrix A another matrix $C(A)$ such that if A and B are two matrices and $C(A)$ and $C(B)$ their respective forms obtained by the algorithm, then $C(A)$ is equal to $C(B)$ if and only if A and B are unitarily equivalent. The solution of this problem for the set of normal matrices is well known; the canonical set consists of all *diagonal* matrices with complex entries arranged in some order agreed on. We shall make use of facts concerning the diagonalization of normal matrices.

The analog of the present problem, where *similarity* is considered instead of unitary equivalence is much simpler (Jordan canonical forms). We cannot expect as simple a canonical set in the case of unitary equivalence. The following example shows how much vaster the set of canonical forms in this case can be as compared to the set of Jordan canonical forms:

Let $n > 2$. Take all $n \times n$ matrices of the form

$$A = \begin{bmatrix} 1 & 1 & a_{13} & a_{14} & a_{15} \cdots a_{1n} \\ 0 & 2 & 1 & a_{24} & a_{25} \cdots a_{2n} \\ 0 & 0 & 3 & 1 & a_{35} \cdots a_{3n} \\ 0 & 0 & 0 & 4 & 1 \quad \cdots a_{4n} \\ & & & \vdots & \\ 0 & 0 & 0 & 0 & 0 \quad \cdots n \end{bmatrix}$$

where the a_{ij} are complex numbers. Under similarity *all* of these matrices are equivalent and their common Jordan form is

$$\mathrm{Diag}(1, 2, 3, \cdots, n);$$

but under unitary transformations two matrices of the above type are equiv-

Received by the editors July 17, 1961.

(1) *Acknowledgment.* This paper is a condensation of a Master's Thesis at the University of Minnesota. The author wishes to acknowledge his indebtedness to Professor G. K. Kalisch for his encouragement in the preparation of this dissertation and to the National Science Foundation (Grant G 14137) for financial support.

alent if and only if they are equal. This means that to just one Jordan canonical form there corresponds an uncountable number of canonical forms under unitary equivalence; in fact each A of the above-mentioned form is its own canonical form if we require that canonical forms be triangular with eigenvalues arranged in ascending order, and the $(i, i+1)$-elements be positive.

The present problem was considered by J. Brenner [1]. On the basis of Brenner's work, D. E. Littlewood made further remarks [2]. A special case was considered by B. E. Mitchell [3]. Another attack on this problem is contained in a dissertaion by Vincent V. McRae [5]. The method which we shall use in this paper will enable us to find canonical forms for matrices A not only under the full group of unitary transformations $\{U\}$, but also under certain subgroups of this group which we call "direct groups." It is the reduction of equivalence under the full unitary group to that under such direct subgroups which provides the fundamental idea involved in Brenner's work [1] and also in the present paper. This reduction is carried out in a stepwise manner to successively "finer" direct groups. Our work differs from Brenner's [1] in that he sketches a double induction based on diagonalizing a block B of the matrix A by multiplying it by unitary blocks U and V on the left and on the right respectively and considering commutators, while in considering a block B of A, we separate out the effect of multiplying B on the left by U from that of multiplying by V on the right. This avoids a great deal of manipulation and permits us to describe more tightly how decisions on unitary equivalence of two matrices can be made in a finite number of steps, and also how to establish canonical forms. In addition, the method used in this paper yields some intermediate results interesting in themselves (such as Theorem 1), and also considers simultaneous unitary equivalence of ordered sets of matrices.

2. **Preliminary remarks and definitions.** By the *norm* of a column vector or a row vector X with components $(a_1, a_2, \cdots, a_n)$ will be meant the nonnegative square root of the quantity $|a_1|^2+|a_2|^2+ \cdots +|a_n|^2$; it will be denoted by $\|X\|$. By a vector we shall always mean a column vector. The symbol A^* will denote the conjugate transpose of the matrix A, so that if X is a vector, then $\|X\|^2=X^*X$. If a matrix A is partitioned into blocks A_{ij}, we shall refer to the arrangement

$$A_{11}, A_{12}, A_{13}, \cdots ; A_{21}, A_{22}, A_{23}, \cdots ; A_{31}, A_{32}, A_{33}, \cdots ; \cdots$$

of the A_{ij} as the *natural ordering* of the blocks.

DEFINITION. If H is a subgroup of the group of $n\times n$ unitary matrices, we say two matrices A and B are *equivalent under* H if $B=U^*AU$ for some member U of H.

DEFINITION. Consider the set G of all $n\times n$ unitary matrices of the form

$$U = \operatorname{Diag}(U_1, U_2, \cdots, U_m),$$

where U_i is any $r_i \times r_i$ unitary matrix and where $r_1+r_2+\cdots+r_m=n$. Then G is a subgroup of the group of $n\times n$ unitary matrices and will be called an *unrestricted direct group*. The sequence of integers $\{r_i\}$ is called the size sequence of G, or, for brevity, the *size* of G.

DEFINITION. We shall make use of subgroups of unitary matrices which are more restricted than those given in the preceding definition: We consider the unrestricted direct group G and let

$$E_1, E_2, \cdots, E_s$$

be a partition of the set of integers $\{1, 2, \cdots, m\}$ into s disjoint subsets. Let H be the set of all members

$$U = \operatorname{Diag}(U_1, U_2, \cdots, U_m)$$

of G with the property that $U_i=U_j$ whenever i and j belong to the same subset E_k. Then H is a subgroup of G and will be called a *direct group*; the sequences $\{r_i\}$ and $\{E_j\}$ will be called the *size* and the partition of H. (If the integers i and j belong to the same set E_k, then r_i and r_j are necessarily equal.)

DEFINITION. If $U=\operatorname{Diag}(U_1, U_2, \cdots, U_m)$, then U_j will be called the jth component of U.

DEFINITION. Let H be a direct group of size $\{r_i\}$ and partition $\{E_j\}$. Then a typical member of H is of the form

$$U = \operatorname{Diag}(U_1, U_2, \cdots, U_k, \cdots, U_m). \tag{1}$$

Let E_t be that member of $\{E_j\}$ which contains the integer k. Let G_0 be an unrestricted direct group of $r_k\times r_k$ unitary matrices with size $\{p_i\}$, $p_1+p_2+\cdots+p_e=r_k$. In the expansion (1) of U replace U_k and all other U_i with i in E_t by

$$V = \operatorname{Diag}(V_1, V_2, \cdots, V_e),$$

a typical member of G_0. The set of all unitary matrices U thus obtained from the members of H forms a subgroup K of H which is itself a direct group; it is called the *refinement of* H by G_0 *in the kth place*. The direct group K is uniquely determined by H, G_0, and the integer k.

NOTE. Unrestricted direct groups are direct groups whose corresponding partitions consist of subsets E_i each of which has only one element. We shall omit the adjective "unrestricted" when no confusion is caused by doing so.

DEFINITION. Let H be a direct group with size $\{r_i\}$ and partition $\{E_j\}$, $i=1, 2, \cdots, m$; $j=1, 2, \cdots, n$. Let the integers e and f be in E_p and in E_q respectively and assume that $r_e=r_f$. Consider the new partition of the set of integers $\{1, 2, \cdots, m\}$ which is obtained from the partition $\{E_j\}$ by uniting E_p and E_q. Call this new partition $\{F_k\}$—after relabelling the sets. The direct group K with size $\{r_i\}$ and partition $\{F_k\}$ will be called a *restriction of* H, or more precisely an (e, f)*-restriction of* H.

We now present Propositions 1, 2, and 3 which are the key to the method used in this paper.

3. **Three propositions.**

PROPOSITION 1. *Let B be an $r\times s$ matrix. Then there exists an $r\times r$ unitary matrix U such that UB has mutually orthogonal row vectors $X_1, X_2, \cdots, X_r$ with $\|X_1\| \geqq \|X_2\| \geqq \cdots \geqq \|X_r\|$. Furthermore, there exists a unique direct group H of $r\times r$ unitary matrices, completely determined by B, such that the set of all unitary matrices U which have the above-stated property is precisely the coset HU_0, where U_0 is any unitary matrix having the property.*

Proof. Consider BB^* which is an $r\times r$ nonnegative-definite Hermitian matrix. There exists a unitary matrix U which transforms BB^* into its diagonal form

$$U(BB^*)U^* = \mathrm{Diag}(c_1, c_2, \cdots, c_r)$$

with $c_1 \geqq c_2 \geqq \cdots \geqq c_r \geqq 0$. In general, the c_i are not all distinct. Let the first r_i of the c_i be equal, then the next r_2 of them equal but distinct from the first r_1, and so on. This gives rise to integers $\{r_i\}$ with $r_1+r_2+ \cdots +r_m=r$. Let H be the unrestricted direct group with size $\{r_i\}$. H is uniquely determined by BB^* and, furthermore, the set of all unitary matrices that diagonalize BB^* with diagonal elements in descending order, is precisely the coset HU.

Next observe that the (i, j)-element of $UBB^*U = (UB)(UB)^*$ is $X_iX_j^*$, where the X_i are the row vectors of UB. It is now easily verified that the matrix U and the direct group H have the properties required in the statement of the proposition.

DEFINITION. If the row vectors of an $r\times s$ matrix B are mutually orthogonal and all have the same norm, then B is called a *row-orthogonal* matrix. A matrix B is row-orthogonal if and only if BB^* is a nonnegative multiple of the identity matrix. For future purposes we also define *column-orthogonal* matrices in an analogous manner. A matrix B is column-orthogonal if and only if B^*B is a nonnegative multiple of the identity.

PROPOSITION 2. *Let B be an $r\times s$ matrix and V any $s\times s$ unitary matrix. Then B and BV give rise to the same direct group H and the same coset HU_0 as occurred in Proposition 1.*

The proof follows from the equation $(BV)(BV)^* = BB^*$.

Considering B^*B instead of BB^* we can prove

PROPOSITION 3. *Let B be an $r\times s$ matrix. Then there exists an $s\times s$ unitary matrix U such that BU has mutually orthogonal column vectors $X_1, X_2, \cdots, X_r$ with $\|X_1\| \geqq \|X_2\| \geqq \cdots \geqq \|X_r\|$. Furthermore, there exists a unique direct group H of $s\times s$ unitary matrices, completely determined by B, such that the set of all unitary matrices having the above property is precisely the coset UH.*

The column analog of Proposition 2 also holds.

Definition. The matrix U of Proposition 1 is called *a row-fixer* of B and the direct group H, *the row-fixed group* of B. A *column-fixer* of B and the *column-fixed group* of B are defined similarly in connection with Proposition 3.

4. A series of algorithms.

Algorithm 1. Let H be a direct group of $n\times n$ matrices with size $\{r_i\}$ and partition $\{E_j\}$ and let A be an $n\times n$ matrix. Partition A into blocks A_{ij} conforming with H, i.e., such that A_{ij} is an $r_i\times r_j$ matrix. Assume that at least one of the A_{ij} is not row-orthogonal, and let A_{rs} be the first A_{ij} in the natural order which is not row-orthogonal. Apply Proposition 1 to A_{rs} and let U_1 be a row-fixer and G the row-fixed group of A_{rs}. Let E_k be that member of $\{E_j\}$ which contains r. Let U be that member of H whose ith component is U_1^* whenever i is in E_k and whose remaining components are all identity matrices. We shall call $A_1=U^*A\,U$ *a transform of* A under *Algorithm* 1. The refinement H_1 of H by G in the rth place will be called the *refinement of H induced by Algorithm* 1 *on A.*

Proposition 4. *Two matrices A and B are equivalent under a direct group H if and only if they give rise to the same refinement H_1 of H under Algorithm* 1 *and their transforms A_1 and B_1 are equivalent under H_1.*

Proof. Assume A and B are equivalent under H. Partition A and B into blocks A_{ij} and B_{ij} conforming with H. Then $B_{ij}=V_i^*A_{ij}V_j$, where the V_k are fixed unitary matrices and $V_i=V_j$ if i and j belong to the same set in the partition of H. Since the row-orthogonality of one of the two matrices A_{ij} and B_{ij} implies that of the other, the first non-row-orthogonal blocks of A and B occur at the same position (r, s). It follows from the above propositions that A_{rs} and B_{rs} have the same fixed group G and hence A and B give rise to the same refinement H_1 of H. Next let U_1 and W_1 be any row-fixers of A_{rs} and B_{rs} respectively; then, by Proposition 2, there exists a member V of G such that $W_1V_r^*=VU_1$. Let A_1 and B_1 be the transforms of A and B under Algorithm 1 obtained by making use of U_1 and W_1. Then the (i, j)-blocks A'_{ij} and B'_{ij} of A_1 and B_1 are given below in all possible cases. Letting E_p denote the set containing the integer r in the partition of H, then:

(1) if $i\notin E_p$ and $j\notin E_p$, then $A'_{ij}=A_{ij}$ and $B'_{ij}=B_{ij}=V_i^*A_{ij}V_j=V_i^*A'_{ij}V_j$;

(2) if $i\in E_p$ and $j\notin E_p$, then $A'_{ij}=U_1A_{ij}$ and $B'_{ij}=W_1B_{ij}=W_1V_r^*A_{ij}V_j$ $=VU_1A_{ij}V_j=VA'_{ij}V_j$;

(3) if $i\notin E_p$ and $j\in E_p$, then $A'_{ij}=A_{ij}U_1^*$ and $B'_{ij}=B_{ij}W_1^*=V_i^*A_{ij}V_rW_1^*$ $=V_i^*A_{ij}(W_1V_r^*)^*=V_i^*A'_{ij}V^*$; and

(4) if $i\in E_p$ and $j\in E_p$, then $A'_{ij}=U_1A_{ij}U_1^*$ and $B'_{ij}=W_1B_{ij}W_1^*$ $=W_1V_r^*A_{ij}V_rW_1^*=VU_1A_{ij}U_1^*V=VA'_{ij}V^*$.

It follows that $B_1=S^*A_1S$, where S is obtained from $\mathrm{Diag}(V_i)$ by replacing V_j by V whenever $j\in E_p$, but, since V belongs to G, this means that A_1

and B_1 are equivalent under H_1, the refinement of H by G in the rth place. This completes the proof of Proposition 4 in one direction. The proof of the converse is trivial.

ALGORITHM 2. Let H be a direct group and A a matrix partitioned into blocks conforming with H, A_{rs} be the first A_{ij} in the natural order which is not column-orthogonal. Apply Proposition 3 to A_{rs} and let U_1 be a column-fixer and G the column-fixed group of A_{rs}. Let E_k be that member of H whose ith component is U_1 whenever i is in E_k and whose other components are all identity matrices. The refinement H_1 of H by G in the sth place will be called the *refinement of H induced by Algorithm* 2 *on A*, and $A_1 = U^*AU$ *a transform of A under Algorithm* 2. The analog of Proposition 4 holds.

We now state a theorem which we consider to be of some interest in itself.

THEOREM 1. *Let H be a direct group of $n\times n$ unitary matrices. There exists an algorithm, called Algorithm* 4 *in the sequel, which, when applied to any $n\times n$ matrix A, associates with it a unique direct subgroup $H(A)$ of H and a matrix A_0 equivalent to A under H such that*

(1) *A_0 when partitioned into blocks conforming with $H(A)$ has the form*

$$\begin{bmatrix} c_{11}U_{11} & c_{12}U_{12} & \cdots & c_{1m}U_{1m} \\ c_{21}U_{21} & c_{22}U_{22} & \cdots & c_{2m}U_{2m} \\ & & \vdots & \\ c_{m1}U_{m1} & c_{m2}U_{m2} & \cdots & c_{mm}U_{mm} \end{bmatrix}$$

where the c_{ij} are nonnegative real numbers and the U_{ij} unitary matrices[2] *and*

(2) *a matrix B is equivalent to A under H if and only if $H(A)=H(B)$ and B_0 is equivalent to A_0 under $H(A)$.*

The proof will follow the description of Algorithm 4.

ALGORITHM 3. Let A_{ij} be the blocks of A conforming with H. If every A_{ij} is row-orthogonal, put

$$A^{(1)} = A, \qquad H^{(1)} = H;$$

otherwise apply Algorithm 1 to A; let A_1 be the transform of A under Algorithm 1 and H_1 the corresponding refinement of H. Repeat the process, i.e., if every block of A_1 when partitioned into blocks conforming with H_1 is row-orthogonal, put

$$A^{(1)} = A_1, \qquad H^{(1)} = H_1;$$

otherwise apply Algorithm 1 with the new group H_1, let A_2 be the transform of H_1, etc. Thus we obtain a sequence of direct groups $\{H_i\}$, where H_{i+1} is a refinement of H_i. Each time that Algorithm 1 is applicable, a proper refine-

(2) The c_{ij} corresponding to nonsquare blocks are, therefore, necessarily zero.

ment of the preceding direct group is obtained, so that after a *finite* number of steps we find a direct group H_m and a matrix A_m, where in the partition of A_m conforming with H_m every block is row-orthogonal. Then let

$$A^{(1)} = A_m \quad \text{and} \quad H^{(1)} = H_m.$$

Clearly $H^{(1)}$ is uniquely determined by this process, and furthermore A and B are equivalent under H if and only if they give rise to the same $H^{(1)}$ and $A^{(1)}$ and $B^{(1)}$ are equivalent under $H^{(1)}$. (This follows from repeated applications of Proposition 4).

Next consider $A^{(1)}$ and $H^{(1)}$ and repeat the process described above with Algorithm 2 instead of Algorithm 1. This will, after a finite number of steps, give rise to a matrix $A^{(2)}$ and a unique direct group $H^{(2)}$ such that $A^{(2)}$, when partitioned into blocks conforming with $H^{(2)}$, has only column-orthogonal blocks. Again, two matrices A and B are equivalent under H if and only if they give rise to the same direct group $H^{(2)}$ and $A^{(2)}$ is equivalent to $B^{(2)}$ under $H^{(2)}$. We will call $H^{(2)}$ the *subgroup* of H *induced by Algorithm* 3 *on* A. The matrix $A^{(2)}$ will be called the *transform* of A *under Algorithm* 3.

Algorithm 4. Let H be a direct group. Find the transform $A^{(2)}$ of A under Algorithm 3 and the corresponding induced subgroup $H^{(2)}$ of H. Having found $A^{(2i)}$ and $H^{(2i)}$ apply Algorithm 3 with the new direct group $H^{(2i)}$ and let $A^{(2i+2)}$ be the transform of $A^{(2i)}$ and $H^{(2i+2)}$ the corresponding induced subgroup of $H^{(2i)}$. Each time that $H^{(2i+2)}$ is not equal to $H^{(2i)}$, it has at least one more component than $H^{(2i+2)}$; hence after a *finite* number of steps we will have $A^{(2m+2)}=A^{(2m)}$ and $H^{(2m+2)}=H^{(2m)}$. Let $A_0=A^{(2m)}$ and $H_0=H(A)=H^{(2m)}$. The subgroup $H^{(2m)}$ of H is uniquely determined by Algorithm 4 and A is equivalent to B under H if and only if $H_0=H(A)=H(B)$ and A_0 is equivalent to B_0 under H_0.

Proof of Theorem 1. Observe that if A_0 and H_0 are the matrix and direct group obtained by Algorithm 4, then in the partition of A_0 into blocks conforming with H_0, the blocks are both row-orthogonal and column-orthogonal and, therefore, nonnegative multiples of unitary matrices.

Algorithm 5. Let H be a direct group of $n\times n$ unitary matrices with size $\{r_i\}$ and partition $\{E_j\}$. Let A be an $n\times n$ matrix whose blocks in the partition conforming with H are nonnegative multiples $c_{ij}U_{ij}$ of unitary matrices U_{ij}. If for each nonzero c_{ij} the integers i and j belong to the same set E_k, let $H_*=H$ and $A_*=A$; otherwise let c_{rs} be the first nonzero c_{ij} in the natural order for which i and j do not belong to the same set. Let r be in E_f and s in E_g. Let U be that member of H whose ith component is U_{rs} for all i in E_f and whose remaining components are identity matrices. Put $A_*=U^*AU$ and let H_* be the (f, g)-restriction of H.

Proposition 5. *If the matrix A_* is obtained from A by Algorithm* 5 *and if H_* is the corresponding restriction of H as in Algorithm* 5, *then the (r, s)-block of U^*A_*U in the partition conforming with H_* is $c_{rs}I$, where I is the identity* *matrix, for all U in H_*; furthermore, two matrices A and B are equivalent under H if and only if they give rise to the same direct group H_* under Algorithm* 5 *and A_* is equivalent to B_* under H_*.*

Proof. That the (r, s)-block of U^*A_*U is equal to $c_{rs}I$ for all U in H_* is easy to see. If A and B are equivalent under H, they have the same ordered set of c_{ij}, so that the integers r and s are the same for A and B; hence they give rise to the same restriction H_* of H. The (r, s)-block of B_* is also $c_{rs}I$ and it follows at once that A_* and B_* are equivalent under H_*. The proof of the converse is trivial.

ALGORITHM 6. Let H be a direct group of $n\times n$ matrices. Let A be an $n\times n$ matrix whose blocks in the partition conforming with H are nonnegative multiples of unitary matrices. Apply Algorithm 5 to A with the group H and obtain A_* and H_*; next apply Algorithm 5 to A_* with the new group H_* and obtain $A_{**}=A_{(2)}$ and $H_{**}=H_{(2)}$. Continue this process, i.e., having found $A_{(k)}$ and $H_{(k)}$, apply Algorithm 5 to $A_{(k)}$, with the direct group $H_{(k)}$ and obtain $A_{(k+1)}$ and $H_{(k+1)}$. Each time that Algorithm 5 is applicable nontrivially, the number of sets in the partition $\{E_k\}$ of the direct group decreases by 1; hence after a *finite* number of steps we obtain $A_{(m+1)}=A_{(m)}$ and $H_{(m+1)}=H_{(m)}$. We shall call $A_{(m)}$ *a transform of A under Algorithm* 6 *and $H_{(m)}$ the subgroup of H induced by Algorithm* 6 *on A.* The following proposition is an immediate consequence of the construction of $A_{(m)}$ and $H_{(m)}$.

PROPOSITION 6. *Assume A, when partitioned to conform with H, has blocks which are nonnegative multiples of unitary matrices. Let A' be a transform of A under Algorithm* 6, *and H' the corresponding induced subgroup of H. Let $\{E_k\}$ be the partition of H'. Let A'_{ij} be the blocks of A' when partitioned according to H'. Then the A'_{ij} are nonnegative multiples $c_{ij}U_{ij}$ of unitary matrices U_{ij} and $c_{ij}=0$ whenever i and j belong to distinct sets of the partition $\{E_k\}$.*

PROPOSITION 7. *If two matrices A and B have nonnegative multiples of unitary matrices as blocks conforming with a direct group H, then they are equivalent under H if and only if they give rise to the same induced subgroup H' of H under Algorithm* 6 *and their transforms A' and B' under the algorithm are equivalent under H'.*

Proof. Apply Proposition 5 repeatedly.

ALGORITHM 7. Let H be a direct group of $n\times n$ unitary matrices with partition $\{E_k\}$. Let A be an $n\times n$ matrix whose blocks in the partition conforming with H are nonnegative multiples $c_{ij}U_{ij}$ of unitary matrices U_{ij}, and such that $c_{ij}=0$ whenever i and j belong to two distinct sets in $\{E_k\}$. Then a member

$$U = \mathrm{Diag}(U_1, \cdots, U_m)$$

of H transforms A into $U^*A\,U$ whose blocks are $c_{ij}U_i^*U_{ij}U_j$, and by assumption $U_i = U_j$ if $c_{ij} \neq 0$. If for each nonzero c_{ij}, the matrix U_{ij} is the identity matrix, put

$$A' = A, \qquad H' = H.$$

Otherwise let c_{rs} be the first nonzero number among the c_{ij}, in the natural order, for which U_{ij} is *not* the identity matrix. Then there exists a unitary matrix V_1 such that $V_1^* U_{rs} V_1$ is diagonal; furthermore, if m is the size of U_{rs}, there exists a unique direct group G of $m \times m$ unitary matrices such that the set of unitary matrices that diagonalize U_{rs} is precisely the coset V_iG. Let U be that member of H whose ith component is V_1 whenever i belongs to the set E_k containing r, and whose remaining components are all identity matrices. Put $A' = U^*AU$ and let H' be the refinement of H by G in the rth place.

PROPOSITION 8. *If H is a direct group and if A and B are two matrices of the form described in Algorithm 7, then A and B are equivalent under H if and only if they give rise to the same refinement H' of H under Algorithm 7 and their transforms A' and B' are equivalent under H'.*

The proof is similar to the proof of Proposition 4.

DEFINITION. Let H be a given direct group of $n \times n$ unitary matrices. Let A be any $n \times n$ matrix. Apply Theorem 1 to A with the direct group H and let H_1 be the resulting subgroup of H and A_1 the resulting matrix. Next apply Algorithm 6 to A_1 with the group H_1 and obtain A_2 and H_2. Then apply Algorithm 7 to A_2 with the group H_2 and let H_3 and A_3 be the resulting direct group and matrix respectively. We shall call A_3 *a first reduced form of A under H*, and H_3 *the first reduced subgroup of H with respect to A.*

PROPOSITION 9. *Any first reduced form of A is equivalent to A under H, and if B is another matrix, then B is equivalent to A under H if and only if A and B give rise to the same first reduced subgroup H' of H and their first reduced forms under H are equivalent under H'.*

The proof follows from Theorem 1 and the earlier propositions.

ALGORITHM 8. Let H be a direct group of $n \times n$ unitary matrices. Given an $n \times n$ matrix A, find a first reduced form $A^{(1)}$ of A under H and the first reduced subgroup $H^{(1)}$ of H with respect to A. Repeat the process, i.e., having found $A^{(j)}$ and $H^{(j)}$, let $A^{(j+1)}$ be a first reduced form of $A^{(j)}$ under $H^{(j)}$ and let $H^{(j+1)}$ be the first reduced subgroup of $H^{(j)}$ with respect to $A^{(j)}$. Each time that $H^{(j+1)} \neq H^{(j)}$, either $H^{(j+1)}$ has at least one more component than $H^{(j)}$ or the number of sets in the partition of $H^{(j+1)}$ is at least one less than that in the partition of $H^{(j)}$. Hence after a finite number of steps we obtain $H^{(k+1)} = H^{(k)}$. We shall call $H^{(k)}$ *the reduced subgroup of H with respect to A, and $A^{(m)}$ the reduced form of A.* By this method the reduced subgroup $H^{(m)}$ of H with respect to A is uniquely determined and we have

THEOREM 2. *Let H be a direct group of $n \times n$ unitary matrices and A an*

$n\times n$ matrix. Let H_0 and A_0 be, respectively, the reduced subgroup of H with respect to A and the reduced form of A under H. Then A_0 is invariant under H_0, i.e., $U^*A_0U=A_0$ for all members U of H_0. Furthermore, two matrices A and B are equivalent under H if and only if they give rise to the same reduced subgroup H_0 of H and their reduced forms under H are equal.

Proof. If for some member U of H_0 we had $U^*A_0U\neq A_0$, then at least one of the Algorithms 4, 6, and 7 would be further applicable to A_0, which would contradict the construction of A_0 and H_0; hence $U^*A_0U=A_0$ for all U in H_0. Repeated applications of Proposition 9 show that A and B are equivalent under H if and only if they give rise to the same reduced subgroup H_0 of H and their reduced forms A_0 and B_0 are equivalent under H_0, but it follows from the first part of this theorem that $B_0=U^*A_0U$ for some U in H_0 if and only if $B_0=A_0$.

Thus we have, by Theorem 2, established canonical forms A_0 for all $n\times n$ matrices A under *any* given direct group H of $n\times n$ unitary matrices. These canonical forms are found by Algorithm 8.

5. **A related result.** Theorem 2 also solves the problem of *simultaneous unitary equivalence* of two finite sets of matrices: Given two ordered sets of $n\times n$ matrices $\{A_i\}$ and $\{B_i\}$, $i=1, 2, \cdots, m$ we are to decide whether or not there exists an $n\times n$ unitary matrix U such that $B_i=U^*A_iU$ for *all* i. This problem is the same as that of deciding whether or not two $mn\times mn$ matrices

$$A = \mathrm{Diag}(A_1, A_2, \cdots, A_m)$$

and

$$B = \mathrm{Diag}(B_1, B_2, \cdots, B_m)$$

are equivalent under the direct group H of $mn\times mn$ unitary matrices whose size is $\{r_i\}$ with $r_i=n$, $i=1, 2, \cdots, m$ and whose partition consists of just one set containing all the integers $1, 2, \cdots, m$. (The components of each member U of H are all equal.)

6. **Triangular canonical forms under the full unitary group.** The direct group H of Theorem 2 can be taken to be the *full* group of $n\times n$ unitary matrices, and we may desire the canonical forms of matrices under unitary equivalence (i.e., under the full group of unitary matrices) to be triangular. This suggests the following procedure:

Triangularization. Let A be an $n\times n$ matrix acting on an n-dimensional unitary space V and let $e_1, e_2, \cdots, e_n$ be the eigenvalues of A arranged in some order agreed on, where each eigenvalue is repeated as many times as its algebraic multiplicity requires. (By this is meant the multiplicity of the eigenvalue as a zero of the characteristic polynomial of A.) Consider the following subspaces of H:

$$
\begin{aligned}
V_1 &= (A - e_2I)(A - e_3I) \cdots (A - e_nI)V, \\
V_2 &= (A - e_3I)(a - e_4I) \cdots (A - e_nI)V, \\
&\vdots \\
V_{n-1} &= (A - e_nI)V, \\
V_n &= V.
\end{aligned}
$$

Each V_i is contained in V_{i+1} for $i=1, 2, \cdots, n-1$. Furthermore, $(A-e_1I)V_1 = 0$ and $(A-e_iI)V_{i+1}=V_i$ for $i=1, 2, \cdots, n-1$. Let m_i be the dimension of V_i. Choose an orthonormal basis $\{X_1, X_2, \cdots, X_n\}$ for V in which the first m_i vectors lie in V_i for $i=1, 2, \cdots, n$. Letting $U=[X_1, X_2, \cdots, X_n]$, the matrix A will take on the triangular form $A_0=U^*AU$ with respect to the new basis.

Let $r_1=m_1$ and $r_i=m_i-m_{i-1}$ for $i=2, 3, \cdots, n$. Then $r_i \geqq 0$ for all i. Take the nonzero r_i and relabel them to form the sequence $\{s_i\}$, where $s_1=r_1$, s_2 is the next nonzero r_i, and so on. Let H be the unrestricted direct group of $n\times n$ unitary matrices with size $\{s_i\}$. We shall call H the triangularizer of A. The eigenvalues, the numbers m_i and therefore the s_i, are invariant under all unitary transformations. Hence, if two matrices A and B are triangularized by the above method giving rise to corresponding triangularizers H_A and H_B, and to triangular forms A_0 and B_0, then A is unitarily equivalent to B if and only if $H_A=H_B$ and A_0 is equivalent to B_0 under H_A. Thus we have the following

COROLLARY TO THEOREM 2. *Let A_0 be a triangular form of the matrix A and let H be its corresponding triangularizer. Let $C(A)$ be the reduced form of A_0 under H. Then the matrix $C(A)$ is the triangular canonical form of A: Two matrices A and B are unitarily equivalent if and only if $C(A)=C(B)$.*

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