THE PYTHAGOREAN THEOREM IN CERTAIN SYMMETRY CLASSES OF TENSORS

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The Hadamard determinant theorem [1] states that if $A = (a_{ij})$ is a positive semi-definite k-square Hermitian matrix then

$$\det A \leqq \prod_{i=1}^{k} a_{ii} \tag{1}$$

with equality if and only if A has a zero row or A is diagonal.

In a recent paper [3] it was conjectured that an analogous result to (1) holds for the permanent of A. We recall that the permanent is defined by

$$\operatorname{per}(A) = \sum_{\sigma \in S_k} \prod_{i=1}^{k} a_{\sigma(i)i}$$

where the summation is over the whole symmetric group of degree k. Recent interest in the permanent function stems from its application to a variety of combinatorial problems [4] and a partly unresolved conjecture of B. L. van der Waerden [2]. In [3] it was suggested that if A is once again a positive semi-definite k-square Hermitian matrix then

$$\operatorname{per}(A) \geqq \prod_{i=1}^{k} a_{ij} \tag{2}$$

with equality if and only if A has a zero row or A is diagonal. We are as yet unable to prove this but the subsequent inequality (3) is a step in this direction.

The first purpose of this note is to exhibit (1) as a case of the Pythagorean Theorem in a suitable symmetry class of tensors. Of course, many proofs of (1) are extant and our purpose in reproving it here is to exhibit a technique that is proving itself useful for examining a wide variety of matrix functions. We then show by a similar approach that

$$\operatorname{per}(A) \geqq \left(\prod_{i=1}^{k} a_{ii} \right) k!/k^{2k} \tag{3}$$

where the inequality is strict unless A has a zero row.

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The technique here is to regard the determinant and permanent functions as analytic expressions for inner products in suitable symmetry classes of tensors. To be explicit, let U be an n-dimensional unitary vector space with an inner product (x, y). Let $U^{(k)}$ be the space of k-tensors over U; i.e., the n^k-dimensional dual space of the vector space $M^{(k)}$ of multilinear functionals ϕ of k-tuples of vectors from U. Certain distinguished "pure" vectors in $U^{(k)}$ are denoted by $f=x_1\otimes \cdots \otimes x_k$ where $x_i \in U$ and f is defined by $f(\phi) = \phi(x_1, \cdots, x_k)$ for each $\phi \in M^{(k)}$. The pure vectors span $U^{(k)}$ and the conjugate bilinear functional defined on pure vectors by

$$(4) \qquad (x_1 \otimes \cdots \otimes x_k, y_1 \otimes \cdots \otimes y_k) = \prod_{i=1}^{k} (x_i, y_i)$$

is extendable to a unitary inner product on $U^{(k)}$. Let T and S be the symmetry operators of $U^{(k)}$ into itself defined by

$$(5) \qquad T(x_1 \otimes \cdots \otimes x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)},$$

$$(6) \qquad S(x_1 \otimes \cdots \otimes x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)},$$

where $\epsilon(\sigma) = \pm 1$ according as σ is even or odd.

It is well known that T and S are Hermitian (with respect to the inner product in (4)) and idempotent. Moreover, one computes easily that

$$(Tx_1 \otimes \cdots \otimes x_k, y_1 \otimes \cdots \otimes y_k) = \frac{1}{k!} \det((x_i, y_j))$$

and

$$(Sx_1 \otimes \cdots \otimes x_k, y_1 \otimes \cdots \otimes y_k) = \frac{1}{k!} \operatorname{per}((x_i, y_j)).$$

To prove (1) we first remark that if A is singular, $\det A = 0$, then the inequality obviously holds and equality requires that some $a_{ii}=0$. But then $0 \leqq a_{ii}a_{jj} - |a_{ij}|^2 = -|a_{ij}|^2$ and hence $a_{ij}=0$, $j=1, \cdots, k$, and row i of A is zero. Hence assume A is nonsingular and let $x_1, \cdots, x_k$ be a set of linearly independent vectors such that $(x_i, x_j) = a_{ij}$, $i, j=1, \cdots, k$. Let $u_1, \cdots, u_k$ be the E. Schmidt orthonormalizing sequence for $x_1, \cdots, x_k$. That is, the space $\langle x_1, \cdots, x_p\rangle$ spanned by $x_1, \cdots, x_p$ is the same as the space $\langle u_1, \cdots, u_p\rangle$ spanned by $u_1, \cdots, u_p$, $p=1, \cdots, k$. Then since $u_{i_1}\otimes \cdots \otimes u_{i_k}$, $1 \leqq i_\alpha \leqq k$, $\alpha=1, \cdots, k$, is an orthonormal basis in $U^{(k)}$ we have from the Pythagorean theorem that

$$\frac{1}{k!}\det A = \frac{1}{k!}\det((x_i, x_j)) = (Tx_1 \otimes \cdots \otimes x_k, Tx_1 \otimes \cdots \otimes x_k)$$
$$= \sum \left| (Tx_1 \otimes \cdots \otimes x_k, u_{i_1} \otimes \cdots \otimes u_{i_k}) \right|^2$$
$$= \left(\frac{1}{k!}\right)^2 \sum \left| \det((x_s, u_{i_t})) \right|^2, \qquad s, t = 1, \cdots, k,$$

where the summation extends over all k^k ordered selections $(i_1, \cdots, i_k)$ from $1, \cdots, k$. Since the determinant vanishes when two columns are the same, the last summation may be taken over sets of distinct ordered choices $(i_1, \cdots, i_k)$, i.e., over all $k!$ permutations of $1, \cdots, k$. Hence

$$\frac{1}{k!}\det A = \left(\frac{1}{k!}\right)^2 \sum_{\sigma \in S_k} \left| \det((x_s, u_{\sigma(t)})) \right|^2$$
$$= \frac{1}{k!} \left| \det((x_s, u_t)) \right|^2$$
$$= \frac{1}{k!} \left| \det \begin{bmatrix} (x_1, u_1) & 0 & \cdots & 0 \\ (x_2, u_1) & (x_2, u_2) & 0 \cdots & 0 \\ \vdots & & & \vdots \\ (x_k, u_1) & (x_k, u_2) & \cdots & (x_k, u_k) \end{bmatrix} \right|^2$$
$$= \frac{1}{k!} \prod_{i=1}^{k} \left| (x_i, u_i) \right|^2$$
$$\leqq \frac{1}{k!} \prod_{i=1}^{k} (x_i, x_i) = \frac{1}{k!} \prod_{i=1}^{k} a_{ii}.$$

Now the equality holds by Schwarz's inequality if and only if x_α is a multiple of u_α. But since $u_1, \cdots, u_k$ is an orthonormal set, it follows that $A = ((x_i, x_j))$ is diagonal.

From (6) we compute that

$$\begin{aligned} \text{per } A &= k!(Sx_1 \otimes \cdots \otimes x_k, x_1 \otimes \cdots \otimes x_k) \\ &= k! \| Sx_1 \otimes \cdots \otimes x_k \|^2 \\ &\geqq k! \left| (Sx_1 \otimes \cdots \otimes x_k, u \otimes \cdots \otimes u) \right|^2 \\ &= k! \left| (x_1 \otimes \cdots \otimes x_k, Su \otimes \cdots \otimes u) \right|^2 \\ &= k! \left| (x_1 \otimes \cdots \otimes x_k, u \otimes \cdots \otimes u) \right|^2 \\ &= k! \prod_{i=1}^{k} \left| (x_i, u) \right|^2, \end{aligned} \tag{7}$$

where u is any unit vector in U.

The problem then is to find for fixed x_i, $i=1, \cdots, k$, a significant lower bound for the expression $\prod_{i=1}^{k} |(x_i, u)|^2$ as a function of the unit vector u. We have the

LEMMA. *If $x_1, \cdots, x_k$ are vectors then there exists a unit vector u such that*

$$|(x_s, u)| \geqq \|x_s\|/k, \qquad s = 1, \cdots, k.$$

Proof. Let $y_i = x_i/\|x_i\|$ or $-x_i/\|x_i\|$ so that $\sum_{i=1}^{k} y_i$ is of maximal length. That is, $\|\sum_{i=1}^{k} y_i\| \geqq \|\sum_{i=1}^{k} \pm x_i/\|x_i\|\,\|$ for all choices of signs on the x_i. We assert that

$$\operatorname{Re}\left(y_s, \sum_{i=1, i\neq s}^{k} y_i\right) \geqq 0, \qquad s = 1, \cdots, k. \tag{8}$$

For let $z_s = \sum_{i=1, i\neq s}^{k} y_i$ and $z = \sum_{i=1}^{k} y_i$ and we have

$$\|z\|^2 = \|y_s + z_s\|^2 \geqq \|-y_s + z_s\|^2,$$

$$\|y_s\|^2 + 2\operatorname{Re}(y_s, z_s) + \|z_s\|^2 \geqq \|y_s\|^2 - 2\operatorname{Re}(y_s, z_s) + \|z_s\|^2$$

and (8) follows.

It is clear that

$$\|z\| \leqq \sum_{i=1}^{k} \|y_i\| = k \tag{9}$$

and thus

$$\begin{aligned} \left|\left(y_s, \frac{z}{\|z\|}\right)\right| &= \left|\left(y_s, \frac{y_s + z_s}{\|z\|}\right)\right| \\ &= \frac{1}{\|z\|} \,|(\|y_s\|^2 + (y_s, z_s))| \\ &\geqq \frac{1}{\|z\|}\operatorname{Re}(1 + (y_s, z_s)) \geqq \frac{1}{\|z\|} \geqq \frac{1}{k}\cdot \end{aligned} \tag{10}$$

In (7) we take $u = z/\|z\|$ to obtain

$$\begin{aligned} \operatorname{per} A &\geqq k! \prod_{i=1}^{k} |(y_i, u)|^2 \|x_i\|^2 \\ &\geqq (k!/k^{2k}) \prod_{i=1}^{k} \|x_i\|^2 \\ &= (k!/k^{2k}) \prod_{i=1}^{k} a_{ii}. \end{aligned}$$

Clearly if any $x_j=0$ then per $A=0$, $a_{jj}=0$ and (3) is equality. If no $x_j=0$ then (9) can be equality only if $z=\theta k y_s$, where $|\theta|=1$, $s=1, \cdots, k$, in which case

$$|(y_s, u)| = \left|\left(y_s, \frac{z}{\|z\|}\right)\right| = k/k = 1 > \frac{1}{k}$$

and hence the inequality in the lemma is strict for $k>1$.

Actually a refinement of the above argument shows that

$$\prod_{s=1}^{k} |(y_s, u)| \geqq \frac{k^{1/2}}{k} \frac{k^{1/2}-1}{k-1} \cdots \frac{k^{1/2}-r}{k-r} \frac{1}{k^{k-r-1}},$$

where $r=[k/(k^{1/2}+1)]$, and therefore,

$$\operatorname{per}(A) \geqq \prod_{i=1}^{k} a_{ii} k! \Big/ \left(\frac{k^{1/2}}{k} \frac{k^{1/2}-1}{k-1} \cdots \frac{k^{1/2}-r}{k-r} \frac{1}{k^{k-r-1}}\right)^2.$$

For

$$\text{(11)} \quad \|z\|^2 = (z, z) = \sum_{s=1}^{k} (y_s, z) = k + \sum_{s=1}^{k} (y_s, z_s) = k + \sum_{s=1}^{k} \operatorname{Re}(y_s, z_s) \geqq k,$$

i.e., $\|z\| \geqq k^{1/2}$. We can assume without loss of generality that Re $(y_1, z) \geqq \cdots \geqq$ Re (y_k, z). Now from (11) we have

$$\|z\| = \sum_{s=1}^{k} \operatorname{Re}\left(y_s, \frac{z}{\|z\|}\right) \geqq k^{1/2}$$

and therefore $\operatorname{Re}(y_1, z/\|z\|) \geqq k^{1/2}/k$, and a fortiori, $|(y_1, z/\|z\|)| \geqq k^{1/2}/k$. Further

$$\left|\sum_{s=2}^{k}\left(y_s, \frac{z}{\|z\|}\right)\right| \geqq \left|\sum_{s=1}^{k}\left(y_s, \frac{z}{\|z\|}\right)\right| - \left|\left(y_1, \frac{z}{\|z\|}\right)\right| \geqq k^{1/2}-1$$

and thus $|(y_2, z/\|z\|)| \geqq (k^{1/2}-1)/(k-1)$.

In general

$$\text{(12)} \quad \left|\left(y_{s+1}, \frac{z}{\|z\|}\right)\right| \geqq \frac{k^{1/2}-s}{k-s}.$$

Comparing (12) with (10) we find, by an elementary computation, that

$$\max\left\{\frac{k^{1/2}-s}{k-s}, \frac{1}{k}\right\} = \begin{cases} \dfrac{k^{1/2}-s}{k-s} & \text{for } s \leqq r, \\[2ex] \dfrac{1}{k} & \text{for } s > r, \end{cases}$$

where r is the greatest integer in $k/(k^{1/2}+1)$.

Thus (12) gives a better lower bound for $s=1, \cdots, r$ while (10) gives a better lower bound for $s=r+1, \cdots, k$. The result follows.

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