# ON THE PROXIMATE LINEAR ORDERS OF ENTIRE DIRICHLET SERIES(1)

BY

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1. Let $f(s)$ $(s=\sigma+it)$ be an entire function defined by a Dirichlet series

$$(1) \qquad \sum_{n=0}^{\infty} a_n e^{\lambda_n s}, \qquad 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n \uparrow \infty,$$

absolutely convergent for all $s$.

The linear order, or order $(R)$, and the lower linear order of $f(s)$ are defined, [4, p. 77] and [3, p. 96], as the numbers $\rho = \limsup_{\sigma=\infty} \log\log M(\sigma)/\sigma$ $(0 \leqq \rho \leqq \infty)$ and

$$\tau = \liminf_{\sigma=\infty} \log\log M(\sigma)/\sigma \qquad (0 \leqq \tau \leqq \infty)$$

with $M(\sigma) = \text{l.u.b.}_{-\infty<t<\infty} |f(\sigma+it)|$. Proximate linear orders for $f(s)$ have been defined by Sunyer i Balaguer [7, p. 28] by extending in the natural way the notion of Lindelöf's proximate order [6, p. 326].

Since $\log M(\sigma)$ is an increasing convex [4, pp. 74–75] and therefore continuous [8, p. 172] function of $\sigma$, the arguments of Shah [5] and this author [2, pp. 20–25] can be applied, with trivial modifications, to the present case.

The following propositions of existence of linear proximate orders $R(\sigma)$ and linear lower proximate orders $T(\sigma)$, are thus easily obtained:

(A) *If* $0<\rho<\infty$, *then for any given number* $a$ $(0<a<\infty)$, *there exists a positive continuous function* $R(\sigma)$ *such that*: (i) *the derivatives* $R'(\sigma)$ *and* $R''(\sigma)$ *exist everywhere but for isolated points where* $R'(\sigma\pm 0)$ *and* $R''(\sigma\pm 0)$ *exist.* (ii) $\lim_{\sigma=\infty} \sigma R'(\sigma) = \lim_{\sigma=\infty} \sigma R''(\sigma) = 0$. (iii) $\lim_{\sigma=\infty} R(\sigma) = \rho$. (iv) $\limsup_{\sigma=\infty} \log M(\sigma)/\exp[\sigma R(\sigma)] = a$.

(B) *If* $0<\tau<\infty$, *then for any given number* $b$ $(0<b<\infty)$, *there exists a continuous positive function* $T(\sigma)$ *satisfying conditions* (i) *and* (ii) *of part* (A) *and such that*: (iii′) $\lim_{\sigma=\infty} T(\sigma)=\tau$. (iv′) $\liminf_{\sigma=\infty} \log M(\sigma)/\exp[\sigma T(\sigma)] = b$.

In what follows we will study the upper and lower limits for $\sigma\to\infty$ of the quotient $\log m(\sigma)/\exp[\sigma P(\sigma)]$ where $m(\sigma)$ is defined as

$$m(\sigma) = \max_{n\geqq 0} |a_n \exp[\lambda_n(\sigma+it)]|, \qquad (n = 0, 1, 2 \cdots),$$

and $P(\sigma)$ is a function having first and second derivatives for all $\sigma$ and such

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that: ($\alpha$) $\lim_{\sigma=\infty} \sigma P'(\sigma)=\lim_{\sigma=\infty} \sigma P''(\sigma)=0$. ($\beta$) $0<p=\lim_{\sigma=\infty} P(\sigma)<\infty$. (For notations and terminology see also [1].)

All our conclusions will be valid if $M(\sigma)$ is substituted for $m(\sigma)$ provided the asymptotic equivalence $\lim_{\sigma=\infty} \log M(\sigma)/\log m(\sigma)=1$ holds. Sufficient conditions to guarantee this equivalence are given in [7, Theorem 5] and [1, Theorem 2].

2. If $x=\exp[\sigma P(\sigma)]$, then the assumptions made on $P(\sigma)$ imply that the inverse function $\sigma=\phi(x)$ exists, as well as $\phi'(x)$ and $\phi''(x)$, for all positive large enough $x$, and has the following properties that will be used later.

With $\varphi(x)$ defined by the condition $\phi(x)=\log x/\varphi(x)$ we have

$$\lim_{x=\infty} \varphi(x) = p,$$

$$\sigma P'(\sigma) = x\varphi(x)\varphi'(x) \log x/[\varphi(x) - x\varphi'(x) \log x], \tag{2}$$

$$x\varphi'(x) \log x = \sigma P(\sigma)P'(\sigma)/[P(\sigma) + \sigma P'(\sigma)].$$

Since $x\rightarrow+\infty$ if and only if $\sigma\rightarrow+\infty$ it follows that

$$\lim_{\sigma+\infty} \sigma P'(\sigma) = \lim_{x=\infty} x\varphi'(x) \log x = 0. \tag{3}$$

From (2) and the definition of $\varphi(x)$ we obtain

$$P'(\sigma) = x\varphi'(x)\varphi(x)^2/[\varphi(x) - x\varphi'(x) \log x]$$

and by differentiation and using (3), it is easily seen that for corresponding values of $\sigma$ and $x$ approaching $+\infty$ we have $\sigma P''(\sigma)\sim[x^2\varphi''(x) \log x]/\varphi(x)$ and therefore

$$\lim_{x=\infty} x^2\varphi(x) \log x = 0. \tag{4}$$

3. We consider next, the properties of the function

$$\Gamma(\lambda, \alpha) = \exp[(\lambda/p) \log \alpha - \lambda \log \lambda/\varphi(\lambda)],$$

where $\lambda$ is a real positive variable and $\alpha$ a real positive parameter. We also consider the related function of two real variables $\lambda, \sigma$ given by $\Gamma(\lambda, \alpha) \exp(\sigma\lambda)$ $(0\leqq\lambda<\infty, -\infty<\sigma<\infty)$. The function

$$y = - \log \Gamma(\lambda, \alpha) \tag{5}$$

is convex for all large values of $\lambda$ since by differentiating with respect to $\lambda$, and writing $\varphi$ for $\varphi(\lambda)$ we obtain

$$y'' = [1/(\lambda\varphi^3)][\varphi^2 - \varphi\lambda^2\varphi'' \log \lambda - 2\varphi\lambda\varphi' \log (\lambda e) + 2(\lambda\varphi' \log \lambda)^2/\lambda] > 0$$

by virtue of (2), (3) and (4), for all large enough $\lambda$. In view of the facts that $y\uparrow\infty$, $y'\uparrow\infty$, $y''>0$ we can conclude that the function of $\sigma$

$$\mu(\sigma, \alpha) = \max_{\lambda \geqq 0} \Gamma(\lambda, \alpha) \exp(\sigma\lambda) \tag{6}$$

does exist. The argument is similar to the standard one used in the case of a Newton-Hadamard polygon [1, pp. 717–718; 6, p. 274] and the strict convexity of (5) implies that for each large $\sigma$ the maximum is obtained for the value of $\lambda = \lambda(\sigma, \alpha)$ uniquely defined by the condition that the slope of the curve (5) be equal to $\sigma$, that is to say

$$\sigma = -(1/p)\log \alpha + (1/\varphi)\log(\lambda e) - (1/\varphi^2)(\lambda\varphi' \log \lambda). \tag{7}$$

Substituting in (6), using the definition of $\Gamma(\lambda, \alpha)$ and writing $\lambda$ for $\lambda(\sigma, \alpha)$, we obtain

$$\log \mu(\sigma, \alpha) = (\lambda/\varphi)[1 - (1/\varphi)(\lambda\varphi' \log \lambda)] \sim \lambda/p. \tag{8}$$

On the other hand, the assumption ($\alpha$) implies that for any $\alpha = \alpha(\sigma)$ bounded in absolute value

$$P(\sigma + \alpha(\sigma)) = P(\sigma) + o(1/\sigma) \tag{9}$$

and consequently from (7) and (9) we deduce

$$\sigma P(\sigma) = [-(1/p)\log \alpha + (1/\varphi)\log(\lambda e) + O(1/\lambda)][P(\log \lambda/\varphi) + o(\varphi/\log \lambda)],$$

and since by the definition of $\varphi(\lambda)$ we know that $P(\log \lambda/\varphi) \equiv \varphi(\lambda)$ we can conclude that $\sigma P(\sigma) = -\log \alpha + \log(\lambda e) + O(1/\lambda)$, which together with (8) implies

$$\lim_{\sigma=\infty} \log \mu(\sigma, \alpha)/\exp[\sigma P(\sigma)] = \alpha/(pe) \tag{10}$$

and, again by (8)

$$\lim_{\sigma=\infty} \lambda(\sigma, \alpha)/\exp[\sigma P(\sigma)] = \alpha/e. \tag{11}$$

4. We denote by $\pi$ the Newton-Hadamard polygon corresponding to the sequence of points with coordinates $\lambda = \lambda_n$, $y = -\log |a_n|$ on the cartesian plane $(\lambda, y)$. (See [1, p. 718; 6, p. 274] for terminology and properties related to $\pi$.) The value of $\log m(\sigma)$ is given for each $\sigma$ by the maximum difference between the ordinates of the line $y = \lambda\sigma$ and the polygon $\pi$ and it is achieved for all the corresponding values of $n$. The non-negative integers $n_i$ such that $\lambda_{n_i}$ are the abscissas of the vertices of $\pi$ are called *principal indices* and they form a strictly increasing sequence $I \equiv \{n_i\}$, $(i = 0, 1, 2, \cdots; n_0 = 0)$, which coincides with the sequence of values taken by the increasing step function [1, p. 718] defined by

$$n(\sigma) = \max_{n \geqq 0} \{n \mid m(\sigma) = |a_n \exp(\sigma\lambda_n)|\}. \tag{12}$$

For any choice of $P(\sigma)$ as defined in §1, and any given strictly increasing sequence $J \equiv \{n_j\}$, $(j=0, 1, 2, \cdots)$ of non-negative integers $n_j$, we will define the non-negative finite or infinite numbers $A$, $B$, $L$, $Q_J$, $l_J$ by the following equalities (in particular for the sequence $I$ we define $Q_I$ and $l_I$):

$$A = \limsup_{\sigma=\infty} \log m(\sigma)/\exp[\sigma P(\sigma)], \tag{13}$$

$$B = \liminf_{\sigma=\infty} \log m(\sigma)/\exp[\sigma P(\sigma)], \tag{14}$$

$$(peL)^{1/p} = \limsup_{n=\infty} \left| a_n \right|^{1/\lambda_n} \exp \phi(\lambda_n), \tag{15}$$

$$(peQ_J)^{1/p} = \liminf_{j=\infty} \left| a_{n_j} \right|^{1/\lambda_{n_j}} \exp \phi(\lambda_{n_j}), \tag{16}$$

$$l_J = \limsup_{j=\infty} \lambda_{n_{j+1}}/\lambda_{n_j}. \tag{17}$$

Under these definitions and notations we will now apply to the present problem, the technique used by Valiron [9, p. 42] in the case of Taylor series, to obtain the following results:

THEOREM 1. *Let $P(\sigma)$ be chosen satisfying the conditions stated above. Then,* $A=L$.

**Proof.** Let us assume first that $0<A<\infty$. With $\alpha=A+\epsilon$ it follows from (10) and (13) that for all $\sigma$ large enough

$$m(\sigma) < \mu[\sigma, (A+\epsilon)pe]. \tag{18}$$

On the other hand with $\alpha=A-\epsilon$ we obtain

$$m(\sigma_k) > \mu[\sigma_k, (A-\epsilon)pe] \quad \text{for some } \sigma_k \uparrow \infty \qquad (k=0, 1, 2, \cdots). \tag{19}$$

The inequality (18) implies that from some $n$ on

$$\left| a_n \right| < \Gamma[\lambda_n, (A+\epsilon)pe] \tag{20}$$

since, otherwise there would be a sequence $M \equiv \{n_m\}$ $(m=0, 1, 2, \cdots)$, such that $|a_{n_m}| \geqq \Gamma[\lambda_{n_m}, (A+\epsilon)pe]$ and for those values of $\sigma=\sigma_m$ such that $\lambda_{n_m}=\lambda[\sigma_m, (A+\epsilon)pe]$ we would have, according to the definition of $\lambda(\sigma, \alpha)$: $m(\sigma_m) \geqq |a_{n_m}\exp(\sigma_m\lambda_{n_m})| \geqq \Gamma[\lambda_{n_m}, (A+\epsilon)pe]\exp(\sigma_m\lambda_{n_m}) = \mu[\sigma_m, (A+\epsilon)pe]$ which contradicts (18). From (19) we conclude that there is a sequence $J \equiv \{n_j\}$ such that

$$\left| a_{n_j} \right| > \Gamma[\lambda_{n_j}, (A-\epsilon)pe] \tag{21}$$

because, otherwise, it would be $|a_n| \leqq \Gamma[\lambda_n, (A-\epsilon)pe]$ from some $n$ on and therefore for all large $\sigma$, $m(\sigma) = |a_{n(\sigma)}\exp(\lambda_{n(\sigma)}\sigma)| \leqq \Gamma[\lambda_{n(\sigma)}, (A-\epsilon)pe]\exp(\lambda_{n(\sigma)}\sigma)$ $\leqq \mu[\sigma, (A-\epsilon)pe]$ contradicting (19).

From the definition of $\Gamma(\lambda, \alpha)$ and (20), (21) and (15) we have $A=L$. If $A=0$ the inequalities (18) and (20) hold for any $\epsilon>0$. It follows that

$L<\epsilon$ and therefore $L=0=A$. If $A=\infty$ then the inequalities (19) and (21) hold with any arbitrarily large number substituted for $A$ and the conclusion $A=L=\infty$ follows immediately.

THEOREM 2. *Let $P(\sigma)$ satisfy the same conditions as above. Then $Q_I \geqq B$.*

**Proof.** Assume first $0<B<\infty$. By (14) and (10) we have $m(\sigma) > \mu[\sigma, (B-\epsilon)pe]$ and therefore the polygon $\pi$ is dominated by the curve $y=-\log \Gamma[\lambda, (B-\epsilon)pe]$ for all real values of $\lambda$ and, in particular, $-\log|a_{n_i}| < -\log \Gamma[\lambda_{n_i}, (B-\epsilon)pe]$ for the sequence $I$ and the result follows.

If $B=\infty$ all the previous inequalities hold with any arbitrarily large number substituted for $B$ and therefore $Q_I=\infty=B$. If $B=0$ the result is trivial.

THEOREM 3. *Under the same assumptions for $P(\sigma)$ and with any $J$ such that $l_J<\infty$ we have $B \geqq Q_J/l_J$.*

**Proof.** If $0<Q_J<\infty$ then by (16) we have $|a_{n_j}| > \Gamma[\lambda_{n_j}, (Q_J-\epsilon)pe]$ for all large $j$.

If the sequence $\sigma_j \uparrow \infty$ is defined by $\lambda_{n_j}=\lambda[\sigma_j, (Q_J-\epsilon)pe]$ then for all $\sigma$ such that $\sigma_j \leqq \sigma < \sigma_{j+1}$ we obtain by (10), with $\alpha=(Q_J-\epsilon)pe$

$$(22)\quad \log m(\sigma) \geqq \log m(\sigma_j) > \log \mu[\sigma_j, (Q_J-\epsilon)pe] = (Q_J-\epsilon)\exp[\sigma_j P(\sigma_j)]u(\sigma_j),$$

where $\lim_{j=\infty} u(\sigma_j)=1$. On the other hand by (11)

$$(23)\quad \exp[\sigma P(\sigma)]/\exp[\sigma_j P(\sigma_j)] \sim \lambda[\sigma, (Q_J-\epsilon)pe]/\lambda[\sigma_j, (Q_J-\epsilon)pe] = \psi(\sigma),$$

and since, by (23) and (17), $1 \leqq \psi(\sigma) < \lambda_{n_{j+1}}/\lambda_{n_j} < l_J+(\epsilon/2)$ for all large $j$, it follows

$$1-\epsilon < \exp[\sigma P(\sigma)]/\exp[\sigma_j P(\sigma_j)] < l_J+\epsilon$$

and by (22), $\log m(\sigma)/\exp[\sigma P(\sigma)] \geqq (Q_J-\epsilon)/(l_J+\epsilon)$ for all large $\sigma$ and since $\epsilon$ is arbitrarily small we have finally $B \geqq Q_J/l_J$. The usual argument shows that the conclusion is equally valid when $Q_J=\infty$ and the proof is complete.

Finally for the case of finite order and regular growth, i.e.: $0<\tau=\rho<\infty$, we have under the same assumptions for $P(\sigma)$ the following

THEOREM 4. *If $p=\rho=\tau$ and $\infty>A\geqq B>0$, then $l_I<\infty$ and $Q_I \geqq B \geqq Q_I/l_I$.*

**Proof.** Obviously in view of Theorems 2 and 3 we need only to prove that $l_I<\infty$. The same arguments used in the proofs of those theorems show that the ordinate $\eta$ of the polygon $\pi$ satisfies $-\log \Gamma[\lambda, (A+\epsilon)pe]=y_1<\eta<y_2 = -\log \Gamma[\lambda, (B-\epsilon)pe]$ for any $\epsilon>0$ arbitrarily small and all large $\lambda$. This implies that the length of each side of $\pi$ is not greater than the segment of the tangent to the graph of $y_2$ parallel to that side and bounded by its intersections with $y_1$ [9, pp. 42–46]. If $\lambda_0$ is the abscissa of the contact point, we will prove that the abscissas $\lambda_0'$ and $\lambda_0''$ of those intersections satisfy

$$(24)\qquad \beta_1\lambda_0 < \lambda_0' < \lambda_0 < \lambda_0'' < \beta_2\lambda_0, \qquad 0<\beta_1<1<\beta_2<\infty,$$

where the constants $\beta_1$ and $\beta_2$ depend only on $\epsilon$. It follows then easily that $l_I < (\beta_2/\beta_1)+\epsilon' < \infty$ for any fixed given values of $\epsilon > 0$ and $\epsilon' > 0$.

To establish (24) we consider the equation of the tangent to $y_2$ at the point of abscissa $\lambda_0$:

$$y = \lambda\lambda_0\phi'(\lambda_0) + \lambda\phi(\lambda_0) - (\lambda/p)\log[(B-\epsilon)pe] - \lambda_0^2\phi'(\lambda_0).$$

The abscissas of the intersections with $y_1$ are the roots of the equation

$$F(\lambda) \equiv [\phi(\lambda) - \phi(\lambda_0)]\lambda + (\lambda/p)\log[(B-\epsilon)/(A+\epsilon)] - (\lambda-\lambda_0)\lambda_0\phi'(\lambda_0) = 0.$$

With $\gamma = (B-\epsilon)/(A+\epsilon)$ and $\lambda = \beta\lambda_0$ this equation is equivalent to the following equation in $\beta$:

$$\begin{aligned} H(\beta,\lambda_0) \equiv F(\beta\lambda_0)/\lambda_0 = [\log(\beta\lambda_0)/\varphi(\beta\lambda_0) - \log\lambda_0/\varphi(\lambda_0)]\beta + (\beta/p)\log\gamma \\ - (\beta-1)[\varphi(\lambda_0) - \lambda_0\varphi'(\lambda_0)\log\lambda_0]/\varphi(\lambda_0)^2 = 0. \end{aligned} \tag{25}$$

Now, for any fixed $\beta > 0$ we have

$$\begin{aligned} &\lim_{\lambda_0=\infty}[\log(\beta\lambda_0)/\varphi(\beta\lambda_0) - \log\lambda_0/\varphi(\lambda_0)] \\ &= \lim_{\lambda_0=\infty}\{\log\beta/\varphi(\beta\lambda_0) + [1/\varphi(\beta\lambda_0) - 1/\varphi(\lambda_0)]\log\lambda_0\} = (1/p)\log\beta \end{aligned} \tag{26}$$

because, by the mean value theorem,

$$\begin{aligned} &[1/\varphi(\beta\lambda_0) - 1/\varphi(\lambda_0)]\log\lambda_0 = (\beta-1)\lambda_0\varphi'(\overset{*}{\lambda}_0)\log\lambda_0/\varphi(\overset{*}{\lambda}_0)^2 \\ &\quad = (\beta-1)[\overset{*}{\lambda}_0\varphi'(\overset{*}{\lambda}_0)\log\overset{*}{\lambda}_0](\lambda_0/\overset{*}{\lambda}_0)(\log\lambda_0/\log\overset{*}{\lambda}_0)/\varphi(\overset{*}{\lambda}_0)^2 \to 0 \qquad \text{for } \overset{*}{\lambda}_0 \to \infty \end{aligned}$$

as $\lambda_0 \to \infty$ and the factor in brackets tends to zero as all the others remain bounded, and $\varphi(\overset{*}{\lambda}_0)^2 \to p^2 > 0$.

We conclude from (25) and (26) that

$$\lim_{\lambda_0=\infty} H(\beta,\lambda_0) = (\beta/p)\log(\beta\gamma) - (\beta-1)/p. \tag{27}$$

This expression is negative for $\beta = 1$, but positive for all $\beta > \beta_2$ for some finite large enough $\beta_2$, and also positive for all $\beta$ such that $0 < \beta < \beta_1$ for some finite $\beta_1$, $(0 < \beta_1 < 1)$, because the limits of (27) as $\beta \to \infty$ and $\beta \to 0$ are respectively $+\infty$ and $1/p$. It is therefore possible to find a constant $C(\epsilon)$ such that for all $\lambda_0 > C(\epsilon)$ we have $F(\beta_r\lambda_0) \equiv \lambda_0 H(\beta_r, \lambda_0) > 0$ $(r = 1, 2)$ and this together with $F(\lambda_0) < 0$ proves both (24) and the theorem.

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