# LOCAL OPERATORS ON TRIGONOMETRIC SERIES

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1. A. Zygmund proved the following theorem in his book [4, p. 146]. Suppose that

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{1}$$

is a Fourier-Stieltjes series of $F(x)$ with $F(x) = \{F(x+0)+F(x-0)\}/2$. A condition necessary and sufficient for $F(x)$ to be absolutely continuous over a closed interval $(a, b)$ is

$$\int_a^b |\sigma_m(x) - \sigma_n(x)|\, dx \to 0, \qquad \text{as } m, n \to \infty$$

where $\sigma_m(x)$ are the arithmetic means of the series (1). We shall generalize this theorem to the successively derived Fourier series and prove some related theorems. With an application of these results, we shall prove a theorem concerning local saturation. The author proved this theorem previously [2]. But the proof given here is simpler than the previous one.

2. Suppose that $f(x)$ is integrable and periodic with the period $2\pi$. Moreover suppose that its Fourier series and conjugate series are

$$S[f] = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{2}$$

$$\tilde{S}[f] = \sum_{n=-\infty}^{\infty} -i\,(\operatorname{sign} n) c_n e^{inx} \tag{3}$$

and their $k$th derived series are

$$S^{(k)}[f] = \sum_{n=-\infty}^{\infty} (in)^k c_n e^{inx}, \tag{4}$$

$$\tilde{S}^{(k)}[f] = \sum_{n=-\infty}^{\infty} -i(\operatorname{sign} n)(in)^k c_n e^{inx} \tag{5}$$

respectively. We denote by $\sigma_m^l[x, S^{(k)}]$ or $\sigma_m^l[x, \tilde{S}^{(k)}]$ the $(C, l)$ means of (4) or (5).

THEOREM 1([1]). *A necessary and sufficient condition for $f^{(k)}(x)$ or $\tilde{f}^{(k)}(x)$,*

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([1]) We suppose $k$ is a positive integer. When $k=0$, the theorem is still valid, if we take $\sigma_m^1[x, S]$ instead of $\sigma_m^0[x, S]$. This remark shall be applied to the following Theorems 2, 3 and 4.

*respectively, to exist and belong to the class C over the closed interval $(a, b)$ is the uniform convergence of $\sigma_m^k[x, S^{(k)}]$ or $\sigma_m^k[x, \tilde{S}^{(k)}]$ over the closed interval $(a, b)$, respectively.*

**Proof.** *Necessity.* Since $c_n = o(1)$ by the Riemann-Lebesgue theorem, $S^k[f]$ is uniformly summable $(C, k)$ to $f^{(k)}(x)$ over $(a, b)$, provided that $f^{(k)}(x)$ exists and is continuous over this interval (see Zygmund [4, p. 367, Theorem 9.20]). Hence the condition is necessary. For the derived conjugate series, the proof is the same.

*Sufficiency.* If $\sigma^k[x, S^{(k)}]$ converges uniformly over $(a, b)$, the limit function $g(x)$ is continuous over $(a, b)$. We denote by $C_0^{(k)}(a, b)$ the class of functions such as continuously differentiable $k$-times and vanishing outside of $(a, b)$. For any $h(x) \in C_0^{(k)}(a, b)$

$$\lim_{m\to\infty} \int_0^{2\pi} h(x)\sigma_m^k[x, S^{(k)}]dx = \int_0^{2\pi} h(x)g(x)dx = (-1)^k \int_0^{2\pi} h^{(k)}(x)G_k(x)dx$$

where $G_k(x)$ is a $k$th primitive of $g(x)$. On the other hand, since the Cesàro kernel is symmetric, we have

$$(-1)^k \int_0^{2\pi} h(x)\sigma_m^k[x, S^{(k)}(f)]dx = \int_0^{2\pi} \sigma_m^k[x, S^{(k)}(h)]f(x)dx$$

$$= \int_0^{2\pi} \sigma_m^k[x, S(h^{(k)})]f(x)dx \to \int_0^{2\pi} h^{(k)}(x)f(x)dx,$$

as $m \to \infty$.

Hence

$$\int_0^{2\pi} h^{(k)}(x)G_k(x)dx = \int_0^{2\pi} h^{(k)}(x)f(x)dx.$$

By a well-known theorem [1, p. 201], $G_k(x) - f(x)$ is an algebraic polynomial of the degree $k-1$ over $(a, b)$, and almost everywhere $f^{(k)}(x) = g(x)$, which is continuous over $(a, b)$.

For the derived conjugate series, since $\sigma_m^k[x, \tilde{S}^{(k)}]$ converges uniformly over $(a, b)$, the limit function $l(x)$ is continuous. For any $h(x) \in C_0^{(k+1)}(a, b)$, we have

$$\lim_{m\to\infty} \int_0^{2\pi} h(x)\sigma_m^k[x, \tilde{S}^{(k)}]dx = \int_0^{2\pi} h(x)l(x)$$

$$= (-1)^{k+1} \int_0^{2\pi} h^{(k+1)}(x)L_{k+1}(x)dx,$$

where $L_{k+1}(x)$ is a $(k+1)$th primitive of $l(x)$. But

$$
\begin{aligned}
(-1)^k \int_0^{2\pi} h(x)\overset{k}{\sigma}_m[x, \tilde{S}^{(k)}(f)]dx &= \int_0^{2\pi} \overset{k}{\sigma}_m[x, \tilde{S}^{(k)}(h)]f(x)dx \\
&= \int_0^{2\pi} \overset{k}{\sigma}_m[x, S(\tilde{h}^{(k)})]f(x)dx \to \int_0^{2\pi} \tilde{h}^{(k)}(x)f(x)dx \\
&= -\int_0^{2\pi} \tilde{h}^{(k+1)}(x)F(x)dx = -\int_0^{2\pi} h^{(k+1)}(x)\tilde{F}(x)dx,
\end{aligned}
$$

by the Parseval relation, where $F(x)$ is an integral of $f(x)$. Hence $\tilde{F}(x) - L_{k+1}(x)$ is a polynomial of degree $k$, and the theorem is proved.

THEOREM 2. *A necessary and sufficient condition for $f^{(k)}(x)$ or $\tilde{f}^{(k)}(x)$, respectively, to exist and belong to the class $L^\infty$ over $(a, b)$ is the uniform boundedness of $\overset{k}{\sigma}_m[x, S^{(k)}]$ or $\overset{k}{\sigma}_m[x, \tilde{S}^{(k)}]$ over $(a, b)$, respectively.*

**Proof.** Since the proof of necessity is almost the same as the proof of necessity part of the above theorem, we omit it.

*Sufficiency.* Since $\overset{k}{\sigma}_m[x, S^{(k)}]$ is uniformly bounded over $(a, b)$, this sequence is weakly compact as functional in $L^\infty(a, b)$. That is, for any $h(x) \in C_0^{(k)}(a, b)$, there is a function $g(x) \in L^\infty(a, b)$ and subsequence $m_n$ such that

$$
\lim_{n\to\infty} \int_0^{2\pi} h(x)\overset{k}{\sigma}_{m_n}[x, S^{(k)}]dx = \int_0^{2\pi} h(x)g(x)dx = (-1)^k \int_0^{2\pi} h^{(k)}(x)G_k(x)dx.
$$

But

$$
\begin{aligned}
\int_0^{2\pi} h(x)\overset{k}{\sigma}_{m_n}[x, S^{(k)}(f)]dx &= (-1)^k \int_0^{2\pi} \overset{k}{\sigma}_{m_n}[x, S^{(k)}(h)]f(x)dx \\
&\to (-1)^k \int_0^{2\pi} h^{(k)}(x)f(x)dx, \qquad \text{as } n \to \infty,
\end{aligned}
$$

and we get sufficiency of the theorem with the same reasoning as the above theorem.

In the similar way, we can get the following theorems.

THEOREM 3. *A necessary and sufficient condition for $f^{(k)}(x)$ or $\tilde{f}^{(k)}(x)$, respectively, to exist and belong to the class $L^p$ $(p > 1)$ over $(a, b)$ is*

$$
\int_a^b \left| \overset{k}{\sigma}_m[x, S^{(k)}] \right|^p dx = O(1) \quad or \quad \int_a^b \left| \overset{k}{\sigma}_m[x, \tilde{S}^{(k)}] \right|^p dx = O(1), \qquad respectively.
$$

THEOREM 4. *A necessary and sufficient condition for $f^{(k)}(x)$ or $\tilde{f}^{(k)}(x)$, respectively, to exist and belong to the class $L$ over $(a, b)$ is*

$$
\int_a^b \left| \overset{k}{\sigma}_m[x, S^{(k)}] - \overset{k}{\sigma}_n[x, S^{(k)}] \right| dx \to 0, \qquad as\ m, n \to \infty
$$

*or*

$$\int_a^b \left| \overset{k}{\sigma}_m[x, \tilde{S}^{(k)}] - \overset{k}{\sigma}_n[x, \tilde{S}^{(k)}] \right| dx \to 0, \qquad \textit{as } m, n \to \infty,$$

*respectively.*

THEOREM 5. *A necessary and sufficient condition for $f^{(k)}(x)$ or $\tilde{f}^{(k)}(x)$, respectively, to exist and belong to the class BV over $(a, b)$ is*

$$\int_a^b \left| \overset{k}{\sigma}_n[x, S^{(k+1)}] \right| dx = O(1) \quad \textit{or} \quad \int_a^b \left| \overset{k}{\sigma}_n[x, \tilde{S}^{(k+1)}] \right| dx = O(1),$$

*respectively.*

3. Concerning the local saturation of Fejér means, the author [2] proved the following theorem.

THEOREM 6([2]). (1) *If*

$$f(x) - \sigma_n(x, f) = o(1/n)$$

*uniformly over $(a, b)$, then $\tilde{f}(x)$ is a constant over $(a, b)$ and vice versa.*

(2) *If*

$$f(x) - \sigma_n(x, f) = O(1/n)$$

*uniformly over $(a, b)$, then $\tilde{f}'(x)$ is bounded over $(a, b)$ and vice versa.*

**Proof.** The proofs of the propositions (1) and (2) are almost the same. So we shall only give the proof of the proposition (2).

If

$$f(x) - \sigma_n(x, f) = O(1/n)$$

uniformly over $(a, b)$, then

$$\sigma_m[x, n\{f(x) - \sigma_n(x, f)\}] = O(1),$$

for every $m$ and uniformly in $x$ in any fixed internal subinterval of $(a, b)$, because([3])

$$f(x) - \sigma_n(x,f) \sim \sum_{k=1}^{n-1} \{1 - (1 - k/n)\} A_k(x) + \sum_{k=n}^{\infty} A_k(x).$$

Letting $n \to \infty$, we have

$$\sigma_m[x, \tilde{S}'(f)] = O(1).$$

Hence we have $\tilde{f}'(x) \in L^\infty$ in $(a, b)$ from the Theorem 2.

([2]) We denote $\sigma_n^1(x, f)$ by $\sigma_n(x, f)$.

([3]) We use here a theorem of Zygmund [4, p. 367, 9.20], again. And the uniformity in $n$ is seen from the proof of that theorem.

Conversely we suppose $\tilde{f}'(x) \in L^\infty$ over $(a, b)$.

Then we get from the Theorem 2

$$\sigma_m[x, \tilde{S}'(f)] = O(1)$$

over $(a, b)$. But

$$n\{f(x) - \sigma_n(x, f)\} \sim \sum_{k=1}^{n-1} kA_k(x) + \sum_{k=n}^{\infty} (n/k)kA_k(x).$$

The sequence

$$l_k = \begin{cases} 1, & \text{for } k = 1, 2, \cdots, n-1, \\ n/k, & \text{for } k = n, n+1, \cdots \end{cases}$$

is a multiplier acting on $\tilde{S}'[f]$. In fact, an easy calculation yields

$$\sum_{k=1}^{\infty} k \left| \Delta^2 l_k \right| = O(1).$$

Hence we get

$$\sigma_m[x, n\{f(x) - \sigma_n(x, f)\}] = O(1)$$

uniformly in $m$ and $n$ over $(a, b)$, by the classical Bohr-Hardy theorem (see [3, p. 105]). This is equivalent to saying that $n\{f(x) - \sigma_n(x, f)\}$ belongs to the class $L^\infty(a, b)$ uniformly in $n$. Since $f(x) - \sigma_n(x, f)$ is continuous over this interval, we have

$$n\{f(x) - \sigma_n(x, f)\} = O(1)$$

uniformly for all $x$ belonging to $(a, b)$.

Thus we get the complete proof of the proposition (2).

REMARK. Theorem 5 is true for $(C, \alpha)$-means $(\alpha \geqq 1)$ instead of $(C, 1)$-means. The proof is the same.

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