# $m$-PSEUDOCOMPACTNESS

BY

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As is pointed out in [1], a topological space $X$ is pseudocompact iff every real continuous image of $X$ is compact. Hence for $X$ to be pseudocompact, it is necessary both that every real continuous image of $X$ be closed and that every such image be bounded. These two conditions imply one another and are each sufficient for pseudocompactness. Thus, we have the more familiar definition that $X$ is pseudocompact iff every real continuous function on $X$ has bounded range. Every countably compact space is pseudocompact, and a near converse is that every pseudocompact normal space is countably compact.

$m$-pseudocompactness generalizes the notion of pseudocompactness in the following way; where $m$ is a cardinal, $R$ denotes the real numbers, and $R^m$ is the cartesian product of $R$ with itself $m$ times ($R^m$ is canonically a topological ring):

DEFINITION. A topological space $X$ is $m$-pseudocompact if every $R^m$-valued continuous function on $X$ has compact range.

We proceed to state and prove generalizations of the above properties of pseudocompactness. We show that $\aleph_0$-pseudocompactness is equivalent to pseudocompactness hence every normal, $\aleph_0$-pseudocompact space is $\omega_0$-compact (i.e., countably compact). Our main theorem is the result of trying to prove that every normal, $\aleph_\alpha$-pseudocompact space is $\omega_\alpha$-compact. (We prove a somewhat weaker theorem.) We also generalize the theorems that a pseudocompact, metric space is compact and that the product of a pseudocompact space with a compact space is pseudocompact.

Auxiliary propositions include some results on the ring $C^m(X)$ of all continuous $R^m$-valued functions on $X$. (This paper was written while reading about the theory of $C(X)=C^1(X)$ in the interesting new book by L. Gillman and M. Jerison, [1].) A brief section of examples and counter-examples is included.

The definitions and known theorems quoted above will be found in [1] or in Kelley's *General topology* [4]. The notation is mainly borrowed from [1]. $A \subset X$ is $C$-embedded (in $X$) iff every function $f \in C(A)$ has at least one extension to a function in $C(X)$. We generalize this in the obvious way to define *$C^m$-embedded.* We use $0_m$ as the zero of $R^m$. If $f \in C^m(X)$, we define $Z_m(f)$ (or

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simply $Z(f)$ if there is no danger of confusion) as the full inverse image of $0_m$. A *zero-set* is defined as a $Z_1(f)$ for $f \in C(X)$; a *cozero-set* is the complement of a zero-set. For convenience, we shall use such abbreviations as $\bigcap \mathcal{S} = \bigcap \{S/S \in \mathcal{S}\}$ (for $\mathcal{S}$ a family of sets)—and $Z(\mathfrak{F}) = \{Z(f)/f \in \mathfrak{F}\}$ (for $\mathfrak{F}$ a family of functions), i.e., $\mathfrak{F} \subset C^m(X)$. The cardinal of the set $S$ will be denoted by $|S|$, and $A^-$ will represent the closure of $A$. We shall generally assume that a topological space is completely regular and Hausdorff.

1. **Structure of the ring $C^m(X)$.** Let $M$ be a set with $|M| = m$. Then $R^m$ can be represented explicitly by $R^M$ (the set of all functions from $M$ into $R$). For each $y \in M$, we have a projection map $\pi_y$: $R^M \rightarrow R$. Thus, if we are given $f \in C^m(X)$, then $\pi_y \circ f \in C(X)$ for all $y \in M$. In other words, $\{\pi_y \circ f/y \in M\}$ is in $C(X)^M$. Conversely, given $\{f_y/y \in M\} \in C(X)^M$, the function $f: X \rightarrow R^M$, determined by $\pi_y \circ f = f_y$, is an evaluation map which is proved continuous in [4]. From this discussion it is easy to see how to prove:

THEOREM 1.1. *$C^m(X)$ is canonically isomorphic to the direct product of $C(X)$ with itself $m$ times.*

Since we shall apply this theorem frequently, it is useful to give a name to the above canonical isomorphism.

DEFINITION. If $f \in C^m(X)$, the set of functions $\mathfrak{F} = \{\pi_y \circ f\}$ shall be called *the projections of $f$ onto $C(X)$*. Conversely, given $\mathfrak{F} \subset C(X)$ with $|\mathfrak{F}| \leqq m$, we shall define $f$ as a *direct product of $\mathfrak{F}$ in $C^m(X)$* if $f \in C^m(X)$ and the projections of $f$ are precisely the set $\mathfrak{F}$. (Explicitly, if we index $\mathfrak{F}$ by $M$, we can take $f$ as the evaluation map from $X$ into $R^{\mathfrak{F}}$.)

From Theorem 1.1, it immediately follows that a subset is $C^m$-embedded iff it is $C$-embedded.

Although $C^m(X)$ is algebraically just the direct product of $C(X)$ with itself $m$ times, it has certain algebraic properties which are not immediately reflected in $C(X)$. For example, if we supply $M$ with the *discrete* topology, Theorem 1.1 shows that $C^m(X) = C(X \times M)$. (Each $f \in C^m(X)$ may be regarded as the mapping $(x, y) \rightarrow (\pi_y \circ f)(x)$ from $X \times M$ to $R$.) Thus, if $Y$ is a topological space with $|Y| = m$, then each map from $M$ onto $Y$ induces an isomorphism from $C(X \times Y)$ into $C(X \times M)$; hence $C(X \times Y)$ may be identified with a subring of $C^m(X)$.

Finally we observe that $C^m(X)$ (as well as $R^m$) has an absolute value. (If $r \in R^m$, then $|r|$ is the element determined by $\pi_y|r| = |\pi_y r|$. If $f \in C^m(X)$ define $|f|$ by $|f|(x) = |f(x)|$.) We trust there will be no confusion between $|f|$, the absolute value of $f$, and $|S|$, the cardinal of $S$.

2. **Some properties of $m$-pseudocompact spaces.** It is obvious that if $X$ is $m$-pseudocompact and $q \leqq m$, then $X$ is $q$-pseudocompact. By definition, 1-pseudocompactness is equivalent to pseudocompactness. On the other hand, if $X$ is pseudocompact and $f \in C^{\aleph_0}(X)$, then $f(X)$ is pseudocompact, metric hence compact. Thus, $X$ is $\aleph_0$-pseudocompact. We have therefore proven:

Theorem 2.1. *$X$ is pseudocompact iff $X$ is $m$-pseudocompact for any (every) countable $m$.*

If $X$ is a pseudocompact subset of the realcompact space $S$, then $X^-$ is both pseudocompact and realcompact hence is compact. Thus, if we assume $X$ is pseudocompact, and $f\in C^m(X)$, we have that $[f(X)]^-$ is compact. It is now easy to see that $X$ is $m$-pseudocompact iff every $f\in C^m(X)$ has *closed* range.

Definition. A family of sets, $\{A_i/i\in I\}$, is said to have $\aleph_\alpha$-i.p. (the $\aleph_\alpha$-intersection property) if every subfamily $J\subset I$ with $|J|\leqq\aleph_\alpha$ has the property that $\bigcap\{A_i/i\in J\}\neq\emptyset$.

Theorem 2.2. *$X$ is $\aleph_\alpha$-pseudocompact iff every collection of zero-sets having f.i.p. has $\aleph_\alpha$-i.p.*

**Proof.** Assume $X$ is $\aleph_\alpha$-pseudocompact. Let $\mathfrak{F}$ be a family of functions in $C(X)$ with $|\mathfrak{F}|=\aleph_\alpha$ and such that $Z(\mathfrak{F})$ has f.i.p. Then let $g\in C^{\aleph_0}(X)$ be a direct product of the functions in $\mathfrak{F}$. Clearly $0_{\aleph_\alpha}\in[g(X)]^-$, in view of the f.i.p. hypothesis; so $0_{\aleph_\alpha}\in g(X)$ by $\aleph_\alpha$-pseudocompactness; hence $\bigcap Z(\mathfrak{F})\neq\emptyset$.

Conversely, if $X$ satisfies the hypothesis that every collection of zero-sets with f.i.p. has $\aleph_\alpha$-i.p., then any continuous image of $X$ also satisfies this hypothesis. In other words, if $f\in C^{\aleph_\alpha}(X)$ and $\mathfrak{O}$ is a cozero cover of $f(X)$ with $|\mathfrak{O}|\leqq\aleph_\alpha$, then $\mathfrak{O}$ has a finite subcover. Now this property implies $f(X)$ is compact by Alexander's subbase theorem as $R^{\aleph_\alpha}$ admits of a cozero subbase $\mathcal{S}$ with $|\mathcal{S}|=\aleph_\alpha$(2).

In [1] it is proven that the product of a pseudocompact space with a compact space is pseudocompact. Glicksberg, in [2], proves that $\beta(X\times Y)=\beta X\times\beta Y$ if $X\times Y$ is pseudocompact. We have observed that if $S$ is pseudocompact and $f\in C^m(S)$, then $[f(S)]^-$ is compact. Using these three facts we can prove:

Theorem 2.3. *If $X$ is $m$-pseudocompact and $Y$ compact, then $X\times Y$ is $m$-pseudocompact.*

**Proof.** We know $X\times Y$ is pseudocompact, $\beta(X\times Y)=\beta X\times Y$. Let $g\in C^m(X\times Y)$ be given. Then $[g(X\times Y)]^-$ is compact, so $g$ has a Stone extension $g^*:\beta X\times Y$ onto $[g(X\times Y)]^-$. It suffices to show $g(X\times Y)=g^*(X\times Y)=g^*(\beta X\times Y)$.

But for all $y\in Y$, we have that $X\times\{y\}$ is homeomorphic to $X$ and hence is $m$-pseudocompact. Thus, the restriction of $g$ to $X\times\{y\}$ has compact range, $g(X\times\{y\})$. But this implies $g^*(\beta X\times\{y\})=g(X\times\{y\})$. Since this is true for all $y\in Y$, we have $g(X\times Y)=g^*(\beta X\times Y)$.

(2) [4, p. 139].

It is well known that every pseudocompact metric space is compact. We generalize and extend this result to arbitrary (completely regular) spaces.

DEFINITION. By the *uniform cardinal* of $X$ we mean the smallest cardinal $u$ for which $X$ admits a uniform structure (compatible with its topology) having a *base* of cardinality $u$.

THEOREM 2.4. *If $X$ is $u$-pseudocompact then $X$ is compact.*

**Proof.** It follows from the uniform metrization theorem that we may assume $X$ is embedded in a product of pseudometric spaces, $\times\{M_i/i\in I\}$, where $|I|=u$ and $M_i=\pi_i(X)$. Each $M_i$ is thus pseudocompact and pseudometric hence compact. In view of the Tychonoff product theorem it suffices to show $X$ is closed in $\times M_i$. But assume $y\in X^-$. Observe that $y$ may be taken as $\{y\}=\bigcap\mathcal{S}$ where $\mathcal{S}$ is a family of *zero-set neighborhoods* of $y$ and $|\mathcal{S}|=u$. Now $\mathcal{S}$ has f.i.p. even when restricted to $X$, so by Theorem 2.2, $\mathcal{S}$ has $u$-i.p. This implies $y\in X$ and so $X$ is closed.

If $u$ is the uniform cardinal of $X$, then, by an application of the uniform metrization theorem, every closed subset of $X$ is a $Z_u(f)$ for some $f\in C^u(X)$. We define the *perfect cardinal*, $p$, of $X$, as the least cardinal such that every closed subset of $X$ is a $Z_p(f)$. If $p=1$, $X$ must be normal (as mentioned in [1]) and in this case $X$ is called *perfectly normal* (by Kelley in [4]).

THEOREM 2.5. *If $X$ is $p$-pseudocompact, then $X$ is* **normal.**

**Proof.** Let $A$ and $B$ be disjoint closed subset of $X$ with $A=Z_p(f)$ and $B=Z_p(g)$. We shall assume $f$ and $g$ are non-negative, i.e., $f=|f|$ and $g=|g|$. Then obviously $0_p\notin[f(B)]^-$ or else we would have $0_p$ in the closure of $(f+g)(X)$ but not in $(f+g)(X)$ itself, a contradiction.

But $R^p$ is completely regular, hence there is a continuous function $h: R^p\rightarrow[0, 1]$, such that $h(0_p)=0$ and $h(f(B)^-)=1$. Therefore $h\circ f$ is 1 on $B$ and 0 on $A$.

The effect of normality on $m$-pseudocompactness is shown by:

THEOREM 2.6. *If $X$ is $m$-pseudocompact, normal, then every closed subset $A\subset X$ is also $m$-pseudocompact.*

**Proof.** Assume $A$ is not $m$-pseudocompact. Then there exists $f\in C^m(A)$ having projections $\mathfrak{F}=\{f_\alpha\}$ such that $Z(\mathfrak{F})$ has f.i.p. and $Z(f)=\bigcap Z(\mathfrak{F})=\emptyset$. Let $f^0\in C^m(X)$ be an extension of $f$. ($A$ is $C$-embedded as $X$ is normal, therefore $A$ is $C^m$-embedded.) Let $\mathfrak{F}^0$ be the projections of $f^0$. Then $Z(\mathfrak{F}^0)$ has f.i.p., hence $B=\bigcap Z(\mathfrak{F}^0)$ is nonempty, closed and disjoint from $A$. Let $g\in C^m(X)$ be such that $g(A)=0_m$ and $0_m\notin[g(B)]^-$. Then take $\mathcal{G}$ as the projections of $|f^0|+|g|$ onto $C(X)$. $Z(\mathcal{G})$ has f.i.p. (as it has f.i.p. on $A$)—however $\bigcap Z(\mathcal{G})$ $=\bigcap Z(\mathfrak{F}^0)\cap Z(g)=\emptyset$ which contradicts the $m$-pseudocompactness of $X$.

It is not true that every closed, even $C$-embedded, subset of an arbitrary $m$-pseudocompact space is $m$-pseudocompact. For example, consider the "big plank" $(\omega_1+1)\times(\omega_2+1)$ with the corner point $(\omega_1, \omega_2)$ removed. Then $S=\{(x, \omega_2)/x\in\omega_1\}$ is closed, $C$-embedded but not $\aleph_1$-pseudocompact. The plank—even without the corner point—is $\aleph_1$-pseudocompact. Proofs of very similar statements are in the last section of this paper (4. Examples and counter-examples).

**3. Relation between $\aleph_\alpha$-pseudocompactness and $\omega_\alpha$-compactness.**

Definition. Let $X$ be a topological space. Then $y\in X$ is a *complete accumulation point* of $S\subset X$, if every neighborhood $N$ of $y$ has the property that $|N\cap S|=|S|$.

We recall that a space $X$ is $\omega_\alpha$-*compact* iff any one of the following three equivalent conditions hold:

(1) Every open cover $\{0_i/i\in I\}$, with $|I|\leqq\aleph_\alpha$, has a finite subcover.

(2) Every family of closed sets with f.i.p. has $\aleph_\alpha$-i.p.

(3) Every infinite subset $A\subset X$ with $|A|\leqq\aleph_\alpha$ has a complete accumulation point in $X$.

By (2) and Theorem 2.2, every $\omega_\alpha$-compact space is $\aleph_\alpha$-pseudocompact. Is the converse true in the presence of normality? The proof of our main result concerning this question depends on some cardinal number analysis for which it is convenient to make the following apparently *ad hoc* definition:

Definition. $m$ is a *regular cardinal modulo* $q$ if for all $k<q$, we have $q^k\leqq m$.

Note. $2^m$ is always regular modulo $m'$ (the immediate successor of $m$). For if $k<m'$ then $k\leqq m$ hence $(m')^k\leqq(2^m)^m=2^m$ (for $m$ transfinite).

If we assume the generalized continuum hypothesis, then every *nonlimit*, transfinite cardinal is regular modulo itself. However, $\aleph_\omega$, for example, is not regular modulo itself in any event because $\aleph_\omega^{\aleph_0}>\aleph_\omega$ (which is pointed out in [3]).

Lemma. *Let $A$ be a set with $|A|\leqq q$. Suppose $m$ is regular modulo $q$. Let $\mathcal{P}=\{P\subset A/|P|<q\}$, then $|\mathcal{P}|\leqq m$.*

**Proof.** Let $k<q$ be given. Let $\mathcal{P}_k=\{P\subset A/|P|=k\}$. Then $|\mathcal{P}_k|\leqq|A|^k$, as is obvious from the definition of $|A|^k$ as the cardinal of a certain set of functions. Thus $|\mathcal{P}_k|\leqq m$, hence $|\mathcal{P}|\leqq m$ is clear.

Theorem 3.1. *Let $X$ be $m$-pseudocompact, normal. Then if $m$ is regular modulo $\aleph_\alpha$, $X$ is $\omega_\alpha$-compact.*

**Proof.** Assume the theorem is false. Then there exists an infinite set $A\subset X$ with no complete accumulation points and $|A|\leqq\aleph_\alpha$. By Theorem 2.6 we may assume $X=A^-$. Let $D$ be the set of all complete accumulation points of $A$ in $\beta X$. Then $D$ is closed, nonempty and $D\cap X=\varnothing$.

Let $\mathcal{S}=\{S\subset A/|A-S|<\aleph_\alpha\}$. By the lemma (essentially) $|\mathcal{S}|\leqq m$. For all $S\in\mathcal{S}$ define $\mathfrak{J}_S=\{(T, S)/T\subset A-S\}$. If $k=|A-S|$, we have $k<\aleph_\alpha$.

Thus, the cardinal of $\mathfrak{Z}_S$ is $2^k \leqq \aleph_\alpha^k \leqq m$ by hypothesis. Let $\mathfrak{Z} = \bigcup \mathfrak{Z}_S$, for $S \in \mathcal{S}$. Clearly, $|\mathfrak{Z}| \leqq m$.

Let $\mathfrak{Z}^*$ be the set of all $(T, S) \in \mathfrak{Z}$ for which there is at least one $f \in C(X)$ with $f(S) = 0$ and $f(T) = 1$. For all $(T, S)$ in $\mathfrak{Z}^*$ choose such an $f$ and call it $\gamma(T, S)$. Define $\mathfrak{F} = \{\gamma(T, S)/(T, S) \in \mathfrak{Z}^*\}$; then $\mathfrak{F} \subset C(X)$. Clearly $Z(\mathfrak{F})$ has f.i.p., so by $m$-pseudocompactness there is a $y \in X$ with $y \in \bigcap Z(\mathfrak{F})$. But $y \notin D$, so we can find $g$ in $C(\beta X)$ such that $g$ is 0 on a *neighborhood* of $D$ and $g$ is 1 on a *neighborhood* of $y$. From the definition of $D$, it is easy to see that $g$ is 0 on a set $S \in \mathcal{S}$. Moreover, as $y \in A^-$, there is a set $T \subset A - S$ such that $g$ is 1 on $T$ and $y \in T^-$. Hence $(T, S) \in \mathfrak{Z}^*$, let $g^* = \gamma(T, S)$. Then $g^*$ is 1 on $T$ so $g^*(y) = 1$, contradicting that $y \in \bigcap Z(\mathfrak{F})$.

COROLLARY 1. *Every pseudocompact normal space is countably compact.*

COROLLARY 2. *If $X$ is $2^{\aleph_\alpha}$-pseudocompact normal, then $X$ is $\omega_{\alpha+1}$-compact.*

## 4. Examples and counter-examples.

4.1. *Dense subsets.* If $X$ is a dense subset of $Y$ such that $X$, with the relative topology, is $m$-pseudocompact, then it is trivial to prove that $Y$ is also $m$-pseudocompact. Therefore, if $X$ is $m$-pseudocompact, any space between $X$ and $\beta X$ is also $m$-pseudocompact. This result leads naturally to a counter-example demonstrating the independence of $\aleph_\alpha$-pseudocompactness and $\omega_\alpha$-compactness. Consider the enlarged "Tychonoff plank": $(\omega_{\alpha+1}+1) \times (\omega_0 + 1)$ with the corner point $(\omega_{\alpha+1}, \omega_0)$ removed (note that $\omega_0 \in (\omega_0 + 1)$, etc.). Now $\omega_{\alpha+1} \times (\omega_0 + 1)$, which is $\aleph_\alpha$-pseudocompact by an application of Theorem 2.3, is dense in the plank, so the plank is $\aleph_\alpha$-pseudocompact. On the other hand, the plank is obviously not even countably compact.

4.2. *Generalized continuum hypothesis.* Suppose we assume the generalized continuum hypothesis $(2^{\aleph_\alpha} = \aleph_{\alpha+1})$. We then claim and sketch a rough proof that every normal $\aleph_\alpha$-pseudocompact space is $\omega_\alpha$-compact. For non-limit cardinals this follows from Corollary 2 of Theorem 3.1. If $\aleph_\alpha$ is a limit cardinal such that every unbounded subset of $\omega_\alpha$ necessarily has cardinality $\aleph_\alpha$, then $\aleph_\alpha$ can be shown to be regular modulo itself so Theorem 3.1 applies (e.g. $\aleph_0$ and the first inaccessible cardinal, if it exists, are regular modulo themselves). Finally we illustrate the remaining case by proving every $\aleph_\omega$-pseudocompact normal space $X$ is $\omega_\omega$-compact: Let $\mathfrak{F}$ be any family of closed sets of $X$ having f.i.p. Then by $\omega_n$-compactness, $\mathfrak{F}$ has $\aleph_n$-i.p. for all finite $n$. Let $i: \omega_\omega \to \mathfrak{F}$ index an arbitrary subset of $\mathfrak{F}$ having cardinal $\aleph_\omega$. By $\aleph_n$-i.p. we have that $S_n = \bigcap \{i(x)/x \leqq \omega_n\}$ is nonvoid; hence applying $\aleph_0$-i.p., $\bigcap S_n \neq \emptyset$. The general case would use an induction hypothesis but otherwise proceed the same way.

The same argument also shows that if $X$ is $\omega_n$-compact for all $n$, then $X$ is $\omega_\omega$-compact. An example of a space which is $\aleph_n$-pseudocompact for all $n$ but not $\aleph_\omega$-pseudocompact is given by $\times \{\omega_n + 1/n \in \omega_0\}$ with the far corner point, $\{\omega_n\}$, removed. (This space is not normal.)

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