SOME HILBERT SPACES OF ENTIRE FUNCTIONS. IV

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Recent work has been concerned with Hilbert spaces whose elements are entire functions and which have these three properties:

(H1) Whenever $F(z)$ is in the space and has a nonreal zero w, the function $F(z)(z - \bar{w})/(z - w)$ is in the space and has the same norm as $F(z)$.

(H2) Whenever w is a nonreal number, the linear functional defined on the space by $F(z) \to F(w)$ is continuous.

(H3) Whenever $F(z)$ is in the space, the function $F^*(z) = \bar{F}(\bar{z})$ is in the space and has the same norm as $F(z)$. If $E(z)$ is an entire function which satisfies the inequality

$$|E(\bar{z})| < |E(z)| \tag{1}$$

for $y > 0$ $(z = x + iy)$, we write $E(z) = A(z) - iB(z)$ where $A(z)$ and $B(z)$ are entire functions which are real for real z and

$$K(w,z) = [B(z)\bar{A}(w) - A(z)\bar{B}(w)]/[\pi(z-\bar{w})].$$

Let $\mathscr{H}(E)$ be the Hilbert space of entire functions $F(z)$ such that

$$\|F\|^2 = \int |F(t)|^2 \, |E(t)|^{-2} dt < \infty,$$

with integration on the real axis, and

$$|F(z)|^2 \leqq \|F\|^2 K(z,z)$$

for all complex z. Then, $\mathscr{H}(E)$ is a Hilbert space of entire functions which satisfies (H1), (H2), and (H3). For each complex number w, $K(w,z)$ belongs to $\mathscr{H}(E)$ as a function of z and

$$F(w) = \langle F(t), K(w,t) \rangle$$

for every $F(z)$ in $\mathscr{H}(E)$. As shown in [7], a Hilbert space, whose elements are entire functions, which satisfies (H1), (H2), and (H3), and which contains a nonzero element, is equal isometrically to some such $\mathscr{H}(E)$.

Conditions are given in Theorems II and III of [9] that one Hilbert space of entire functions be contained isometrically in another. These involve matrix valued entire functions

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$$M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix},$$

whose entries are real for real z and satisfy

$$\begin{aligned} A(z)D(z) - B(z)C(z) &= 1, \\ \operatorname{Re}\,[A(z)\bar{D}(z) - B(z)\bar{C}(z)] &\geqq 1, \\ [B(z)\bar{A}(z) - A(z)\bar{B}(z)]/(z-\bar{z}) &\geqq 0, \\ [D(z)\bar{C}(z) - C(z)\bar{D}(z)]/(z-\bar{z}) &\geqq 0 \end{aligned} \tag{2}$$

for all complex z. If $E(a,z)$ and $E(b,z)$ are entire functions which satisfy (1) and have no real zeros, and if $\mathscr{H}(E(a))$ is contained isometrically in $\mathscr{H}(E(b))$, there is a unique matrix valued entire function $M(a,b,z)$, satisfying (2), such that

$$(A(b,z),B(b,z)) = (A(a,z),B(a,z))\,M(a,b,z) \tag{3}$$

for all complex z.

THEOREM I. *If $E(a,z)$, $E(b,z)$, and $E(c,z)$ are entire functions which satisfy* (1) *and have no real zeros, and if $\mathscr{H}(E(a))$ and $\mathscr{H}(E(b))$ are contained isometrically in $\mathscr{H}(E(c))$, then either $\mathscr{H}(E(a))$ contains $\mathscr{H}(E(b))$ or $\mathscr{H}(E(b))$ contains $\mathscr{H}(E(a))$.*

If μ is a non-negative measure on the Borel sets of the real line, conditions are given by Theorem V of [8] that $\mathscr{H}(E)$ be contained isometrically in $L^2(\mu)$. One might ask whether the space $\mathscr{H}(E(c))$ of Theorem I can be replaced by $L^2(\mu)$. Although this is not the case in general, as finite dimensional examples will show, it is in the presence of a suitable growth restriction, for instance if we have functions $F(z)$ of exponential type which satisfy.

$$\int (1 + t^2)^{-1} \log^+ |F(t)|\,dt < \infty. \tag{4}$$

THEOREM II. *Let $E(a,z)$ and $E(b,z)$ be entire functions of exponential type with no real zeros, which satisfy* (1) *and* (4). *Let μ be a non-negative measure on the Borel sets of the real line. If $\mathscr{H}(E(a))$ and $\mathscr{H}(E(b))$ are contained isometrically in $L^2(\mu)$, then either $\mathscr{H}(E(a))$ contains $\mathscr{H}(E(b))$ or $\mathscr{H}(E(b))$ contains $\mathscr{H}(E(a))$.*

Because of (3), Theorem I has implications for the factorization of matrix valued entire functions satisfying (2).

THEOREM III. *Let $M(a,b,z)$, $M(a,c,z)$, $M(b,d,z)$, and $M(c,d,z)$ be matrix valued entire functions which satisfy* (2). *If*

$$M(a,b,z)M(b,d,z) = M(a,c,z)M(c,d,z) \tag{5}$$

for all complex z, then either

$$M(a,b,z)^{-1}\, M(a,c,z) \text{ or } M(a,c,z)^{-1}\, M(a,b,z)$$

is a matrix valued entire function which satisfies (2).

The conclusion of Theorem II may also be obtained under different hypotheses.

THEOREM IV. *Let* $E(b,z)$ *and* $E(c,z)$ *be entire functions which satisfy* (1) *and have no real zeros. Let* μ *be a non-negative measure on the Borel sets of the real line. If* $\mathscr{H}(E(b))$ *and* $\mathscr{H}(E(c))$ *are contained isometrically in* $L^2(\mu)$ *and if the intersection of* $\mathscr{H}(E(b))$ *and* $\mathscr{H}(E(c))$ *contains a nonzero element, then either* $\mathscr{H}(E(b))$ *contains* $\mathscr{H}(E(c))$ *or* $\mathscr{H}(E(c))$ *contains* $\mathscr{H}(E(b))$.

In [10] isometric inclusions of spaces of entire functions were obtained from first order differential equations. Let

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

be a matrix valued function of $t > 0$, where $\alpha(t)$, $\beta(t)$, $\gamma(t)$ are real valued, absolutely continuous functions of $t > 0$ such that

$$\alpha'(t) \geqq 0, \quad \gamma'(t) \geqq 0, \quad \beta'(t)^2 \leqq \alpha'(t)\, \gamma'(t) \tag{6}$$

a.e. for $t > 0$,

$$\lim_{t \searrow 0} \alpha(t) = 0, \tag{7}$$

and

$$\lim_{t \to \infty} [\alpha(t) + \gamma(t)] = \infty. \tag{8}$$

We will also study the special case in which

$$\lim_{t \searrow 0} \beta(t) = 0 \text{ and } \lim_{t \searrow 0} \gamma(t) = 0. \tag{9}$$

A real number $b > 0$ is said to be singular with respect to $m(t)$ if it belongs to an open interval (a,c) in which $\alpha'(t)$, $\beta'(t)$, $\gamma'(t)$ are equal a.e. to constant multiples of a single function and

$$\beta'(t)^2 = \alpha'(t)\, \gamma'(t)$$

a.e. Otherwise, a number $b > 0$ is said to be regular with respect to $m(t)$. Let

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In stating Theorem I of [10], we will suppose that $m(t)$ satisfies (6), (7), and (8), and that $\alpha(t) > 0$ for $t > 0$. Suppose that there exists a family $(E(t,z))$ of entire functions satisfying (1) with no real zeros, such that $E(t,0) = 1, t > 0$, such that for each complex number w, $E(t,w)$ is a continuous function of $t > 0$,

$$(A(b,w),B(b,w))I - (A(a,w),B(a,w))I = w \int_a^b (A(t,w),B(t,w))\, dm(t) \tag{10}$$

whenever $0 < a < b < \infty$, and

$$\lim_{a \searrow 0} K(a,w,w) = 0. \tag{11}$$

Then, when $a < b$ are regular points with respect to $m(t)$, $\mathscr{H}(E(a))$ is contained isometrically in $\mathscr{H}(E(b))$. If the regular points, with respect to $m(t)$, are unbounded, there is a unique non-negative measure μ on the Borel sets of the real line such that every $\mathscr{H}(E(a))$, with a regular, is contained isometrically in $L^2(\mu)$. In this case, the union of the spaces $\mathscr{H}(E(a))$, with a regular, is dense in $L^2(\mu)$. But if the regular points are unbounded, there is a largest one, call it b. Although there are many non-negative measures μ on the Borel sets of the real line such that $\mathscr{H}(E(b))$ is contained isometrically in $L^2(\mu)$, there is a unique one such that

$$\frac{y}{\pi}\int \frac{|E(b,t)|^2\, d\mu(t)}{(t-x)^2+y^2} = \lim_{c\to\infty} \frac{y}{\pi}\int \frac{|E(b,t)|^2\, |E(c,t)|^{-2}\, dt}{(t-x)^2+y^2}$$

for $y > 0$. In this case, $\mathscr{H}(E(b))$ fills $L^2(\mu)$.

The above construction is interesting because of Theorem II of [10]. Let $E(z)$ be an entire function which satisfies (1) and has no real zeros, such that $E(0) = 1$. Let v be a non-negative measure on the Borel sets of the real line such that $\mathscr{H}(E)$ is contained isometrically in $L^2(v)$. Then, there is a matrix valued function $m(t)$, as above, and there is a corresponding family $(E(t,z))$ in which the function $E(z)$ appears; that is, $E(z) = E(b,z)$ for some regular number $b > 0$. Furthermore, the construction can be made so that μ coincides with the given v. As a result, we are in possession of spaces of entire functions contained isometrically in $\mathscr{H}(E)$, and also of spaces containing $\mathscr{H}(E)$ but contained isometrically in $L^2(v)$. A consequence of Theorems I and IV is that we have obtained all the spaces associated with $\mathscr{H}(E)$ and $L^2(v)$ in this way.

THEOREM V. *Let $m(t)$ be a matrix valued function of $t > 0$ which satisfies* (6), (7), *and* (8) *with $\alpha(t) > 0$ for $t > 0$. We suppose given a corresponding family $(E(t,z))$ of entire functions satisfying* (1), (10) *and* (11) *with corresponding measure μ. Let $b > 0$ be regular with respect to $m(t)$ and let $E(z)$ be an entire function which satisfies* (1) *and has no real zeros.*

(A) *If $\mathscr{H}(E)$ is contained isometrically in $\mathscr{H}(E(b))$, then $\mathscr{H}(E)$ is equal isometrically to $\mathscr{H}(E(a))$ for some regular number a with $0 < a \leqq b$.*

(B) *If $\mathscr{H}(E)$ contains $\mathscr{H}(E(b))$ and is contained isometrically in $L^2(\mu)$, then $\mathscr{H}(E)$ is equal isometrically to $\mathscr{H}(E(c))$ for some regular number c with $b \leqq c < \infty$.*

In particular, these results apply to the situation of Theorems IV and VIII of [9], which give a more strongly stated special case of Theorem I of [10]. A related construction is given by Theorem VI of [9]. Let $m(t)$ be a matrix valued function of $t \geqq 0$ which satisfies (6), (7), and (9). Then, for each complex number w, there is a unique continuous matrix valued function $M(t,w)$ of $t \geqq 0$ such that

$$M(a,w)I - I = w \int_0^a M(t,w)dm(t) \tag{12}$$

for $a \geqq 0$. For each fixed $a \geqq 0$, $M(a,z)$ is a matrix valued entire function of z which satisfies (2) and $M(a,0) = 1$. Similarly, for each fixed $a \geqq 0$ and each complex number w, there is a unique continuous matrix valued function $M(a,t,w)$ of $t \geqq a$ such that

$$M(a,b,w)I - I = w \int_a^b M(a,t,w)dm(t) \tag{13}$$

whenever $b \geqq a$. For each fixed a and b, with $a \leqq b$, $M(a,b,z)$ is a matrix valued entire function of z which satisfies (2) and $M(a,b,0) = 1$. As in the proof of Theorem IV of [9], the uniqueness of these constructions implies that

$$M(b,z) = M(a,z)\,M(a,b,z) \tag{14}$$

for all complex z, whenever $a \leqq b$. The construction is useful because of Theorem VII of [9]. Let $M(z)$ be a given matrix valued entire function which satisfies (2). Then, there is some choice of matrix valued function $m(t)$, satisfying (6), (7), and (9), such that for some $b \geqq 0$,

$$M(z) = M(0)M(b,z)$$

for all complex z, where the family $(M(t,z))$ corresponds to $m(t)$ by (12). Then, (14) yields a factorization of $M(z)$ into matrix valued entire functions which satisfy (2). We will now show that these are the only such factorizations of $M(z)$.

THEOREM VI. *Let $m(t)$ be a matrix valued function of $t \geqq 0$ which satisfies* (6), (7), *and* (9), *and let $(M(t,z))$ be the corresponding family of matrix valued entire functions defined by* (12). *Let $b \geqq 0$. If $M(z)$ is a matrix valued entire function which satisfies* (2) *and if $M(z)^{-1}M(b,z)$ also satisfies* (2), *then*

$$M(z) = M(0)\,M(a,z) \tag{15}$$

for some number a with $0 \leqq a \leqq b$.

A similar result can be obtained for the construction of Theorem XI of [9]. Let $m(t)$ be a matrix valued function of $t \geqq 0$ which satisfies (6), (7), (8), and (9) with $\alpha(t) > 0$ for $t > 0$. Let $(M(t,z))$ be the corresponding family of matrix valued entire functions defined by (12). Then, the function $E(a,z) = A(a,z) - iB(a,z)$ satisfies (1) for every $a > 0$ and has no real zeros. The family $(E(t,z))$ so defined satisfies (10) and (11). Let μ be the corresponding non-negative measure on the

Borel sets of the real line, defined as in Theorem I of [10] or the here equivalent condition of Theorem XI of [9], where it is shown that

$$\int (1+t^2)^{-1}\, d\mu(t) < \infty. \tag{16}$$

As we have said, $\mathcal{H}(E(a))$ is contained isometrically in $L^2(\mu)$ when a is regular. Furthermore, it is known from Theorem IV of [8] that each $E(a,z)$ has exponential type and satisfies (4). By Theorem XII of [9], if ν is a non-negative measure on the Borel sets of the real line which satisfies (16) and does not vanish identically, then $\nu = \mu$ for some such choice of $m(t)$. Therefore, we know the existence of spaces of entire functions contained isometrically in the given $L^2(\nu)$. A consequence of Theorem II is that this construction yields the only spaces with the stated properties.

THEOREM VII. *Let $m(t)$ be a matrix valued function of $t \geqq 0$ which satisfies* (6), (7), (8), *and* (9) *with $\alpha(t) > 0$ for $t > 0$. Let $(M(t,z))$ be the corresponding family of matrix valued entire functions defined by* (12), *and let $E(t,z) = A(t,z) - iB(t,z)$. If $E(z)$ is an entire function of exponential type with no real zeros which satisfies* (1) *and* (4), *and if $\mathcal{H}(E)$ is contained isometrically in $L^2(\mu)$, then $\mathcal{H}(E)$ is equal isometrically to $\mathcal{H}(E(a))$ for some regular number $a > 0$.*

In applications, it is at times necessary to consider situations similar to those above except that the axiom (H3) is not satisfied. Only a small amount of additional information is needed to handle these cases.

THEOREM VIII. *Let $E(a,z)$ and $E(b,z)$ be entire functions which satisfy* (1) *and have no real zeros, such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$. Then,*

$$\tau = \lim_{y\to\infty} y^{-1} \log |E(b,iy)/E(a,iy)| \tag{17}$$

exists and $\tau \geqq 0$. If $-\tau \leqq h \leqq \tau$, then $e^{ihz}F(z)$ is in $\mathcal{H}(E(b))$ whenever $F(z)$ is in $\mathcal{H}(E(a))$.

THEOREM IX. *Let $E(b,z)$ be an entire function which satisfies* (1) *and has no real zeros. Let $\mathcal{H}$ be a closed subspace of $\mathcal{H}(E(b))$ such that $F(z)/(z-w)$ belongs to $\mathcal{H}$ whenever $F(z)$ belongs to $\mathcal{H}$ and $F(w) = 0$. If $\mathcal{H}$ contains a nonzero element, then there is a function $E(a,z)$ which satisfies* (1) *and has no real zeros, such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$, and there is a number h as in Theorem* VIII *with this property: an entire function $F(z)$ in $\mathcal{H}(E(b))$ belongs to $\mathcal{H}$ if, and only if, $e^{-ihz}F(z)$ belongs to $\mathcal{H}(E(a))$.*

THEOREM X. *Let $m(t)$ be a matrix valued function of $t > 0$ which satisfies* (6), (7), *and* (8) *with corresponding family $(E(t,z))$ of entire functions which satisfy* (1), (10), *and* (11). *If $0 < a \leqq b < \infty$, then*

$$\int_a^b [\alpha'(t)\gamma'(t)-\beta'(t)^2]^{1/2}\,dt = \lim_{y\to\infty} y^{-1}\log|E(b,iy)/E(a,iy)|. \tag{18}$$

Results for spaces of entire functions give information about certain kinds of integral transforms because of Theorem III of [10]. To avoid the complications of the general theorem, we will develop a special case which is notationally easy and yet is still useful for some applications.

THEOREM XI. *Let $m(t)$ be a matrix valued function of $t > 0$ which satisfies* (6), (7), *and* (8) *with corresponding family of entire functions satisfying* (1), (10), *and* (11) *and with corresponding measure μ. Suppose that there are no singular points with respect to $m(t)$ and that*

$$\alpha'(t) = u(t)\bar{u}(t), \quad \beta'(t) = u(t)\bar{v}(t) = v(t)\bar{u}(t), \quad \gamma'(t) = v(t)\bar{v}(t) \tag{19}$$

a.e. where $u(t)$ and $v(t)$ are measurable functions of $t > 0$.

(A) *If $a > 0$, then*

$$\int_0^a |A(t,w)\,u(t) + B(t,w)\,v(t)|^2\,dt < \infty$$

for every complex number w. For each element $f(t)$ of $L^2(0,\infty)$ which vanishes a.e. for $t \geqq a$, define a corresponding "eigentransform" $F(z)$ by

$$\pi F(w) = \int f(t)\,[A(t,w)\,\bar{u}(t) + B(t,w)\,\bar{v}(t)]\,dt \tag{20}$$

for all complex w. Then, $F(z)$ is an entire function, it belongs to $\mathscr{H}(E(a))$, and

$$\pi\int |F(t)|^2\,d\mu(t) = \int |f(t)|^2\,dt. \tag{21}$$

Every element $G(z)$ of $\mathscr{H}(E(a))$ is equal to the eigentransform $F(z)$ of some such element $f(t)$ of $L^2(0,\infty)$.

(B) *Let $a > 0$, let $f(t)$ and $g(t)$ be elements of $L^2(0,\infty)$ which vanish a.e. for $t \geqq a$, and let $F(z)$ and $G(z)$ be the corresponding eigentransforms. A necessary and sufficient condition that $G(z) = zF(z)$ for all complex z is that*

$$f(x) = \int_x^a g(t)\;[u(x)\bar{v}(t) - v(x)\,\bar{u}(t)]\,dt \tag{22}$$

for almost all x and that

$$\int_0^a g(t)\,\bar{u}(t)\,dt = 0.$$

(C) *If $f(t)$ is in $L^2(0,\infty)$, the corresponding eigentransform $F(x)$, defined by*

$$\pi F(x) = \lim_{a\to\infty}\int_0^a f(t)\,[A(t,x)\,\bar{u}(t) + B(t,x)\,\bar{v}(t)]\,dt,$$

exists with convergence in the metric of $L^2(\mu)$ *and* (21) *holds. Every element* $G(x)$ *of* $L^2(\mu)$ *is equal, a.e. with respect to* μ, *to the eigentransform* $F(x)$ *of an element* $f(t)$ *of* $L^2(0,\infty)$.

(D) *Let* $f(x)$ *and* $g(x)$ *be in* $L^2(0,\infty)$ *and let* $F(x)$ *and* $G(x)$ *be the corresponding eigentransforms in* $L^2(\mu)$. *A necessary and sufficient condition that* $G(x) = xF(x)$ *a.e. with respect to* μ *is that*

$$f(x) = v(x) \int_0^x g(t)\,\bar{u}(t)dt - u(x) \int^x g(t)\bar{v}(t)\,dt \tag{23}$$

a.e. where $\int^x g(t)\bar{v}(t)dt$ *denotes the choice of an absolutely continuous function whose derivative is* $g(x)\bar{v}(x)$ *a.e.*

Our theorems on spaces of entire functions now have a number of applications to integral transforms of the form (22), of which the following result is typical.

THEOREM XII. *If* $u(x)$ *and* $v(x)$ *are functions of* x *in* $L^2(0,1)$, *satisfy*

$$\bar{u}(x)\,v(x) = \bar{v}(x)\,u(x) \tag{24}$$

a.e. and are essentially linearly independent when restricted to any subinterval of $(0,1)$, *consider the corresponding bounded linear transformation* T *of* $L^2(0,1)$ *into itself, defined by* $T: g \to f$ *if*

$$f(x) = \int_x^1 g(t)\left[u(x)\bar{v}(t) - v(x)\,\bar{u}(t)\right]dt \tag{25}$$

for almost all values of x. *Let* $\mathcal{M}$ *be a closed subspace of* $L^2(0,1)$ *which is invariant under* T *in the sense that* Tg *belongs to* $\mathcal{M}$ *whenever* g *belongs to* $\mathcal{M}$. *Then, there is a number* a, *with* $0 \leqq a \leqq 1$, *such that* $\mathcal{M}$ *coincides with the set of functions* $f(x)$ *of* $L^2(0,1)$ *which vanish a.e. for* $x \geqq a$.

The same conclusion is available from the work of Kalisch [15] when $u(x)$ and $v(x)$ satisfy additional smoothness conditions. The particular integral transforms which we study are related to certain kinds of Sturm-Liouville equations. Let $p(x)$ and $r(x)$ be measurable, real valued functions defined for $0 \leqq x \leqq 1$, with $p(x) > 0$ and with $p(x)^{-1}$ and $r(x)$ absolutely integrable. As shown by Stone [19], there exist absolutely continuous, real valued functions $u(x)$ and $v(x)$ defined for $0 \leqq x \leqq 1$ such that

$$(p(x)\,u'(x))' = r(x)\,u(x),$$

$$(p(x)\,v'(x))' = r(x)\,v(x),$$

$$p(x)\,u'(x)\,v(x) - p(x)\,v'(x)\,u(x) = 1$$

in a suitable a.e. interpretation, and these functions may be chosen to satisfy boundary conditions at the origin. In the presence of boundary conditions at $x = 1$, the differential equation

$$g(x) = -(p(x)f'(x))' + r(x)f(x)$$

is equivalent to the integral equation (25) for functions $f(x)$ and $g(x)$ in $L^2(0,1)$. Theorem XI then gives a version of the eigenfunction expansions obtained by Kodaira [16] and Titchmarsh [20] for Sturm-Liouville equations. Theorem XII can be used to give an alternative proof of uniqueness in the inverse Sturm-Liouville problem studied by Levinson [17]. These results for differential equations have in fact inspired much of the present work, though it can also be thought of as a generalization of the theory of orthogonal polynomials which Shohat and Tamarkin [18] apply to the Hamburger moment problem.

Our proofs will use some properties of the generalized Hilbert transform of [10], which are best given in matrix notation.

LEMMA 1. *If $M(z)$ is a matrix valued entire function of z which satisfies* (2), *there is a unique Hilbert space $\mathscr{H}(M)$, whose elements are pairs*

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix} = (F(z), G(z))^{-}$$

of entire functions with this property: for each complex number w and for each pair of complex numbers u and v,

$$\frac{M(z)I\bar{M}(w) - I}{2\pi(z-\bar{w})} \begin{pmatrix} u \\ v \end{pmatrix}$$

belongs to $\mathscr{H}(M)$ as a function of z and

$$\begin{pmatrix} u \\ v \end{pmatrix}^{-} \begin{pmatrix} F(w) \\ G(w) \end{pmatrix} = \left\langle \begin{pmatrix} F(t) \\ G(t) \end{pmatrix}, \frac{M(t)I\bar{M}(w) - I}{2\pi(t-\bar{w})} \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle \tag{26}$$

for every $(F(z), G(z))^{-}$ in $\mathscr{H}(M)$. If

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix} \text{ is in } \mathscr{H}(M), \text{ then } \begin{pmatrix} [F(z) - F(w)]/(z-w) \\ [G(z) - G(w)]/(z-w) \end{pmatrix}$$

is in $\mathscr{H}(M)$ as a function of z for every complex number w. If $(F_1(z), G_1(z))^{-}$ and $(F_2(z), G_2(z))^{-}$ are in $\mathscr{H}(M)$, then

$$2\pi \begin{pmatrix} F_2(\bar{w}) \\ G_2(\bar{w}) \end{pmatrix}^{-} I \begin{pmatrix} F_1(w) \\ G_1(w) \end{pmatrix}$$

$$= \left\langle \begin{pmatrix} F_1(t) \\ G_1(t) \end{pmatrix}, \begin{pmatrix} [F_2(t) - F_2(\bar{w})]/(t-\bar{w}) \\ [G_2(t) - G_2(\bar{w})]/(t-\bar{w}) \end{pmatrix} \right\rangle \tag{27}$$

$$- \left\langle \begin{pmatrix} [F_1(t) - F_1(w)]/(t-w) \\ [G_1(t) - G_1(w)]/(t-w) \end{pmatrix}, \begin{pmatrix} F_2(t) \\ G_2(t) \end{pmatrix} \right\rangle$$

for all complex w. If $(\bar{F}(z), \bar{G}(z))^{-}$ *is in* $\mathscr{H}(M)$, *then* $(F(\bar{z}), G(\bar{z}))^{-}$ *is in* $\mathscr{H}(M)$ *and has the same norm.*

Vertical pairs are necessary in Lemma 1 for the most consistent matrix notation. Since such pairs can be awkward in print, we use the adjoint notation

$$\begin{pmatrix} u \\ v \end{pmatrix}^{-} = (\bar{u}, \bar{v})$$

of [10] to transpose. Property (27) of the generalized Hilbert transform, which was previously overlooked, has the following consequence.

LEMMA 2. *If* $M(z)$ *is a matrix valued entire function which satisfies* (2) *and if u and v are numbers such that* $(\bar{u}, \bar{v})^{-}$ *belongs to* $\mathscr{H}(M)$, *then* $\bar{u}v = \bar{v}u$.

LEMMA 3. *Let* $M(b,z)$ *be a matrix valued entire function which satisfies* (2), *and let u and v be numbers which satisfy* $\bar{u}v = \bar{v}u$ *and are not both zero. A necessary and sufficient condition that* $(\bar{u}, \bar{v})^{-}$ *belong to* $\mathscr{H}(M(b))$ *is that*

$$M(b,z) = M(a,z)\, M(a,b,z), \tag{28}$$

where $M(a,b,z)$ *is a matrix valued entire function satisfying* (2) *and*

$$A(a,z) = 1 - \beta z, \quad B(a,z) = \alpha z,$$

$$C(a,z) = -\gamma z, \quad D(a,z) = 1 + \beta z,$$

and α, β, γ *are real numbers, not all zero, such that*

$$\alpha \geqq 0, \ \gamma \geqq 0, \ \beta^2 = \alpha\gamma, \ \alpha v = \beta u, \ \beta v = \gamma u.$$

This lemma is used in obtaining a strengthened version of Theorems II and III of [9].

LEMMA 4. *Let* $E(a,z)$ *be an entire function which satisfies* (1), *and let* $M(a,b,z)$ *be a matrix valued function which satisfies* (2). *We suppose that there are no numbers u and v, not both zero, such that* $A(a,z)u + B(a,z)v$

belongs to $\mathcal{H}(E(a))$ *and* $(\bar{u},\bar{v})^-$ *belongs to* $\mathcal{H}(M(a,b))$. *Then, the entire functions* $A(b,z)$ *and* $B(b,z)$ *defined by* (3) *are real for real* z, *and* $E(b,z) = A(b,z) - iB(b,z)$ *satisfies* (1). *The space* $\mathcal{H}(E(a))$ *is contained isometrically in the space* $\mathcal{H}(E(b))$. *If* $(\bar{F}(z), \bar{G}(z))^-$ *is in* $\mathcal{H}(M(a,b))$, *then* $A(a,z)\,F(z) + B(a,z)\,G(z)$ *is in* $\mathcal{H}(E(b))$, *it is orthogonal to* $\mathcal{H}(E(a))$, *and*

$$\|(\bar{F}(t),\bar{G}(t))^-\|^2 = 2\,\|A(a,t)F(t) + B(a,t)G(t)\|^2. \tag{30}$$

Every element of $\mathcal{H}(E(b))$ *which is orthogonal to* $\mathcal{H}(E(a))$ *is uniquely of this form. If* $E(a,z)$ *has no real zeros, then neither does* $E(b,z)$.

LEMMA 5. *If* $E(a,z)$ *and* $E(b,z)$ *are entire functions which satisfy* (1) *and have no real zeros, and if* $\mathcal{H}(E(a))$ *is contained isometrically in* $\mathcal{H}(E(b))$, *then* (3) *holds for a unique matrix valued entire function* $M(a,b,z)$ *as in Lemma* 2.

LEMMA 6. *Let* $E(a,z)$ *and* $E(b,z)$ *be entire functions which satisfy* (1) *and have no real zeros, and let* $\mathcal{H}(E(a))$ *be contained isometrically in* $\mathcal{H}(E(b))$. *If* $F(z)$ *is in* $\mathcal{H}(E(b))$ *and if* $L(z)$ *is an element of* $\mathcal{H}(E(b))$ *orthogonal to* $\mathcal{H}(E(a))$, *then there is an entire function* $f(z)$ *such that*

$$f(w)\,G(w) = \left\langle \frac{F(t)\,G(w) - G(t)\,F(w)}{t - w},\ L(t) \right\rangle$$

for every $G(z)$ *in* $\mathcal{H}(E(a))$ *and for all complex* w. *The function* $f(z)$ *has exponential type and satisfies* (4).

The key Lemma 6 is adopted from the ideas of [2] and [3]. Its successful application depends on information about the modulus of entire functions of minimal exponential type, given by Heins [14] and stated below first as a lemma on subharmonic functions.

LEMMA 7. *Let* $u(x + iy)$ *be a non-negative, continuous, subharmonic function defined in the complex plane and periodic of period* $2\pi i$. *Let* Z *be the interior of the set of zeros of this function. If*

$$Q(x)^2 = \int_0^{2\pi} u(x + iy)^2\,dy$$

with the positive choice of root and if $p(x)$ *is the probability that* $x + iy$ *not lie in* Z *on each vertical line, then* $Q(x)^2$ *is a convex function of*

$$\int^x \exp\ \left(\int^u p(t)^{-1}\,dt\right) du$$

in the region where $Q(x) \neq 0$.

We include a formal proof of the lemma by the Carleman method. Although the steps are easily justified if $u(x+iy)$ has continuous first and second partial derivatives in x and y, the general case requires an approximation procedure as in Heins [14]. A consequence of the lemma is that any two nonconstant entire functions of minimal exponential type must be simultaneously very large in some part of the plane.

LEMMA 8. *If $f(z)$ and $g(z)$ are entire function of minimal exponential type such that*

$$\min(|f(z)|, |g(z)|) \leqq |y|^{-1} \tag{31}$$

for all complex z, then either $f(z)$ or $g(z)$ vanishes identically.

An apparently stronger assertion now follows by use of the Ahlfors-Heins theorem.

LEMMA 9. *If $f(z)$ and $g(z)$ are entire functions of exponential type which are real for real z and satisfy* (4), *and if*

$$|y| \leqq |f(z)|^{-1} + |g(z)|^{-1} \tag{32}$$

for all complex z, then either $f(z)$ or $g(z)$ vanishes identically.

The only other information needed is a variant of Lemma 11 of [9].

LEMMA 10. *Let $E(a,z)$ and $E(b,z)$ be entire functions which satisfy* (1) *and have no real zeros, such that $\mathscr{H}(E(a))$ is contained isometrically in $\mathscr{H}(E(b))$. If $F(z)$ is in $\mathscr{H}(E(b))$ and if*

$$F(iy) = o(E(a, i|y|)) \tag{33}$$

as $|y| \to \infty$, then $F(z)$ is in $\mathscr{H}(E(a))$.

Proof of Lemma 1. To show the existence of $\mathscr{H}(M)$, consider first the special case in which $M(z)$ is a constant. Then, $M(z)I\bar{M}(w) - I$ vanishes identically for all choices of w. Property (26) implies that $\mathscr{H}(M)$ contains no nonzero element. Since the conclusions of the lemma are obvious in this case, we may suppose in the remainder of the proof that $M(z)$ is not a constant.

The construction of $\mathscr{H}(M)$ depends on properties of the generalized Hilbert transform of [10]. The discussion there requires that

$$A(iy) - iB(iy) = o[yD(iy) + iyC(iy)], \tag{34}$$

$$D(iy) + iC(iy) = o[yA(iy) - iyB(iy)] \tag{35}$$

as $y \to +\infty$. These are the corrected statements of conditions (34) and (42) of [10], where factors of y are inadvertently omitted; both conditions are special cases of condition (4) of [9]. If (34) and (35) are not fulfilled, it is always possible to make a change of variable. For any fixed real number α, the matrix valued entire function

$$M_\alpha(z) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} M(z)$$

satisfies (2). The definitions imply that

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix} \to \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

is a linear isometric transformation of $\mathscr{H}(M)$ onto $\mathscr{H}(M_\alpha)$. If one of these spaces can be constructed, the other is obtained by a rotation of coordinates. The required properties of the spaces are easily seen to be preserved under this transformation. Suppose that one condition, say (35), is not satisfied by $M(z)$. The hypothesis (2) implies that

$$f(z) = [A(z) - iB(z)]^{-1}[D(z) + iC(z)]$$

is defined and analytic for $y > 0$ and that $\operatorname{Re} f(z) \geqq 0$. By the Poisson representation of a function positive and harmonic in a half-plane, there is a number $a \geqq 0$ such that

$$\operatorname{Re} f(x + iy) = ay + \frac{y}{\pi}\int \frac{\operatorname{Re} f(t)\, dt}{(t-x)^2+y^2}$$

for $y > 0$. By the Lebesgue dominated convergence theorem,

$$a = \lim_{y\to+\infty} y^{-1} \operatorname{Re} f(x + iy).$$

If $M(z)$ does not satisfy (35), then $a > 0$ and

$$A(iy) - iB(iy) = o[D(iy) + iC(iy)].$$

Since

$$\begin{aligned} A_\alpha(z) - iB_\alpha(z) &= \cos\alpha\, [A(z) - iB(z)] - i\sin\alpha\, [D(z) + iC(z)], \\ D_\alpha(z) + iC_\alpha(z) &= -i\sin\alpha[A(z) - iB(z)] + \cos\alpha[D(z) + iC(z)], \end{aligned}$$

we have

$$\lim_{y\to+\infty} [A_\alpha(iy) - iB_\alpha(iy)]^{-1}[D_\alpha(iy) + iC_\alpha(iy)] = i\cot\alpha.$$

Therefore, $M_\alpha(z)$ satisfies (34) and (35) if α is not an integral multiple of $\pi/2$. A similar argument applies to the reciprocal of $f(z)$ if (34) is not satisfied. To avoid complications of terminology, we will suppose for the construction of $\mathscr{H}(M)$ that $M(z)$ satisfies (34) and (35).

Because of (2), the function $A(z) - iB(z)$ has no zeros for $y \geqq 0$ and satisfies the inequality

$$|A(\bar{z}) - iB(\bar{z})| \leqq |A(z) - iB(z)|$$

for $y > 0$, which implies that

$$|[A(z) - iB(z)]^{-1}[A(z) + iB(z)]| \leqq 1$$

for $y > 0$. By the representation of functions analytic in a half-plane, Boas [1, p. 92], these inequalities are strict unless $[A(z) - iB(z)]^{-1}[A(z) + iB(z)]$ is a constant of absolute value 1. Therefore, $A(z) - iB(z)$ has no zeros if it does not satisfy (1). By Theorem IV of [8], the hypothesis (2) implies that $A(z) - iB(z)$ has exponential type and satisfies (4). If this function has no zeros,

$$A(z) - iB(z) = [A(0) - iB(0)] \exp(iaz)$$

where a is a constant which is real because of (4). Since

$$[A(z) - iB(z)]^{-1}[A(z) + iB(z)] = [A(0) - iB(0)]^{-1}[A(0) + iB(0)]\, e^{-2iaz}$$

is a constant of $A(z) - iB(z)$ does not satisfy (1), $a = 0$ in this case and $A(z) - iB(z)$ is a constant. We will now show that $D(z) + iC(z)$ is a constant in this case. By Theorem IV of [8], this function has exponential type and satisfies (4). By Boas [1, p. 97], its indicator diagram is a vertical line segment. The hypothesis (2) implies that

$$|D(\bar{z}) + iC(\bar{z})| \leqq |D(z) + iC(z)|$$

for $y > 0$. Therefore, (35) implies that

$$D(iy) + iC(iy) = o(|y|)$$

as $|y| \to \infty$, and the function $D(z) + iC(z)$ has minimal exponential type. By Boas [1, p. 83], these estimates on the imaginary axis make $D(z) + iC(z)$ a constant. We have shown that the function $A(z) - iB(z)$ satisfies (1) unless $M(z)$ is a constant. A similar argument using (34) will show that $D(z) + iC(z)$ satisfies (1) unless $M(z)$ is a constant. Since the case in which $M(z)$ is a constant has already been treated, we must suppose that $A(z) - iB(z)$ and $D(z) + iC(z)$ satisfy (1).

Theorem IX of [10] is now applicable because of (34). For every complex number w,

$$\frac{1 - A(z)\bar{D}(w) + B(z)\bar{C}(w)}{\pi(z - \bar{w})}$$

belongs to $\mathscr{H}(A - iB)$ as a function of z. If $F(z)$ is in $\mathscr{H}(A - iB)$, the corresponding entire function $G(z)$, defined by

$$G(w) = \left\langle F(t), \frac{1 - A(t)\bar{D}(w) + B(t)\bar{C}(w)}{\pi(t - \bar{w})} \right\rangle_{A - iB} \tag{36}$$

for all complex w, belongs to $\mathscr{H}(D + iC)$ and

$$\| F \|_{A - iB} = \| G \|_{D + iC} \quad . \tag{37}$$

Because of (35), we may apply Theorem IX of [10] with $A(z) - iB(z)$ and $D(z) + iC(z)$ interchanged. For every complex number w,

$$\frac{1 - D(z)\bar{A}(w) + C(z)\bar{B}(w)}{\pi(z - \bar{w})}$$

belongs to $\mathscr{H}(D + iC)$ as a function of z. If $G(z)$ is in $\mathscr{H}(D + iC)$, the corresponding entire function $F(z)$ defined by

$$F(w) = -\left\langle G(t), \frac{1 - D(t)\bar{A}(w) + C(t)\bar{B}(w)}{\pi(t - \bar{w})} \right\rangle_{D + iC} \tag{38}$$

for all complex w, belongs to $\mathscr{H}(A - iB)$, and (37) holds. Although the equivalence of (36) and (38) is not explicit in [10], it is a consequence of Theorem X there. Let $\mathscr{H}(M)$ be the Hilbert space of such pairs $(\bar{F}(z), \bar{G}(z))^-$ with

$$\| (\bar{F}(t), \bar{G}(t))^- \|_M^2 = \| F(t) \|_{A - iB}^2 + \| G(t) \|_{D + iC}^2 \; .$$

Property (26) follows from this definition by a straightforward substitution. By Theorem XI of [10], difference quotients do belong to $\mathscr{H}(M)$, which is closed under the stated conjugation. If

$$\begin{pmatrix} F_i(z) \\ G_i(z) \end{pmatrix} = \frac{M(z) I M(w_i) - I}{2\pi(z - \bar{w}_i)} \begin{pmatrix} u_i \\ v_i \end{pmatrix} ,$$

$i = 1, 2$, where u_i, v_i, w_i are complex numbers, formula (27) follows from (26) using (2). The general case of (27) follows by linearity and continuity since finite linear combinations of these functions are dense in $\mathscr{H}(M)$ as a result of (26).

This completes the proof of existence of a space $\mathscr{H}(M)$ with the stated property. Uniqueness is a consequence of (26) by standard arguments. For if two spaces $\mathscr{H}_1(M)$ and $\mathscr{H}_2(M)$ have this property,

$$\frac{M(z) I \bar{M}(w) - I}{2\pi(z - \bar{w})} \begin{pmatrix} u \\ v \end{pmatrix}$$

belongs to each space for every choice of complex numbers u, v, and w. Therefore, the finite linear combinations of such functions are common to both spaces. A short calculation from (26) will show that such combinations have the same norm in each space. Since such combinations are dense in each space as a result of (26), we can now easily show by approximation that $\mathscr{H}_1(M)$ and $\mathscr{H}_2(M)$ are isometrically equal.

Proof of Lemma 2. Let $F_1(z) = F_2(z) = u$ and $G_1(z) = G_2(z) = v$ in (27), which then reads $\bar{v}u - \bar{u}v = 0$.

Proof of Lemma 3, the necessity. By hypothesis, $(\bar{u},\bar{v})^-$ belongs to $\mathscr{H}(M(b))$. By multiplying u and v by a constant, we may suppose with no loss of generality that it has norm 1. Let

$$\alpha = 2\pi u\bar{u}, \quad \beta = 2\pi u\bar{v} = 2\pi v\bar{u}, \quad \gamma = 2\pi v\bar{v}.$$

The matrix valued entire function $M(a,z)$, defined as in the statement of the lemma, is easily verified to satisfy (2), and $(\bar{u},\bar{v})^-$ is an element of norm 1 in $\mathscr{H}(M(a))$ which spans this space. Therefore, $\mathscr{H}(M(a))$ is contained isometrically in $\mathscr{H}(M(b))$. It follows from (26) in these two spaces that for every choice of complex numbers u, v, and w,

$$\frac{M(b,z)I\bar{M}(b,w) - M(a,z)I\bar{M}(a,w)}{2\pi(z-\bar{w})}\begin{pmatrix} u \\ v \end{pmatrix}$$

belongs to $\mathscr{H}(M(b))$ as a function of z and is orthogonal to $\mathscr{H}(M(a))$. If $(\bar{F}(z),\bar{G}(z))^-$ is in $\mathscr{H}(M(b))$ and is orthogonal to $\mathscr{H}(M(a))$, then

$$\begin{pmatrix} u \\ v \end{pmatrix}^- \begin{pmatrix} F(w) \\ G(w) \end{pmatrix} = \left\langle \begin{pmatrix} F(t) \\ G(t) \end{pmatrix}, \frac{M(b,t)I\bar{M}(b,w)-M(a,t)I\bar{M}(a,w)}{2\pi(t-\bar{w})}\begin{pmatrix} u \\ v \end{pmatrix}\right\rangle. \tag{39}$$

In particular, we obtain the inequality

$$\begin{pmatrix} u \\ v \end{pmatrix}^- \frac{M(b,w)I\bar{M}(b,w) - M(a,w)I\bar{M}(a,w)}{2\pi(w-\bar{w})}\begin{pmatrix} u \\ v \end{pmatrix}$$
$$= \left\| \frac{M(b,t)I\bar{M}(b,w) - M(a,t)I\bar{M}(a,w)}{2\pi(t-\bar{w})}\begin{pmatrix} u \\ v \end{pmatrix}\right\|^2 \geqq 0.$$

Because of (2), $M(a,b,z) = M(a,z)^{-1}M(b,z)$ is a well-defined matrix valued entire function whose entries are real for real z. We have seen that

$$\begin{pmatrix} u \\ v \end{pmatrix}^- M(a,w)\frac{M(a,b,w)I\bar{M}(a,b,w) - I}{2\pi(w-\bar{w})}\bar{M}(a,w)\begin{pmatrix} u \\ v \end{pmatrix} \geqq 0$$

for every choice of numbers u and v. Since $M(a,w)$ is an invertible matrix, we obtain the matrix inequality

$$\frac{M(a,b,w)\,I\bar{M}(a,b,w) - I}{2\pi(x-\bar{w})} \geqq 0$$

for all complex w. A short calculation will show that this inequality implies (2) for $M(a,b,z)$, since the entries of this matrix are real for real z. This completes the proof of necessity for Lemma 3, but additional properties of the construction are required for the proof of sufficiency. If $(\bar{F}(z),\bar{G}(z))^-$ is in $\mathscr{H}(M(a,b))$, then $M(a,z)(\bar{F}(z),\bar{G}(z))^-$ is in $\mathscr{H}(M(b))$, is orthogonal to $\mathscr{H}(M(a))$, and has the same norm as $(\bar{F}(z),\bar{G}(z))^-$. When $(\bar{F}(z),\bar{G}(z))^-$ is a finite linear combination of functions of the form

$$\frac{M(a,b,z)\, I\, \bar{M}(a,b,w) - I}{2\pi(z-\bar{w})}\, \bar{M}(a,w) \begin{pmatrix} u \\ v \end{pmatrix} ,$$

this fact is easily verified from (39) using the identity (26) in the space $\mathscr{H}(M(a,b))$. The general case follows by linearity and continuity since such combinations are dense in $\mathscr{H}(M(a,b))$ as a result of (26). It is not hard to show from (39) that the transformation

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix} \to M(a,z) \begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

takes $\mathscr{H}(M(a,b))$ onto the orthogonal complement of $\mathscr{H}(M(a))$ in $\mathscr{H}(M(b))$. What is needed later is that $(\bar{u},\bar{v})^-$ does not belong to $\mathscr{H}(M(a,b))$. For if this were true $(\bar{u},\bar{v})^- = M(a,z)(\bar{u},\bar{v})^-$ would belong to the orthogonal complement in $\mathscr{H}(M(b))$ of $\mathscr{H}(M(a))$. This is not the case since $(\bar{u},\bar{v})^-$ belongs to $\mathscr{H}(M(a))$ by construction and is not zero by hypothesis.

Proof of Lemma 3, the sufficiency. By the proof of necessity, we may write $M(a,b,z) = M(a,c,z)\, M(c,b,z)$, where $M(c,b,z)$ is a matrix valued entire function which satisfies (2) and $(\bar{u},\bar{v})^-$ does not belong to $\mathscr{H}(M(c,b))$, and where $M(a,c,z)$ is defined by

$$A(a,c,z) = 1 - \beta_1 z, \quad B(a,c,z) = \alpha_1 z,$$
$$C(a,c,z) = -\gamma_1 z, \quad D(a,c,z) = 1 + \beta_1 z,$$

using real numbers $\alpha_1, \beta_1, \gamma_1$ which satisfy (29). Then,

$$M(b,z) = M(c,z) M(c,b,z)$$

where

$$M(c,z) = M(a,c) M a,c,z)$$

and

$$A(c,z) = 1 - \beta_2 z, \quad B(c,z) = \alpha_2 z,$$
$$C(c,z) = -\gamma_2 z, \quad D(c,z) = 1 + \beta_2 z,$$

and

$$\alpha_2 = \alpha + \alpha_1, \quad \beta_2 = \beta + \beta_1, \quad \gamma_2 = \gamma + \gamma_1,$$

are real numbers which satisfy (29). Since α, β, γ are not all zero by hypothesis, the numbers $\alpha_2, \beta_2, \gamma_2$ are not all zero. As in the proof of necessity, $(\bar{u},\bar{v})^-$ belongs to $\mathscr{H}(M(c))$ and spans this space. On the other hand, $(\bar{u},\bar{v})^-$ is not in $\mathscr{H}(M(c,b))$ and is not of the form $M(c,z)(\bar{F}(z), \bar{G}(z))^-$ with $(\bar{F}(z), \bar{G}(z))^-$ in $\mathscr{H}(M(c,b))$ since this would imply $F(z) = u$ and $G(z) = v$. Let $\mathscr{H}$ be the Hilbert space of pairs

$$\begin{pmatrix} \lambda u \\ \lambda v \end{pmatrix} + M(c,z) \begin{pmatrix} F(z) \\ G(z) \end{pmatrix} ,$$

where λ is complex and $(\bar{F}(z), \bar{G}(z))^{-}$ is in $\mathscr{H}(M(c,b))$, using the norm

$$\left\| \begin{pmatrix} \lambda u \\ \lambda v \end{pmatrix} + M(c,t) \begin{pmatrix} F(t) \\ G(t) \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} \lambda u \\ \lambda v \end{pmatrix} \right\|^2_{(M,c)} + \left\| \begin{pmatrix} F(t) \\ G(t) \end{pmatrix} \right\|^2_{(Mc,b)} .$$

The definition is unambiguous by what has been said about the location of $(\bar{u},\bar{v})^{-}$. A short calculation using (26) in $\mathscr{H}(M(c))$ and in $\mathscr{H}(M(c,b))$ will show that (26) holds in $\mathscr{H}$ with $M(z) = M(b,z)$. By Lemma 1, $\mathscr{H}$ is equal isometrically to $\mathscr{H}(M(b))$, which therefore contains $(\bar{u},\bar{v})^{-}$.

Proof of Lemma 4. The isometric inclusion of $\mathscr{H}(E(a))$ in $\mathscr{H}(E(b))$ will be proved using Theorem II of [9]. Since (3) holds, we need only show that condition (4) of [9] holds, or the equivalent condition given by Lemma 8 of [9], if u and v are numbers such that $A(a,z)u + B(a,z)v$ belongs to $\mathscr{H}(E(a))$. By Theorem I of [9], we have $\bar{u}v = \bar{v}u$ in this case, and the desired conclusion follows from Lemma 3 since $(\bar{u},\bar{v})^{-}$ does not belong to $\mathscr{H}(M(a,b))$ by hypothesis. The proof of (30) is easiest when $(\bar{F}(z), \bar{G}(z))^{-}$ is a finite linear combination of functions of the form

$$\frac{M(a,b,z)I\bar{M}(a,b,w) - I}{\pi(z-\bar{w})} \begin{pmatrix} \bar{A}(a,w) \\ \bar{B}(a,w) \end{pmatrix}, \tag{40}$$

for

$$(A(a,z), B(a,z)) \frac{M(a,b,z)I\bar{M}(a,b,w) - I}{\pi(z-\bar{w})} \begin{pmatrix} \bar{A}(a,w) \\ \bar{B}(a,w) \end{pmatrix} = K(b,w,z) - K(a,w,z)$$

belongs to $\mathscr{H}(E(b))$ and is orthogonal to $\mathscr{H}(E(a))$ by the isometric property of the inclusion. In this case, formula (30) may be verified using (26). The general case of (30) follows by linearity and continuity once it is shown that finite linear combinations of functions of the form (40) are dense in $\mathscr{H}(M(a,b))$. In other words, it must be shown that an element $(\bar{F}(z), \bar{G}(z))^{-}$ of $\mathscr{H}(M(a,b))$, which is orthogonal to all such combinations, vanishes identically. Because of (26), we have $A(a,z)F(z) + B(a,z)G(z) = 0$ identically in this case. For every complex number w,

$$\begin{pmatrix} [F(z) - F(w)]/(z-w) \\ [G(z) - G(w)]/(z-w) \end{pmatrix}$$

belongs to $\mathscr{H}(M(a,b))$ as a function of z by Lemma 1. By what we have already shown,

$$A(a,z)\frac{F(z) - F(w)}{z-w} + B(a,z)\frac{G(z) - G(w)}{z-w}$$

belongs to $\mathscr{H}(E(b))$ and is orthogonal to $\mathscr{H}(E(a))$. Since this function may be written in the form $K(a,\bar{w},z)$ where

$$c = \pi F(w) / B(a,w) = -\pi G(w) / A(a,w),$$

it also belongs to $\mathscr{H}(E(a))$, and hence vanishes identically. In other words, $A(a,z)\, F(w) + B(a,z)\, G(w) = 0$ for all complex z and w, and we may conclude that $F(z)$ and $G(z)$ vanish identically since $E(a,z)$ satisfies (1).

It remains to show that the transformation

$$(\bar{F}(z), \bar{G}(z))^{-} \rightarrow A(a,z) F(z) + B(a,z) G(z)$$

takes $\mathscr{H}(M(a,b))$ onto the orthogonal complement of $\mathscr{H}(E(a))$ in $\mathscr{H}(E(b))$. Since this transformation satisfies (30), its range is closed. It is sufficient to show that an element $L(z)$ of $\mathscr{H}(E(b))$ which is orthogonal to $\mathscr{H}(E(a))$ and to the range of the transformation vanishes identically. Since $K(b,w,z) - K(a,w,z)$ is known to be in the range for every complex number w,

$$L(w) = \langle L(t),\ K(a,w,t) \rangle + \langle L(t), K(b,w,t) - K(a,w,t) \rangle = 0.$$

The function $L(z)$ vanishes identically by the arbitrariness of w. If $E(a,z)$ has no real zeros, $K(a,w,w) > 0$ for all real w. Since $K(b,w,w) \geqq K(a,w,w,)$, as shown in the proof of Theorem VII of [8], it follows that $K(b,w,w) > 0$ and that $E(b,z)$ has no real zeros.

Proof of Lemma 5. The lemma follows from Theorem III of [9] since condition (4) of [9] is equivalent, via Lemma 8 of [9] and the present Lemma 3, to the requirement that $(\bar{u},\bar{v})^{-}$ not belong to $\mathscr{H}(M(a,b))$.

Proof of Lemma 6. Since $\mathscr{H}(E(a))$ is contained in $\mathscr{H}(E(b))$, $F(z)\, G(w) - G(z) F(w)$ belongs to $\mathscr{H}(E(b))$ as a function of z for every complex number w. Since this function vanishes when $z = w$ and since $E(b,z)$ has no real zeros,

$$[F(z)\, G(w) - G(z)\, F(w)] / (z - w)$$

belongs to $\mathscr{H}(E(b))$. We may write

$$F(z) = F_1(z) + F_2(z),$$

where $F_1(z)$ is in $\mathscr{H}(E(a))$ and $F_2(z)$ is orthogonal to $\mathscr{H}(E(a))$. The above argument will show that $[F_1(z)\, G(w) - G(z)\, F_1(w)] / (z - w)$ belongs to $\mathscr{H}(E(a))$, since $E(a,z)$ has no real zeros by hypothesis. Since $L(z)$ is assumed orthogonal to $\mathscr{H}(E(a))$, it is sufficient to prove the lemma in the case that $F(z) = F_2(z)$ is orthogonal to $\mathscr{H}(E(a))$. By Lemmas 4 and 5, we may write

$$F(z) = A(a,z)\, P(z) + B(a,z) Q(z),$$

where $(\bar{P}(z), \bar{Q}(z))^{-}$ is in $\mathscr{H}(M(a,b))$. Furthermore, we know from Lemma 1 that

$$\begin{pmatrix} [P(z) - P(w)] / (z - w) \\ [Q(z) - Q(w)] / (z - w) \end{pmatrix}$$

belongs to $\mathscr{H}(M(a,b))$ as a function of z for every complex number w. By Lemma 4,

$$A(a,z)\frac{P(z) - P(w)}{z-w} + B(a,z)\frac{Q(z) - Q(w)}{z-w}$$

is in $\mathscr{H}(E(b))$ and is orthogonal to $\mathscr{H}(E(a))$. Since

$$\begin{aligned}\frac{F(z)G(w) - G(z)F(w)}{z-w} &= \left[A(a,z)\frac{P(z) - P(w)}{z-w} + B(a,z)\frac{Q(z) - Q(w)}{z-w}\right]G(w)\\ &+ \left[\frac{A(a,z)G(w) - G(z)A(a,w)}{z-w}\right]P(w)\\ &+ \left[\frac{B(a,z)G(w) - G(z)B(a,w)}{z-w}\right]Q(w),\end{aligned}$$

where the last two terms on the right are in $\mathscr{H}(E(a))$,

$$\begin{aligned}&\left\langle\frac{F(t)\,G(w) - G(t)\,F(w)}{t-w}, L(t)\right\rangle\\ &= G(w)\left\langle A(a,t)\frac{P(t) - P(w)}{t-w} + B(a,t)\frac{Q(t) - Q(w)}{t-w}, L(t)\right\rangle.\end{aligned}$$

Therefore, the desired entire function $f(z)$ exists and

$$f(w) = \left\langle A(a,t)\frac{P(t) - P(w)}{t-w} + B(a,t)\frac{Q(t) - Q(w)}{t-w}, L(t)\right\rangle$$

for all complex w. By Lemma 4 we may write

$$L(z) = A(a,z)\,R(z) + B(a,z)\,S(z),$$

where $(\bar{R}(z), \bar{S}(z))^{-}$ is in $\mathscr{H}(M(a,b))$ and

$$2f(w) = \left\langle \begin{pmatrix}[P(t) - P(w)]/(t-w)\\ [Q(t) - Q(w)]/(t-w)\end{pmatrix}, \begin{pmatrix}R(t)\\ S(t)\end{pmatrix}\right\rangle \tag{41}$$

for all complex w. By Theorem IV of [8], the property (2) of $M(a,b,z)$ implies that the entire entries of this matrix have exponential type and satisfy (4). An estimate from (26) using the Schwarz inequality will show that $P(z)$ and $Q(z)$ have exponential type and satisfy (4). By the proof of Lemma 1, $\mathscr{H}(M(a,b))$ inner products can be evaluated as integrals along the real axis with respect to a suitable measure. Estimates from (41), similar to those for the lemma of [2], will now show that $f(z)$ has exponential type and satisfies (4).

Formal proof of Lemma 7. From the definition of $Q(x)$, we have

$$Q(x)\ Q'(x) = \int_0^{2\pi} u(x+iy)\ \frac{\partial u}{\partial x}(x+iy)\ dy,$$

$$Q'(x)^2 + Q(x)\,Q''(x) = \int_0^{2\pi} \frac{\partial u}{\partial x}(x+iy)\ \frac{\partial u}{\partial x}(x+iy)\,dy$$

$$+ \int_0^{2\pi} u(x+iy)\frac{\partial^2 u}{\partial x^2}(x+iy)\ dy,$$

where by the Schwarz inequality

$$Q(x)^2 Q'(x)^2 \leqq Q(x)^2 \int_0^{2\pi} \frac{\partial u}{\partial x}(x+iy)\frac{\partial u}{\partial x}(x+iy)\,dy\ .$$

Since

$$\frac{\partial^2 u}{\partial x^2}(x+iy) + \frac{\partial^2 u}{\partial y^2}(x+iy) \geqq 0$$

because $u(x+iy)$ is subharmonic,

$$Q(x)Q''(x) \geqq -\int_0^{2\pi} u(x+iy)\ \frac{\partial^2 u}{\partial y^2}(x+iy)$$

$$\geqq \int_0^{2\pi}\frac{\partial u}{\partial y}(x+iy)\frac{\partial u}{\partial y}(x+iy)\ dy$$

on integration by parts. If an absolutely continuous function $f(x)$ of real x vanishes outside of a finite interval (a,b) and has a square integrable derivative, then

$$\pi^2\int |f(t)|^2\,dt \leqq (b-a)^2\int |f'(t)|^2\,dt.$$

For each fixed x, $u(x+iy)$ vanishes outside of a union of y-intervals whose total length is $2\pi p(x)$ in each period. Application of this inequality on vertical line segments yields

$$\int_0^{2\pi} u(x+iy)^2\,dy \leqq 4p(x)^2\int_0^{2\pi}\frac{\partial u}{\partial y}(x+iy)\ \frac{\partial u}{\partial y}(x+iy)\ dy.$$

Therefore, $Q''(x) \geqq 4^{-1}p(x)^{-2}Q(x)$, and

$$[Q(x)Q'(x)]' \geqq Q'(x)^2 + 4^{-1}p(x)^{-2}Q(x)^2 \geqq p(x)^{-1}Q(x)Q'(x),$$

from which the lemma follows.

Proof of Lemma 8. If

$$u_1(z) = \overset{+}{\log}\,|f(\exp z)|\,,\quad u_2(z) = \overset{+}{\log}\,|g(\exp z)|\,,$$

then each $u_i(x + iy)$ satisfies the hypotheses of Lemma 7. Let $Q_i(x)$ and $p_i(x)$ be the corresponding quantities as in the statement of the lemma. Because of (31), either $u_1(x + iy) = 0$ or $u_2(x + iy) = 0$ whenever $|\sin y| \geqq \exp(-x)$, from which it follows that

$$p_1(x) + p_2(x) \leqq 1 + e^{-x}$$

and that

$$p_1(x)^{-1} + p_2(x)^{-1} \geqq 4e^x/(e^x + 1).$$

Now argue by contradiction, supposing that neither $f(z)$ nor $g(z)$ is a constant. By Liouville's theorem, $Q_1(x)$ and $Q_2(x)$ are unboundedly large on the right. As a result of Lemma 7, there is a number $k > 0$ for which the inequalities

$$Q_i(x)^2 \geqq 2k \int_0^x \exp\left(\int_0^u p_i(t)^{-1}\, dt\right) du,$$

$i = 1,2$, hold when $x \geqq a$ is sufficiently large. By the convexity of the exponential function

$$\begin{aligned} Q_1(x)^2 &+ Q_2(x)^2 \\ &\geqq 4k \int_0^x \exp\left(\tfrac{1}{2}\int_0^u p_1(t)^{-1}dt + \tfrac{1}{2}\int_0^u p_2(t)^{-1}dt\right) du \\ &\geqq 4k \int_0^x \exp\left(2\int_0^u (e^t + 1)^{-1} e^t\, dt\right) du \\ &\geqq k \int_0^x (e^u + 1)^2\, du \end{aligned}$$

when $x \geqq a$. It follows that

$$\liminf e^{-2x}[Q_1(x)^2 + Q_2(x)^2] \geqq \tfrac{1}{2}k > 0$$

in contradiction of the minimal type hypotheses in the form

$$\limsup e^{-x}\, Q_i(x) = 0,$$

$i = 1,2$. To get out of this dilemma, we must grant that one function, say $g(z)$, is a constant. If $g(z)$ is not zero, then (31) implies that $f(z)$ goes to zero at both ends of the imaginary axis and so vanishes identically by Boas [1, p. 83].

Proof of Lemma 9. If $f(z)$ and $g(z)$ are of minimal exponential type, the lemma follows from Lemma 8 after a change of variable. Otherwise, one function, say $f(z)$, has positive type τ. By the Ahlfors-Heins theorem, Boas [1, p. 116],

$$\lim r^{-1} \log|f(re^{i\theta})| = \tau|\sin\theta|$$

holds as $r \to \infty$ for almost all θ. We use this deep fact only to conclude that

$$\lim f(re^{i\theta})^{-1} = 0$$

holds as $r \to \infty$ for almost all θ, from which our hypothesis (32) now implies that

$$\lim g(re^{i\theta}) = 0$$

holds as $r \to \infty$ for almost all θ. Since $g(z)$ has exponential type for hypothesis, the Phragmen-Lindelöf principle, Boas [1, p. 3], applies for angles less than π. With several applications of it, we find that $g(z)$ remains bounded in the complex plane. By Liouville's theorem, $g(z)$ is a constant, which is zero by the last limit.

Proof of Lemma 10. If $L(z)$ is an element of $\mathscr{H}(E(b))$ orthogonal to $\mathscr{H}(E(a))$, the corresponding entire function $f(z)$, given by Lemma 6, has exponential type and satisfies (4). By the Schwarz inequality,

$$|f(z)G(z)| \leqq |y|^{-1} [\|F\| \, |G(z)| + \|G\| \, |F(z)|] \, \|L\|$$

holds for all complex z. Since we may choose $G(z) = K(a,w,z)$ where w is any complex number, we have

$$|f(z)| \leqq |y|^{-1}[\|F\| + K(a,z,z)^{-\frac{1}{2}}|F(z)|] \, \|L\|$$

for all complex z. Since (33) holds by hypothesis, $f(z)$ goes to zero at both ends of the imaginary axis. Since the indicator diagram of $f(z)$ is a vertical line segment because of (4), this function has minimal exponential type and vanishes identically by Boas [1, p. 83]. By the arbitrariness of $L(z)$, the expression

$$[F(z)G(w) - G(z)F(w)]/(z - w)$$

belongs to $\mathscr{H}(E(a))$ as a function of z for every complex number w. As in the proof of Lemma 11 of [9], the hypothesis (33) now implies that $F(z)$ belongs to $\mathscr{H}(E(a))$.

Proof of Theorem I. If $F(z)$ is in $\mathscr{H}(E(a))$ and if $G(z)$ is in $\mathscr{H}(E(b))$, the function $[F(z)G(w) - G(z)F(w)]/(z - w)$ is in $\mathscr{H}(E(c))$ since $E(c, z)$ has no real zeros. We will show that it must belong either to $\mathscr{H}(E(a))$ or to $\mathscr{H}(E(b))$. In doing so, we can suppose that neither $F(z)$ nor $G(z)$ vanishes identically, since the conclusion is obvious in that case. If $L(z)$ is in $\mathscr{H}(E(c))$ and is orthogonal to $\mathscr{H}(E(a))$, and if $T(z)$ is in $\mathscr{H}(E(c))$ and is orthogonal to $\mathscr{H}(E(b))$, consider the functions $f(z)$ and $g(z)$ defined as in Lemma 6 so that

$$(42) \qquad \begin{aligned} g(w)F(w) &= \left\langle \frac{G(t)F(w) - F(t)G(w)}{t - w}, L(t) \right\rangle, \\ f(w)G(w) &= \left\langle \frac{F(t)G(w) - G(t)F(w)}{t - w}, T(t) \right\rangle \end{aligned}$$

for all complex w. The Schwarz inequality yields estimates

$$|g(z)F(z)| \leqq |y|^{-1}[\|G\|\,|F(z)| + \|F\|\,|G(z)|]\,\|L\|,$$
$$|f(z)G(z)| \leqq |y|^{-1}[\|F\|\,|G(z)| + \|G\|\,|F(z)|]\,\|T\|,$$

which imply that

$$|y| \leqq \|G\|\;\|L\|\;|g(z)|^{-1} + \|F\|\;\|T\|\;|f(z)|^{-1}$$

for all complex z, if $L(z)$ and $T(z)$ do not vanish identically. By Lemma 6, $f(z)$ and $g(z)$ are entire functions of exponential type which satisfy (4). If they are real for real z, we may conclude from Lemma 9 that either $f(z)$ or $g(z)$ vanishes identically. This is the case if $F(z)$, $G(z)$, $L(z)$, and $T(z)$ are all real for real z, and the general case is easily reduced to this one using (H3). By the arbitrariness of $L(z)$ and $T(z)$, $[F(z)G(w) - G(z)F(w)]/(z-w)$ belongs to $\mathscr{H}(E(a))$ as a function of z for all complex w, or it belongs to $\mathscr{H}(E(b))$ as a function of z for all complex w, whenever $F(z)$ is in $\mathscr{H}(E(a))$ and $G(z)$ is in $\mathscr{H}(E(b))$. Furthermore, it is clear from the proof that one of the spaces, either $\mathscr{H}(E(a))$ or $\mathscr{H}(E(b))$, must contain all such functions $[F(z)G(w) - G(z)F(w)]/(z-w)$. For definiteness let us suppose it is $\mathscr{H}(E(a))$.

If $g(z)$ is an element of $\mathscr{H}(E(b))$ whose product by z is in $\mathscr{H}(E(b))$, we have $[F(z)wG(w) - zG(z)F(w)]\,/\,(z-w)$ in $\mathscr{H}(E(a))$ for every $F(z)$ in $\mathscr{H}(E(a))$ and every complex number w. Since $E(a,z)$ has no real zeros by hypothesis, such a choice of $F(z)$ is possible with $F(0) = 1$. When $w = 0$, we find that $G(z)$ is in $\mathscr{H}(E(a))$. If such elements $G(z)$ are dense in $\mathscr{H}(E(b))$, we may conclude immediately that $\mathscr{H}(E(b))$ is contained in $\mathscr{H}(E(a))$. Otherwise, let $G_0(z)$ be the choice of an element of norm 1 in $\mathscr{H}(E(b))$ which is orthogonal to the domain of multiplication by z in this space. By Theorem I of [9], $G_0(z)$ spans the orthogonal complement of this domain. Therefore, the orthogonal complement of $G_0(z)$ in $\mathscr{H}(E(b))$ is contained in $\mathscr{H}(E(a))$. If $G_0(z)$ belongs to $\mathscr{H}(E(a))$, then $\mathscr{H}(E(b))$ is contained in $\mathscr{H}(E(a))$. We must still study the case in which $G_0(z)$ does not belong to $\mathscr{H}(E(a))$.

Let us write $G_0(z) = [G_0(z) - S(z)] + S(z)$, where $G_0(z) - S(z)$ is in $\mathscr{H}(E(a))$ and $S(z)$ is orthogonal to $\mathscr{H}(E(a))$. We have seen that

$$[F(z)G_0(w) - G_0(z)F(w)]/(z-w)$$

belongs to $\mathscr{H}(E(a))$ for every $F(z)$ in $\mathscr{H}(E(a))$ and every complex number w. Since $E(a,z)$ has no real zeros by hypothesis, the same is true if $G_0(z)$ is replaced by any element of $\mathscr{H}(E(a))$. It follows that $[F(z)S(w) - S(z)F(w)]/(z-w)$ belongs to $\mathscr{H}(E(a))$ for every $F(z)$ in $\mathscr{H}(E(a))$ and every complex number w. The argument of [9, p. 132] will now show that there exist complex numbers u and v such that $S(z) = A(a,z)u + B(a,z)v$ for all complex z. Furthermore, these numbers satisfy $\bar{u}v = \bar{v}u$ and are not zero both. The space $\mathscr{H}(E(b))$ is contained in the closed span of $S(z)$ and $\mathscr{H}(E(a))$, and this span is itself a Hilbert space of entire functions satisfying (H1), (H2), and (H3). By the theorem of [7], there is an entire function $E(z)$ which satisfies (1) and such that this closed span is equal isometrically to $\mathscr{H}(E)$. The function $E(z)$ can have no real zeros since $\mathscr{H}(E)$ contains $\mathscr{H}(E(a))$ isometrically and $E(a,z)$ has no real zeros by hypothesis. In order to avoid new notation, let us simply suppose for the rest of the proof that $E(c,z)$ coincides with $E(z)$.

If $F(z)$ is in $\mathscr{H}(E(a))$ and if $G(z)$ is in $\mathscr{H}(Eb))$, the function

$$\frac{zG(z)F(w) - F(z)wG(w)}{z - w} = G(z)F(w) + w\,\frac{G(z)F(w) - F(z)G(w)}{z - w}$$

belongs to $\mathscr{H}(E(c))$ as a function of z for every complex number w. Since $S(z)$ is orthogonal to $\mathscr{H}(E(a))$ and since $[G(z)F(w) - F(z)G(w)]/(z - w)$ is known to be in $\mathscr{H}(E(a))$,

$$\left\langle \frac{tG(t)F(w) - F(t)wG(w)}{t - w}, S(t)\right\rangle = F(w)\langle G(t), S(t)\rangle$$

for all complex w, where we can have $\langle G(t), S(t)\rangle = 1$ if $G(z)$ is suitably chosen. In this case, we have

$$G(iy)^{-1}F(iy) = o(|y|)$$

as $|y| \to \infty$, by the Lebesgue dominated convergence theorem. If $T(z)$ is in $\mathscr{H}(E(c))$ and is orthogonal to $\mathscr{H}(E(b))$, we have seen that (42) holds for an entire function $f(z)$ of exponential type which satisfies (4) and

$$f(z)| \leqq |y|^{-1}\,\|F\|\,\|T\| + y|^{-1}|G(z)^{-1}F(z)|\,\|G\|\,\|T\|.$$

Since $f(z)$ goes to zero at both ends of the imaginary axis, it has minimal exponential type by Boas [1, p. 97] and vanishes identically by Boas [1, p. 83]. By the arbitrariness of $T(z)$, the expression $[F(z)G(w) - G(z)F(w)]/(z - w)$ belongs to $\mathscr{H}(E(b))$ as a function of z for every complex number w. In our derivation of this fact, we supposed that $\langle G(t), S(t)\rangle = 1$, but this hypothesis can now be removed by linearity.

If $F(z)$ is an element of $\mathscr{H}(E(a))$ whose product by z is in $\mathscr{H}(E(a))$, then $F(z)$ must belong to $\mathscr{H}(E(b))$ by an argument already used above with a and b interchanged. If such elements are dense in $\mathscr{H}(E(a))$, the inclusion of $\mathscr{H}(E(a))$ in $\mathscr{H}(E(b))$ follows. Otherwise, let $F_0(z)$ be an element of norm 1 in $\mathscr{H}(E(a))$ which is orthogonal to the domain of multiplication by z in this space. By Theorem I of [9], we can conclude that the orthogonal complement of $F_0(z)$ in $\mathscr{H}(E(a))$ is contained in $\mathscr{H}(E(b))$. If $F_0(z)$ belongs to $\mathscr{H}(E(b))$, then certainly $\mathscr{H}(E(a))$ is contained in $\mathscr{H}(E(b))$. Otherwise (and we shall see this impossible) the orthogonal complement of $F_0(z)$ in $\mathscr{H}(E(a))$ coincides with the intersection of $\mathscr{H}(E(a))$ and $\mathscr{H}(E(b))$. Since $[F_0(z)G_0(w) - G_0(z)F_0(w)]/(z-w)$ is known to belong to this intersection, which is orthogonal to both $F_0(z)$ and $G_0(z)$, we have

$$i(\bar{w}-w)\left\| \frac{F_0(t)G_0(w) - G_0(t)F_0(w)}{t-w} \right\|^2$$

$$= 2\operatorname{Re} i\left\langle F_0(t)G_0(w) - G_0(t)F_0(w), \frac{F_0(t)G_0(w) - G_0(t)F_0(w)}{t-w} \right\rangle = 0$$

and hence $F_0(z)G_0(w) - G_0(z)F_0(w)$ vanishes identically if w is not real. It follows that $\mathscr{H}(E(a))$ coincides with $\mathscr{H}(E(b))$, a case which has already been discarded in our presentation of the argument.

Proof of Theorem II. The proof is formally the same as for Theorem I except that inner products are taken in $L^2(\mu)$ instead of $\mathscr{H}(E(c))$. The entire functions $L(z)$ and $T(z)$ must be replaced by measurable functions $L(x)$ and $T(x)$ in $L^2(\mu)$. The only new verification to be made is that the conclusion of Lemma 6 holds. For instance, if $F(z)$ is in $\mathscr{H}(E(b))$ and if $L(x)$ is an element of $L^2(\mu)$ orthogonal to $\mathscr{H}(E(a))$, we must show that there is an entire function $f(z)$ such that

$$f(w)G(w) = \int \frac{F(t)G(w) - G(t)F(w)}{t-w} L(t)\,d\mu(t)$$

for every $G(z)$ in $\mathscr{H}(E(a))$ and all complex w, and that $f(z)$ so defined has exponential type and satisfies (4). The existence of $f(z)$ follows from the identity

$$G_1(w)\int \frac{F(t)G_2(w) - G_2(t)F(w)}{t-w} L(t)\,d\mu(t)$$

$$= G_2(w)\int \frac{F(t)G_1(w) - G_1(t)F(w)}{t-w} L(t)\,d\mu(t),$$

which holds whenever $G_1(z)$ and $G_2(z)$ are in $\mathscr{H}(E(a))$ because

$$[G_1(z)G_2(w) - G_2(z)G_1(w)]/(z-w)$$

belong to $\mathscr{H}(E(a))$ as a function of z for all choices of w. Simple estimates show that $f(z)$ is entire because $F(z)$ and $G(z)$ are entire by context and because the integrals are absolutely convergent by the Schwarz inequality. Since $F(z)$ and $G(z)$ are of exponential type and satisfy (4) as a result of the hypotheses on $E(a,z)$ and $E(b,z)$, the function $f(z)\,G(z)$ has exponential type and satisfies (4) by the proof of the lemma of [2]. Since $G(z)$ has exponential type and satisfies (4), the same follows for $f(z)$ using the representation theorem of Boas [1, p. 92].

Proof of Theorem III. Since $M(a,b,z)$ and $M(b,d,z)$ are matrix valued entire functions which satisfy (2) by hypothesis, $M(a,d,z) = M(a,b,z)\ M(b,d,z)$ is a matrix valued entire function which satisfies (2). Let u and v be complex numbers, not both zero, such that $\bar{u}v = \bar{v}u$. If there exist complex numbers u_1 and v_1, not both zero, such that $(\bar{u}_1, \bar{v}_1)^-$ belongs to $\mathscr{H}(M(a,d))$, choose u and v so that $uv_1 \neq u_1 v$. By Lemma 2, $(\bar{u},\bar{v})^-$ cannot then belong to $\mathscr{H}(M(a,d))$. The function $E(a,z) = 1 - \pi u\bar{v}z - i\pi u\bar{u}z$ satisfies (1), and $u = A(a,z)u + B(a,z)v$ is an element of norm 1 in $\mathscr{H}(E(a))$ which spans this space. If $A(d,z)$ and $B(d,z)$ are the entire functions defined by

$$(A(d,z), B(d,z)) = (A(a,z), B(a,z))\, M(a,d,z),$$

they are real for for real z, and $E(d,z) = A(d,z) - iB(d,z)$ satisfies (1) by Lemma 4. The space $\mathscr{H}(E(a))$ is contained isometrically in the space $\mathscr{H}(E(d))$. Since $(\bar{u},\bar{v})^-$ does not belong to $\mathscr{H}(M(a,d))$ by construction, it does not belong to $\mathscr{H}(M(a,b))$ by Lemma 3. The entire functions $A(b,z)$ and $B(b,z)$, defined by (3), are real for real z, and $E(b,z) = A(b,z) - iB(b,z)$ satisfies (1). By Lemma 4, $\mathscr{H}(E(a))$ is contained isometrically in $\mathscr{H}(E(b))$. By this construction, we have

$$(A(d,z), B(d,z)) = (A(b,z), B(b,z))\, M(b,d,z)$$

for all complex z, but we cannot conclude that $\mathscr{H}(E(b))$ is contained isometrically in $\mathscr{H}(E(d))$ as there may be numbers u_2 and v_2, not both zero, such that $(\bar{u}_2, \bar{v}_2)^-$ belongs to $\mathscr{H}(M(b,d))$ and $A(b,z)u_2 + B(b,z)v_2$ belongs to $\mathscr{H}(E(b))$. By Lemma 2, we then have $\bar{u}_2 v_2 = \bar{v}_2 u_2$. By Lemma 3, we may write

$$M(b,d,z) = Mb, b_+, z) M(b_+, d, z),$$

where $M(b_+, d, z)$ is a matrix valued entire function which satisfies (2) and

$$A(b,b_+,z) = 1 - \beta_2 z, \quad B(b,b_+,z) = \alpha_2 z,$$

$$C(b,b_+,z) = -\gamma_2 z, \quad D(b,b_+,z) = 1 + \beta_2 z,$$

where $\alpha_2, \beta_2, \gamma_2$ are real numbers, not all zero, such that

$$\alpha_2 \geqq 0, \ \gamma_2 \geqq 0, \ \beta_2^2 = \alpha_2\gamma_2, \quad \alpha_2 v_2 = \beta_2 u_2, \quad \beta_2 v_2 = \gamma_2 u_2.$$

As in the proof of the lemma, we make the construction in such a way that $(\bar{u}_2, \bar{v}_2)^-$ does not belong to $\mathscr{H}(M(b_+,d))$. The entire functions $A(b_+,z)$ and $B(b_+,z)$, defined by

$$(A(b_+,z), B(b_+,z)) = (A(b,z), B(b,z))\, M(b,b_+,z),$$

are real for real z, and $E(b_+,z) = A(b_+,z), - iB(b_+,z)$ satisfies (1). The space $\mathscr{H}(E(b))$ coincides with $\mathscr{H}(E(b_+))$ as a set, and

$$A(b,z)u_2 + B(b,z)v_2 = A(b_+,z)u_2 + B(b_+,z)v_2$$

has a smaller norm in the second of these spaces. The orthogonal complement $\mathcal{M}$ of this function in the first space coincides isometrically with the orthogonal complement of the same function in the second space. By this construction, we have

$$(A(d,z), B(d,z)) = (A(b_+,z), B(b_+,z))\, M(b_+,d,z)$$

for all complex z. If u_3 and v_3 are complex numbers such that $(\bar{u}_3, \bar{v}_3)^-$ belongs to $\mathcal{H}(M(b_+,d))$, then $\bar{u}_3 v_3 = \bar{v}_3 u_3$ by Lemma 2 and $u_3 v_2 \neq v_3 u_2$ by the choice of $M(b_+,d,z)$. Since $A(b_+,z)u_2 + B(b_+,z)v_2$ belongs to $\mathcal{H}(E(b_+))$ by construction, $A(b_+,z)u_3 + B(b_+,z)v_3$ does not belong to $\mathcal{H}(E(b_+))$ unless both u_3 and v_3 are zero, by Lemma 7 of [7]. By Lemma 4, $\mathcal{H}(E(b_+))$ is contained isometrically in $\mathcal{H}(E(d))$. Since $M(a,b,z)$ and $M(b,b_+,z)$ are matrix valued entire functions which satisfy (2), so is

$$M(a,b_+,z) = M(a,b,z)\, M(b,b_+,z).$$

Since

$$M(a,d,z) = M(a,b_+,z)\, M(b_+,d,z),$$

where $(\bar{u}, \bar{v})^-$ does not belong to $\mathcal{H}(M(a,d))$ by construction, we may conclude that $(\bar{u}, \bar{v})^-$ does not belong to $\mathcal{H}(Ma,b_+))$ by Lemma 3. Since

$$(A(b_+,z), B(b_+,z)) = (A(a,z), B(a,z))\, M(a,b_+,z),$$

$\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b_+))$ by Lemma 4. By Theorem II of [8], $\mathcal{M}$ is a Hilbert space of entire functions which satisfies (H1), (H2), and (H3) when considered in the metric of $\mathcal{H}(E(b_+))$. It cannot be the zero subspace of $\mathcal{H}(E(b_+))$, for if $\mathcal{H}(E(b_+))$ had dimension 1, $\mathcal{H}(E(a))$ would fill this space and $M(a,b,z)$ would be a constant as a result of Lemma 4. It would follow that both factors $M(a,b,z)$ and $M(b,b_+,z)$ are constant by Theorem IX of [9], whereas $\alpha_2, \beta_2, \gamma_2$ are not all zero by construction. The only alternative, then, is that $\mathcal{M}$ contains a nonzero element. By [7], $\mathcal{M}$ is equal isometrically to $\mathcal{H}(E(b_-))$, where $E(b_-,z)$ is an entire function which satisfies (1). This function has no real zeros because of Theorem II of [8]. By Lemma 5, there is a unique matrix valued entire function $M(b_-,b_+,z)$, satisfying (2), such that

$$(A(b_+,z), B(b_+,z)) = (A(b_-,z), B(b_-,z))\, M(b_-,b_+,z)$$

for all complex z. By Theorem I of [8], we may choose $E(b_-,z)$ in such a way that $M(b_-,b_+,0) = 1$. By construction, the function $A(b_+,z)u_2 + B(b_+,z)v_2$ spans the orthogonal complement of $\mathcal{H}(E(b_-))$ in $\mathcal{H}(E(b_+))$. By Lemma 4, $(\bar{u}_2, \bar{v}_2)^-$ belongs to $\mathcal{H}(M(b_-,b_+))$ and spans this space, and

$$A(b_-,b_+,z) = 1 - \beta z\,, \quad B(b_-,b_+,z) = \alpha z\,,$$

$$C(b_-,b_+,z) = -\gamma z\,, \quad D(b_-,b_+,z) = 1 + \beta z\,,$$

where α, β, γ are real numbers, not all zero, such that $\alpha \geqq 0$, $\gamma \geqq 0$, $\beta^2 = \alpha\gamma$, $\alpha v_2 = \beta u_2$, $\beta v_2 = \gamma u_2$. Since $\mathscr{H}(E(b_-))$ is contained isometrically in $\mathscr{H}(E(b))$, we have similarly

$$(A(b,z), B(b,z)) = (A(b_-,z), B(b_-,z))\, M(b_-,b,z),$$

where

$$\begin{aligned} A(b_-,b,z) &= 1 - \beta_1 z, \quad B(b_-,b,z) = \alpha_1 z, \\ C(b_-,b,z) &= -\gamma_1 z, \quad D(b_-,b,z) = 1 + \beta_1 z, \end{aligned}$$

and $\alpha_1, \beta_1, \gamma_1$ are real numbers, not all zero, which satisfy

$$\alpha_1 \geqq 0, \quad \gamma_1 \geqq 0, \quad \beta_1^2 = \alpha_1\gamma_1, \quad \alpha_1 v_2 = \beta_1 u_2, \quad \beta_1 v_2 = \gamma_1 u_2 .$$

Since

$$(A(b_+,z), B(b_+,z)) = (A(b_-,z),\ B(b_-,z))\, M(b_-,b,z)\, M(b,b_+,z),$$

we have

$$M(b_-,b_+,z) = M(b_-,b,z)\, M(b,b_+,z)$$

by the uniqueness part of Lemma 5, and hence

$$\alpha = \alpha_1 + \alpha_2, \quad \beta = \beta_1 + \beta_2, \quad \gamma = \gamma_1 + \gamma_2 .$$

Since $\mathscr{H}(E(a))$ and $\mathscr{H}(E(b_-))$ are contained isometrically in $\mathscr{H}(E(b_+))$, either $\mathscr{H}(E(a))$ contains $\mathscr{H}(E(b_-))$ or $\mathscr{H}(E(b_-))$ contains $\mathscr{H}(E(a))$ by Theorem I. Since $\mathscr{H}(E(a))$ has dimension 1 by construction and since $\mathscr{H}(E(b_-))$ contains a nonzero element, $\mathscr{H}(E(a))$ is contained in $\mathscr{H}(E(b_-))$. By Lemma 5, there is a unique matrix valued entire function $M(a,b_-,z)$, satisfying (2), such that

$$(A(b_-,z), B(b_-,z)) = (A(a,z), B(a,z))\, M(a,b_-,z)$$

for all complex z. Since

$$\begin{aligned} (A(b,z), B(b,z)) &= (A(a,z), B(a,z))\, M(a,b_-,z)\, M(b_-,b,z), \\ M(a,b,z) &= M(a,b_-,z)\, M(b_-,b,z) \end{aligned}$$

by the uniqueness part of Lemma 5.

A similar construction can be made with b replaced by c. There are entire functions $E(c_-,z)$ and $E(c_+,z)$ which satisfy (1) and

$$\begin{aligned} (A(c_-,z),\ B(c_-,z)) &= (A(a,z),\ B(a,z))M(a,c_-,z), \\ (A(c,z),\ B(c,z)) &= (A(c_-,z),\ B(c_-,z))M(c_-,c,z), \\ (A(c_+,z),\ B(c_+,z)) &= (A(c,z),\ \ B(c,z))M(c,c_+,z), \\ (A(d,z),\ B(d,z)) &= (A(c_+,z),\ B(c_+,z))M(c_+,d,z), \end{aligned}$$

where $M(a,c_-,z)$, $M(c_-,c,z)$, $M(c,c_+,z)$, $M(c_+,d,z)$ are matrix valued entire functions which satisfy (2) and

$$\begin{aligned} M(a,c,z) &= M(a,c_-,z)\ M(c_-,c,z), \\ M(c,d,z) &= M(c,c_+,z)\ M(c_+,d,z) \end{aligned}$$

for all complex z. The diagram of isometric inclusions

$$\mathcal{H}(E(a)) \subset \mathcal{H}(E(c_-)) \subset \mathcal{H}(E(c_+)) \subset \mathcal{H}(E(d))$$

holds, and $M(c_-, c, z)$, $M(c, c_+, z)$, $M(c_-, c_+, z) = M(c_-, c, z)M(c, c_+, z)$ are linear functions with value 1 at the origin.

If $\mathcal{H}(E(b_+))$ is contained in $\mathcal{H}(E(c_-))$, then by Lemma 5,

$$(A(c_-, z),\ B(c_-, z)) = (A(b_+, z),\ B(b_+, z))M(b_+, c_-, z),$$

where $M(b_+, c_-, z)$ is a matrix valued entire function which satisfies (2). In this case,

$$M(b, c, z) = M(b, b_+, z)M(b_+, c_-, z)M(c_-, c, z)$$

is a matrix valued entire function which satisfies (2) and

$$(A(c, z),\ B(c, z)) = (A(b, z),\ B(b, z))M(b, c, z)$$

for all complex z. Since

$$(A(c, z),\ B(c, z)) = (A(a, z),\ B(a, z))M(a, b, z)M(b, c, z)$$

for all complex z, we have

$$M(a, c, z) = M(a, b, z)M(b, c, z) \tag{43}$$

by the uniqueness part of Lemma 5, and

$$M(b, c, z) = M(a, b, z)^{-1} M(a, c, z)$$

satisfies (2) in this case. Otherwise, $\mathcal{H}(E(b_+))$ contains $\mathcal{H}(E(c_-))$ properly by Theorem I. By the same theorem, $\mathcal{H}(E(b_+))$ contains $\mathcal{H}(E(c_+))$ or $\mathcal{H}(E(c_+))$ contains $\mathcal{H}(E(b_+))$. Since the orthogonal complement of $\mathcal{H}(E(c_-))$ in $\mathcal{H}(E(c_+))$ has dimension 0 or 1, $\mathcal{H}(E(b_+))$ contains $\mathcal{H}(E(c_+))$ in this case.

If $\mathcal{H}(E(c_+))$ is contained in $\mathcal{H}(E(b_-))$, a similar argument will show that $M(a,c,z)^{-1}\ M(a,b,z)$ satisfies (2). Otherwise, $\mathcal{H}(E(c_+))$ contains $\mathcal{H}(E(b_+))$. Because of double inclusions, we have now only the case to consider in which $\mathcal{H}(E(b_+))$ is equal to $\mathcal{H}(E(c_+))$. We may suppose that the inclusions of $\mathcal{H}(E(b_-))$ in $\mathcal{H}(E(b_+))$ and $\mathcal{H}(E(c_-))$ in $\mathcal{H}(E(c_+))$ are proper since otherwise we fall into a case already considered. Either $\mathcal{H}(E(b_-))$ contains $\mathcal{H}(E(c_-))$ or $\mathcal{H}(E(c_-))$ contains $\mathcal{H}(E(b_-))$, by Theorem I. Since the orthogonal complement of $\mathcal{H}(E(b_-))$ in $\mathcal{H}(E(b_+))$ and the orthogonal complement of $\mathcal{H}(E(c_-))$ in $\mathcal{H}(E(c_+))$ have dimension 1, we must then have $\mathcal{H}(E(b_-))$ equal to $\mathcal{H}(E(c_-))$.

By Theorem I of [8], there are constant matrices $M(b_-, c_-, z)$ and $M(b_+, c_+, z)$, satisfying (2), such that

$$(A(c_-, z),\ B(c_-, z)) = (A(b_-, z),\ B(b_-, z))M(b_-, c_-, z),$$
$$(A(c_+, z),\ B(c_+, z)) = (A(b_+, z),\ B(b_+, z))M(b_+, c_+, z)$$

and hence

$$(A(c_+,z),\ B(c_+,z)) = (A(b_-,z),\ B)b_-,z))M(b_-,b_+,z)M(b_+,c_+,z),$$
$$(A(c_+,z),\ B(c_+,z)) = (A(b_-,z),\ B(b_-,z))M(b_-,c_-,z)M(c_-,c_+,z)$$

for all complex z. By the uniqueness part of Lemma 5,

$$M(b_-,b_+,z)M(b_+,c_+,z) = M(b_-,c_-,z)M(c_-,c_+,z).$$

Since $M(b_-,b_+,z)$ and $M(c_-,c_+,z)$ are constructed with value 1 at the origin, $M(b_-,c_-,z) = M(b_+,c_+,z)$. If we denote this constant matrix by P, it has real entries of determinant 1, and

$$M(c_-,c_+,z) = P^{-1}M(b_-,b_+,z)\,P$$

for all complex z. Since we have linear matrix valued functions which satisfy (2) and have value 1 at the origin,

$$M(b_-,b,z) = 1 + M'(b_-,b,z)\,z,$$
$$M(b,b_+,z) = 1 + M'(b,b_+,z)\,z,$$

where $M'(b_-,b,z)$ and $M'(b,b_+,z)$ are constant matrices which satisfy the matrix inequalities

$$M'(b_-,b,z)\,I \geqq 0 \text{ and } M'(b,b_+,z)\,I \geqq 0\ .$$

The same statement is true when b is replaced by c. Since $M(b_-,b_+,z)$ is linear, $M'(b_-,b,z)$ and $M'(b,b_+,z)$ are linearly dependent and $M'(b_-,b_+,z) = M'(b_-,b,z) + M'(b,b_+,z)$. Again the statement is true when b is replaced by c.

Since

$$M'(c_-,c_+,z) = P^{-1}M'(b_-,b_+,z)\,P,$$

we have an identity

$$P^{-1}M'(b_-,b,z)\,PI + P^{-1}M'(b,b_+,z)\,PI = M'(c_-,c,z)I + M'(c,c_+,z)\,I,$$

in which the four matrix terms are non-negative and are scalar multiples of a single matrix. Since the real numbers are totally ordered, either

$$P^{-1}M'(b_-,b,z)PI \leqq M'(c_-,c,z)I$$

and

$$P^{-1}M'(b,b_+,z)PI \geqq M'(c,c_+,z)I,$$

or

$$P^{-1}M'(b_-,b,z)PI \geqq M'(c_-,c,z)I$$

and

$$P^{-1}M'(b,b_+,z)PI \leqq M'(c,c_+,z)I.$$

If the first pair of inequalities holds, then

$$\begin{aligned} M(b,c,z) &= M(b_-,b,z)^{-1}M(b_-,c,z) \\ &= M(a,b,z)^{-1}M(a,c,z) \end{aligned}$$

satisfies (2). If the second pair of inequalities holds, then

$$\begin{aligned} M(c,b,z) &= M(c_-,c,z)^{-1}M(c_-,b,z) \\ &= M(a,c,z)^{-1}M(a,b,z) \end{aligned}$$

satisfies (2).

Proof of Theorem IV. The intersection of $\mathscr{H}(E(b))$ and $\mathscr{H}(E(c))$ is a Hilbert space of entire functions which satisfies (H1), (H2), and (H3) when considered in the metric of $L^2(\mu)$, since $\mathscr{H}(E(b))$ and $\mathscr{H}(E(c))$ each have these properties. Since the intersection contains a nonzero element by hypothesis, it is equal isometrically to $\mathscr{H}(E(a))$, where $E(a,z)$ is an entire function which satisfies (1). Since $E(b,z)$ has no real zeros by hypothesis, $F(z)/(z-w)$ belongs to $\mathscr{H}(E(b))$ whenever $F(z)$ belongs to $\mathscr{H}(E(b))$ and $F(w)=0$. Since $\mathscr{H}(E(c))$ has this property for the same reason, so does $\mathscr{H}(E(a))$. It follows that $E(a,z)$ has no real zeros and that Lemma 5 applies. Let $M(a,b,z)$ and $M(a,c,z)$ be the unique matrix valued entire functions satisfying (2) such that (3) and (43) hold for all complex z. If u_b and v_b are complex numbers, not both zero, such that $(\bar{u}_b,\bar{v}_b)^-$ belongs to $\mathscr{H}(M(a,b))$, then $A(a,z)u_b + B(a,z)v_b$ does not belong to $\mathscr{H}(E(a))$. The same statement is true if b is replaced by c. As a result of Lemma 2, each such pair of numbers is unique within a constant factor if it exists. For our construction, it is convenient to suppose that neither u_b nor v_b is zero if such a pair $(\bar{u}_b,\bar{v}_b)^-$ exists, and that neither u_c nor v_c is zero if $(\bar{u}_c,\bar{v}_c)^-$ exists. Such a situation can always be obtained by a rotation, as in the proof of Lemma 1, at the cost of altering $E(a,z)$ according to Theorem I of [8]. To prove the theorem, we must show that $\mathscr{H}(E(a))$ is equal either to $\mathscr{H}(E(b))$ or to $\mathscr{H}(E(c))$, or in other words that $M(a,b,z)$ or $M(a,c,z)$ is a constant. We will argue by contradiction, supposing that neither of these functions is constant.

Since $\mathscr{H}(E(b))$ is contained isometrically in $L^2(\mu)$, Theorem V of [8] applies. There is a unique function $W(b,z)$, defined and analytic for $y>0$, such that $|W(b,z)| \leqq 1$ and

$$\frac{y}{\pi}\int \frac{|E(b,t)|^2\,d\mu(t)}{(t-x)^2+y^2} = \operatorname{Re}\frac{E(b,z) + E^*(b,z)\,W(b,z)}{E(b,z) - E^*(b,z)\,W(b,z)}$$

for $y>0$. The same statement is true if b is replaced by a or c. By the proof of necessity for Theorem IX of [8],

$$\frac{1+W(a,z)}{1-W(a,z)} = \frac{[D(a,b,z) + iC(a,b,z)] + [D(a,b,z) - iC(a,b,z)]\,W(b,z)}{[A(a,b,z) - iB(a,b,z)] - [A(a,b,z) + iB(a,b,z)]\,W(b,z)}$$

for $y>0$. The same formula holds with b replaced by c. Since we suppose that that $M(a,b,z)$ is not constant and that $(0,1)^-$ does not belong to $\mathscr{H}(M(a,b))$, the function $A(a,b,z)-iB(a,b,z)$ satisfies (1) by the proof of Lemma 1. Since $|W(b,z)| \leqq 1$, the function

$$f(b,z) = \frac{[A(a,b,z) - iB(a,b,z)] + [A(a,b,z) + iB(a,b,z)]\,W(b,z)}{[A(a,b,z) - iB(a,b,z)] - [A(a,b,z) + iB(a,b,z)]\,W(b,z)}$$

is defined and analytic for $y > 0$ and $\operatorname{Re} f(b,z) > 0$. By the Poisson representation of a function positive and harmonic in a half-plane, there is a non-negative measure ν on the Borel sets of the real line and a number $p \geqq 0$ such that

$$\frac{y}{\pi}\int \frac{|A(a,b,t) - iB(a,b,t)|^2 d\nu(t)}{(t-x)^2 + y^2} + py = \operatorname{Re} f(b,z) \tag{44}$$

for $y > 0$. In writing the representation in this form, we use the fact that $A(a,b,z) - iB(a,b,z)$ has no real zeros as a result of (2) for $M(a,b,z)$. Because of Theorems I and II of [10], we actually have $p = 0$ and $\mathcal{H}(A(a,b) - iB(a,b))$ contained isometrically in $L^2(\nu)$ according to Theorem V of [8]. By the proof of necessity for Theorem IX of [8],

$$\frac{y}{\pi}\int \frac{d\nu(t)}{(t-x)^2 + y^2} = \operatorname{Re}\frac{1 + W(a,z)}{1 - W(a,z)} \tag{45}$$

for $y > 0$. A similar argument and construction can now be made with b replaced by c. Because of (45), this substitution does not change the measure ν in (44). Therefore, $\mathcal{H}(A(a,b) - iB(a,b))$ and $\mathcal{H}(A(a,c) - iB(a,c))$ are contained isometrically in the same $L^2(\nu)$. By Theorem IV of [8], $A(a,b,z) - iB(a,b,z)$ and $A(a,c,z) - iB(a,c,z)$ have exponential type and satisfy (4). By Theorem II, either $\mathcal{H}(A(a,b) - iB(a,b))$ contains $\mathcal{H}(A(a,c) - iB(a,c))$ or $\mathcal{H}(A(a,c) - iB(a,c))$ contains $\mathcal{H}(A(a,b) - iB(a,b))$; for definiteness we suppose the second alternative holds. By our choice of u_b and v_b, we see as in Lemma 1 that $D(a,b,z) + iC(a,b,z)$ satisfies (1) and the generalized Hilbert transform takes $\mathcal{H}(A(a,b) - iB(a,b))$ isometrically onto $\mathcal{H}(D(a,b) + iC(a,b))$. By our choice of u_c and v_c, $D(a,c,z) + iC(a,c,z)$ satisfies (1) and the generalized Hilbert transform takes

$$\mathcal{H}(A(a,c) - iB(a,c))$$

isometrically onto $\mathcal{H}(D(a,c) + iC(a,c))$. These two transformations are consistent because of Theorem X of [10], and we may conclude that $\mathcal{H}(M(a,b))$ is contained isometrically in $\mathcal{H}(M(a,c))$. By the proof of necessity for Lemma 3, the matrix valued entire function $M(b,c,z) = M(a,b,z)^{-1}M(a,c,z)$ satisfies (2). Since

$$(A(c,z),\ B(c,z)) = (A(b,z),\ B(b,z))M(b,c,z),$$

$\mathcal{H}(E(b))$ is contained in $\mathcal{H}(E(c))$. As the argument was posed, this conclusion is a contradiction, to escape which our only alternative is to grant the theorem.

Proof of Theorem V. In case (A), let V be the set of regular points t with the property that $\mathcal{H}(E)$ is contained in $\mathcal{H}(E(t))$. Our hypothesis is that b belongs to V. If t is in V, the inequality

$$0 < K(z,z) \leqq K(t,z,z)$$

follows from the isometric nature of the inclusion. Since (11) holds by hypothesis, the set V is bounded away from zero. Since the regular points are clearly an open set by definition, V has a least element a. If $(0,a)$ is an interval of singular points for $m(t)$, $\mathscr{H}(E(a))$ has dimension 1 by Theorem III of [10]. Since $\mathscr{H}(E)$ is contained in $\mathscr{H}(E(a))$ and since $\mathscr{H}(E)$ contains a nonzero element, it must coincide with $\mathscr{H}(E(a))$ in this case. Otherwise, there are regular points s with $s < a$. Since $\mathscr{H}(E)$ is not contained in $\mathscr{H}(E(s))$ by the choice of a, $\mathscr{H}(E(s))$ is contained in $\mathscr{H}(E)$ by Theorem I. If a is not the right-hand end point of an interval of singular points, the union of such spaces $\mathscr{H}(E(s))$ is dense in $\mathscr{H}(E)$ by Theorem III of [10], and so $\mathscr{H}(E)$ coincides with $\mathscr{H}(E(a))$ in this case. Otherwise, there is a regular point a_- with $a_- < a$, such that (a_-, a) is an interval of singular points. Since the orthogonal complement of $\mathscr{H}(E(a_-))$ in $\mathscr{H}(E(a))$ has dimension 0 or 1 by Theorem III of [10], $\mathscr{H}(E)$ coincides with $\mathscr{H}(E(a))$ in this case also.

In case (B), let U be the set of regular points t with the property that $\mathscr{H}(E(t))$ is contained in $\mathscr{H}(E)$. Our hypothesis is that b belongs to U. If t is in U, the inequality

$$K(t,z,z) \leqq K(z,z) < \infty$$

follows from the isometric nature of the inclusion. Since when z is not real,

$$\lim_{t\to\infty} K(t,z,z,) = \infty$$

by Theorem I of [10], the set V is bounded. Since the regular points are an open set of real numbers, V has a largest element c. If c is the largest regular point for $m(t)$, then $\mathscr{H}(E(c))$ fills $L^2(\mu)$ by Theorem I of [10]. Since $\mathscr{H}(E(c))$ is contained in $\mathscr{H}(E)$, which in turn is contained in $L^2(\mu)$, it follows that $\mathscr{H}(E)$ equals $\mathscr{H}(E(c))$ in this case. Otherwise, there are regular points s with $s > c$. Since $\mathscr{H}(E(s))$ is not contained in $\mathscr{H}(E)$ by the choice of c, $\mathscr{H}(E)$ is contained in $\mathscr{H}(E(s))$ by Theorem IV. If c is not the left-hand end point of an interval of singular points, the intersection of such spaces $\mathscr{H}(E(s))$ is $\mathscr{H}(E)$ by Theorem III of [10], and it follows that $\mathscr{H}(E)$ is equal to $\mathscr{H}(E(c))$ in this case. Otherwise, there is a regular point c_+ with $c_+ > c$ such that (c,c_+) is an interval of singular points. Since the orthogonal complement of $\mathscr{H}(E(c))$ is $\mathscr{H}(E(c_+))$ has dimension 0 or 1 by Theorem III of [10], $\mathscr{H}(E)$ is equal to $\mathscr{H}(E(c))$ in this case also.

Proof of Theorem VI. For ease of notation our proof will suppose that $\alpha(t) + \gamma(t) = t$, as can always be obtained by a reparametrization. Let U be the set of numbers $u \geqq 0$ such that $M(u,z)^{-1}M(z)$ satisfies (2), and let V be the set of numbers $v \geqq 0$ such that $M(z)^{-1}M(v,z)$ satisfies (2). If u is in U and v is in V, the product $M(u,z)^{-1}M(v,z)$ satisfies (2). If $u \leqq v$, this fact is in agreement with our previous knowledge that $M(u,v,z)$ satisfies (2). But if $v \leqq u$, it is the assertion that $M(v,u,z)^{-1}$ satisfies (2). Since $M(v,u,z)$ has value 1 at the origin and since its z-derivative there is

$$M'(v,u,0) = \begin{pmatrix} \beta(v) - \beta(u), & \alpha(u) - \alpha(v) \\ \gamma(v) - \gamma(u), & \beta(u) - \beta(v) \end{pmatrix},$$

the function $M(v,u,z)^{-1}$ has value 1 at the origin and z-derivative there equal to $-M'(v,u,0)$. If $M(v,u,z)^{-1}$ satisfies (2), we must have

$$\alpha(v) - \alpha(u) \geqq 0 \text{ and } \gamma(v) - \gamma(u) \geqq 0,$$

by Theorem IX of [9]. In our parametrization, it follows that $u \leqq v$, and hence $u = v$ since the reverse inequality also holds. In other words, we have $u \leqq v$ whenever u is in U and v is in V. Since 0 is in U and b is in V by hypothesis, there is a number a in $[0,b]$ such that $u \leqq a \leqq v$ whenever u is in U and v is in V, by the completeness of the real numbers. If s is in $[0,b]$, we know that $M(b,z) = M(s,z)\, M(s,b,z)$, where $M(s,z)$ and $M(s,b,z)$ are matrix valued entire functions which satisfy (2), whereas $M(z)$ and $M(z)^{-1}M(b,z)$ satisfy (2) by hypothesis. By Theorem III, either $M(s,z)^{-1}M(z)$ or $M(z)^{-1}M(s,z)$ satisfies (2), or in other words, s belongs to U or V. It follows that U contains every number s with $0 \leqq s < a$ and that V contains every number s with $a < s \leqq b$. Therefore, there is a sequence (u_n) with u_n in U for every n, $u_n \leqq u_{n+1}$, and $a = \lim u_n$. Since $M(u_n,z)^{-1}\, M(z)$ satisfies (2) for every n, and since $M(a,z) = \lim M(u_n, z)$, for all complex z, $M(a,z)^{-1}\, M(z)$ satisfies (2) and a belongs to U. A similar approximation from above will show that a belongs to V. Since both $M(a,z)^{-1}M(z)$ and its reciprocal are now known to satisfy (2), this function is a constant by Theorem IX of [9], and the theorem follows.

Proof of Theorem VII. The argument for Theorem V, A or B, applies with an appeal to Theorem II instead of Theorem I or Theorem IV.

Proof of Theorem VIII. By Lemma 5, formula (3) holds for a matrix valued entire function $M(a,b,z)$ which satisfies (2). As in the proof of Theorem VII of [8], the function

$$W(a,b,z) = \frac{D(a,b,z) + iC(a,b,z) - A(a,b,z) + iB(a,b,z)}{D(a,b,z) + iC(a,b,z) + A(a,b,z) - iB(a,b,z)}$$

is defined and analytic for $y \geqq 0$, $|W(a,b,z)| \leqq 1$ there, and

$$\frac{y}{\pi}\int \frac{|E(a,t)/E(b,t)|^2\, dt}{(t-x)^2 + y^2} = \operatorname{Re}\frac{E(a,z) + E^*(a,z)\, W(a,b,z)}{E(a,z) - E^*(a,z)\, W(a,b,z)}$$

for $y > 0$. Since (2) holds for $M(a,b,z)$, we have

$$\left|\frac{E(a,z)}{E(b,z)}\right|^2 \leqq \operatorname{Re}\frac{E(a,z) + E^*(a,z)\, W(a,b,z)}{E(a,z) - E^*(a,z)\, W(a,b,z)}$$

when $y > 0$. It follows that there is some real number τ such that

$$\log|E(b,z)/E(a,z)| = \tau y + \frac{y}{\pi}\int \frac{\log|E(b,t)/E(a,t)|\, dt}{(t-x)^2 + y^2} \tag{46}$$

for $y > 0$, with absolute convergence of the integral. See the proof of the representation theorem in a half-plane, given by Boas [1, p. 92]. Although he makes an exponential type hypothesis natural in applications to entire functions, the same argument applies under the present estimate. Formula (17) follows by the Lebesgue dominated convergence theorem. Since $|E(a,z)/E(b,z)|^2$ has been given an explicit majorization by a Poisson integral, the Lebesgue dominated convergence theorem also implies that

$$|E(a,iy)/E(b,iy)| = o(y)$$

as $y \to +\infty$, and hence that $\tau \geqq 0$. If $F(z)$ is in $\mathscr{H}(E(a))$, the definition of this space yields the estimate

$$|F(z)/E(a,z)|^2 \leqq (4\pi y)^{-1} \|F\|^2$$

for $y > 0$, where

$$\|F\|^2 = \int |F(t)/E(a,t)|^2 dt < \infty.$$

If $F(z)$ does not vanish identically, as we can suppose with no loss of generality, we may conclude that

$$\log|F(z)/E(a,z)| \leqq \frac{y}{\pi} \int \frac{\log|F(t)/E(a,t)|\,dt}{(t-x)^2+y^2}$$

for $y > 0$. Let $G(z) = e^{ihz} F(z)$ where h is fixed and $-\tau \leqq h \leqq \tau$. Since $F(z)$ is in $\mathscr{H}(E(b))$ and since $|G(z)| = |F(z)|$ for all real z,

$$\int |G(t)/E(b,t)|^2 dt < \infty,$$

whereas the last inequality and (46) imply that

$$\log|G(z)/E(b,z)| \leqq \frac{y}{\pi} \int \frac{\log|G(t)/E(b,t)|\,dt}{(t-x)^2+y^2}$$

for $y > 0$. Since u^2 is a convex function of $\log u$, Jensen's inequality yields

$$|G(z)/E(b,z)|^2 \leqq \frac{y}{\pi} \int \frac{|G(t)/E(b,t)|^2\,dt}{(t-x)^2+y^2}$$

for $y > 0$, as in Boas [1, p. 100]. Approximation from large semicircles establishes Cauchy's formula

$$G(z)/E(b,z) = \frac{1}{2\pi i} \int \frac{G(t)/E(b,t)\,dt}{t-z}$$

for $y > 0$, and hence also

$$0 = \frac{1}{2\pi i} \int \frac{G(t)/E(b,t)dt}{t-\bar{z}}$$

for $y < 0$. By (H3), the same formulas hold when $G(z)$ is replaced by $G^*(z)$. As in [7], it follows that

$$G(w) = \int G(t)\bar{K}(b,w,t)\,|E(b,t)|^{-2}\,dt$$

for all complex w, and that $G(z)$ belongs to $\mathscr{H}(E(b))$.

Proof of Theorem IX. Clearly, $\mathscr{H}$ is a Hilbert space of entire functions which satisfies (H1) and (H2) when considered in the metric of $\mathscr{H}(E(b))$. For each complex number w, let $K(w,z)$ be the unique element of $\mathscr{H}$ such that

$$F(w) = \langle F(t),\ K(w,t) \rangle$$

for every $F(z)$ in $\mathscr{H}$. Since $\mathscr{H}$ contains a nonzero element by hypothesis, $K(w,w) > 0$ when w is not real by Lemma 1 of [7]. The same inequality holds when w is real since $F(z)/(z-w)$ belongs to $\mathscr{H}$ whenever $F(z)$ belongs to $\mathscr{H}$ and $F(w) = 0$, by hypothesis. By the same lemma, $\bar{K}(z,w) = K(w,z)$ for all complex z and w. By Lemma 3 of [7], $K(w,z)$ satisfies an identity which is equivalent to the existence of entire functions $A(z)$ and $B(z)$ such that

$$2\pi(z - \bar{w})\ K(w,z) = B(z)\ \bar{A}(w) - A(z)\ \bar{B}(w)$$

for all complex z and w. When $z = \bar{w}$, we find that $B(z)A^*(z) = A(z)B^*(z)$. These functions may be chosen in a number of ways and in particular so that $A(0) = 1$ and $B(0) = 0$. Since

$$\begin{aligned} 2\pi i(\bar{w} - z)K(w,z) = {} & [A(z) - iB(z)][A(w) - iB(w)]^- \\ & - [A(z) + iB(z)][A(w) + iB(w)]^- \end{aligned}$$

and since $K(w,w) > 0$ for all complex w, we see that $A(z) - iB(z)$ has no zeros for $y \geqq 0$ and that $A(z) + iB(z)$ has no zeros for $y \leqq 0$. Therefore,

$$\frac{A^*(z) - iB^*(z)}{A(z) - iB(z)} = \frac{A^*(z) + iB^*(z)}{A(z) + iB(z)}$$

is an entire function which has no zeros. Since it has value 1 at the origin by construction, we may write it as $U(z)^2$, where $U(z)$ is an entire function with value 1 at the origin, and which also satisfies $U(z)^{-1} = U^*(z)$. It follows that

$$A(a,z) = A(z)\,U(z) \text{ and } B(a,z) = B(z)\,U(z)$$

are entire functions which are real for real z, and that $E(a,z) = A(a,z) - iB(a,z)$ satisfies (1) and has no real zeros. Since

$$K(a,w,z) = K(w,z)\,U(z)\,\bar{U}(w)$$

for all complex z and w, the transformation $F(z) \to F(z)\,U(z)$ take $\mathscr{H}$ isometrically onto $\mathscr{H}(E(a))$ by the proof of Lemma 3. Since $\mathscr{H}$ is contained in $\mathscr{H}(E(b))$, $K(w,z)$ belongs to $\mathscr{H}(E(b))$ as a function of z for every complex number w.

As in the proof of Theorem VIII, there is a number $\tau_+ \geqq 0$ such that

$$(47)\quad \log\left|\frac{A(z)-iB(z)}{E(b,z)}\right| = -\tau_+ y + \frac{y}{\pi}\int \log\left|\frac{A(t)-iB(t)}{E(b,t)}\right| \frac{dt}{(t-x)^2+y^2}$$

holds for $y > 0$, with absolute convergence of the integral. Because of (H3), the same formula holds when $A(z)$ is replaced by $A^*(z)$ and $B(z)$ is replaced by $B^*(z)$, if τ_+ is replaced by $\tau_- \geqq 0$. If $2h = \tau_+ - \tau_-$, we obtain

$$\log|U(z)| = hy + \frac{y}{\pi}\int \frac{\log|U(t)|\,dt}{(t-x)^2+y^2}$$

for $y > 0$. Since $U(z)$ has absolute value 1 on the real axis and value 1 at the origin, $U(z) = e^{-ihz}$. Therefore, if $F(z)$ is in $\mathscr{H}(E(a))$, the function $e^{ihz}F(z)$ belongs to $\mathscr{H}$ and hence to $\mathscr{H}(E(b))$. By (H3), $e^{-ihz}F^*(z)$ also belongs to $\mathscr{H}(E(b))$. If $F(z)$ is real for real z, this function must now satisfy the defining inequality of $\mathscr{H}(E(b))$ and hence belong to this space. Since $\mathscr{H}(E(a))$ satisfies (H3), the inclusion of $\mathscr{H}(E(a))$ in $\mathscr{H}(E(b))$ follows. The inclusion is isometric because $F(z) \to e^{ihz}F(z)$ takes $\mathscr{H}(E(a))$ isometrically into $\mathscr{H}(E(b))$ by construction and because e^{ihz} has absolute value 1 on the real axis. Since (47) holds and $U(z)$ has been determined, we may deduce (46) with $2\tau = \tau_+ + \tau_-$, and so $-\tau \leqq h \leqq \tau$.

Proof of Theorem X. By Theorem VIII,

$$\tau(a,b) = \lim_{y\to+\infty} y^{-1}\log|E(b,iy)/E(a,iy)|$$

exists. From (3) we have

$$\begin{aligned} 2E(b,z) = {} & E(a,z)[A(a,b,z) - iB(a,b,z) + D(a,b,z) + iC(a,b,z)] \\ & + E^*(a,z)[A(a,b,z) - iB(a,b,z) - D(a,b,z) - iC(a,b,z)] \end{aligned}$$

where $|E^*(a,z)| < |E(a,z)|$ for $y > 0$. Therefore,

$$|E(b,z)/E(a,z)| \leqq |A(a,b,z) - iB(a,b,z)| + |D(a,b,z) + iC(a,b,z)|$$

for $y > 0$, and the inequality

$$\tau(a,b) \leqq \int_a^b [\alpha'(t)\gamma'(t) - \beta'(t)^2]^{1/2}dt$$

follows from Theorem X of [9]. The reverse inequality follows from the proof of that theorem.

Proof of Theorem XI. Since $m(t)$ has no singular points by hypothesis, the space $L_0^2(m)$ of [10] coincides with $L^2(m)$. Because of (19),

$$(f(t), g(t)) \to f(t)u(t) + g(t)v(t)$$

is a linear isometric transformation of $L^2(m)$ onto $L^2(0,\infty)$. The theorem now follows from Theorem III of [10]. A factor of π is inadvertently omitted from the left-hand side of formula (11) of [10]. It should be placed so as to match the corresponding factor in formula (10) of that paper.

Proof of Theorem XII. Since $u(t)$ and $v(t)$ are in $L^2(0,1)$ and since (24) holds, the matrix valued function $m(t)$, defined by

$$\begin{aligned}\alpha(a) &= \int_0^a u(t)\,\bar{u}(t)\,dt,\\ \beta(a) &= \int_0^a u(t)\bar{v}(t)\,dt \quad = \int_0^a v(t)\,\bar{u}(t)\,dt,\\ \gamma(a) &= \int_0^a v(t)\bar{v}(t)\,dt\end{aligned}$$

satisfies (6) for $0 \leqq t \leqq 1$. The same formulas may be used to define $m(t)$ for $t \geqq 1$ if $u(t)$ and $v(t)$ are suitably extended. For example, $u(t) = 1$ and $v(t) = t$ are a possible choice for $t \geqq 1$. The function $m(t)$ so defined satisfies (6), (7), (8), and (9). Since $u(t)$ and $v(t)$ are assumed linearly independent when restricted to any subinterval of $[0,1]$, the function $m(t)$ has no singular points in this interval. If $(M(t,z))$ is the corresponding family of matrix valued entire functions defined by (12), we write $E(t,z) = A(t,z) - iB(t,z)$. Since $\alpha(a) > 0$ for $a > 0$ by our linear independence hypothesis, $E(a,z)$ satisfies (1) when $a > 0$. The equation (12) for the family $(M(t,z))$ implies (10) and (11) for the family $(E(t,z))$. If $f(t)$ belongs to $L^2(0,1)$, let its eigentransform $F(z)$ be defined according to part A of Theorem XI. If $g(t)$ is in $L^2(0,1)$ with eigentransform $G(z)$, and if $f = Tg$, then

$$\begin{aligned}\pi w F(w) &= w\int_0^1 f(t)\,[A(t,w)\,\bar{u}(t) + B(t,w)\bar{v}(t)]\,dt\\ &= w\int_0^1\int_0^1 g(x)\,[u(t)\bar{v}(x) - v(t)\,\bar{u}(x)]\,dx\\ &\qquad\qquad \times [A(t,w)\,\bar{u}(t) + B(t,w)\bar{v}(t)]\,dt\\ &= w\int_0^1 g(x)\int_0^x [u(t)\bar{v}(x) - v(t)\,\bar{u}(x)]\\ &\qquad\qquad \times [A(t,w)\,\bar{u}(t) + B(t,w)\bar{v}(t)]\,dt\,dx\\ &= \int_0^1 g(x)\,[A(x,w)\,\bar{u}(x) + B(x,w)\bar{v}(x)\\ &\qquad\qquad - A(x,0)\,\bar{u}(x) - B(x,0)\bar{v}(x)]\,dx\\ &= \pi G(w) - \pi G(0)\end{aligned}$$

for all complex w by (12) and (20). Interchange of the integrals is justified because the double integral converges absolutely by the Schwarz inequality. Let $\mathscr{H}$ be the set of all eigentransforms $F(z)$ of elements $f(t)$ of $\mathscr{M}$. Since $\mathscr{M}$ is a closed subspace of $L^2(0,1)$ by hypothesis and since (21) holds, $\mathscr{H}$ is a closed subspace of $\mathscr{H}(E(1))$. Since Tg belongs to $\mathscr{M}$ whenever g belongs to $\mathscr{M}$ by hypothesis, $[G(z) - G(0)]/z$ belongs to $\mathscr{H}$ whenever $G(z)$ belongs to $\mathscr{H}$ by what we have shown above. By Theorems III and IV of [8], $[F(z) - F(w)]/(z - w)$ belongs to $\mathscr{H}(E(1))$ whenever $F(z)$ belongs to $\mathscr{H}(E(1))$, for every complex number w. Since the transformation $R(w): F(z) \to [F(z) - F(w)]/(z - w)$ has a closed graph in $\mathscr{H}(E(1))$ as a result of (H2), it is bounded. The closed graph theorem can be avoided here at the cost of explicit estimates from Theorem III of [8]. It now follows that $[F(z) - F(w)]/(z - w)$ belongs to $\mathscr{H}$ whenever $F(z)$ belongs to $\mathscr{H}$, for every complex number w. For let Ω be the set of numbers w with this property. Since the resolvent identity

$$(z - w) R(z) R(w) = R(z) - R(w)$$

holds for all complex z and w, the power series

$$R(z) = R(w) \sum (z - w)^n R(w)^n$$

converges in the operator norm when $|z - w| < \| R(w) \|^{-1}$. It follows that Ω is both open and closed. Since this set contains the origin, it is the complex plane. We use this fact only to conclude that $F(z)/(z - w)$ belongs to $\mathscr{H}$ whenever $F(z)$ belongs to $\mathscr{H}$ and $F(w) = 0$. If $\mathscr{H}$ contains no nonzero element, then every element of $\mathscr{M}$ vanishes a.e. because of (21). If $\mathscr{H}$ contains a nonzero element, let $E(a, z)$ and h be defined for $\mathscr{H}$ as in Theorem IX with $b = 1$. The number τ, defined by (17) with $b = 1$, is equal to zero because of (18) and (19). Therefore, $h = 0$ and $\mathscr{H}$ coincides with $\mathscr{H}(E(a))$. The theorem now follows from part A of Theorem XI.

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