SEMI-DISCRETE ANALYTIC FUNCTIONS(1)

BY

G. J. KUROWSKI(2)

1. **Introduction.** Several papers have been written concerning discrete analogues for analytic functions (see, for example, [3; 4; 5; 7 and 8]). In each of these, either a discrete analogue to the Cauchy-Riemann equations or, in the case of Ferrand [7], a discrete version of Morera's theorem is used to define discrete analytic functions.

In [4], Isaacs considers two forms of discrete analogues. He calls those lattice-functions which satisfy the difference equation

$$f(z+1)-f(z)=[f(z+i)-f(z)]/i \tag{1.1}$$

monodiffric functions of the first kind and calls those which satisfy

$$f(z+1)-f(z-1)=[f(z+i)-f(z-i)]/i \tag{1.2}$$

monodiffric functions of the second kind. The discrete analytic functions considered by Ferrand [7; 8] and Duffin [3] satisfy a difference equation equivalent to (1.2).

Of concern here are single-valued functions of one continuous and one discrete variable defined on a semi-lattice, a uniformly spaced sequence of lines parallel to the real-axis. Such functions are called *semi-discrete.*

Definitions for the appropriate semi-discrete analogues of analytic functions are obtained from the classic Cauchy-Riemann equations on replacing the y-derivative by either a nonsymmetric difference

$$\frac{\partial f(z)}{\partial x}=[f(z+ih)-f(z)]/ih, \qquad z=x+ikh, \tag{1.3}$$

or a symmetric difference

$$\frac{\partial f(z)}{\partial x}=[f(z+ih/2)-f(z-ih/2)]/ih, \qquad z=x+ikh/2. \tag{1.4}$$

Presented to the Society, March 17, 1961; received by the editors April 11, 1961 and, in revised form, December 20, 1961.

(1) The work on this paper was sponsored partly by the office of Ordnance Research, U. S. Army, Contract DA-36-061-ORD-490 and partly by a fellowship granted by the International Telephone and Telegraph Corporation. This paper is based upon the thesis submitted by the author in partial fulfillment of the requirements for the Ph. D. at Carnegie Institute of Technology.

(2) Duke University, Research Associate, "Special Research in Numerical Analysis," sponsored by the Army Research Office, U. S. Army, Contract DA-31-124-ORD-10-8.

Semi-discrete functions which satisfy (1.3) or (1.4) are called, respectively, *semi-discrete analytic functions of the first, second kind.*

Helmbold [1] considers functions on a semi-lattice which satisfy the following semi-discrete analogue of Laplace's equation:

$$\frac{d^2u(x,k)}{dx^2} + [u(x,k+1) - 2u(x,k) + u(x,k-1)] = 0. \tag{1.5}$$

He calls these functions *semi-discrete harmonic.* It is shown that semi-discrete analytic (II) functions; that is, those which satisfy (1.4) for $h = 1$, have real and imaginary parts which satisfy (1.5).

With path integration defined on the semi-lattice, analogues for Cauchy's integral theorem and formula are presented. The derivative and indefinite integral of a semi-discrete analytic function are also shown to be semi-discrete analytic. The family of semi-discrete analytic functions is not closed under the usual multiplication; consequently, a modified "multiplication" having this property is discussed. Appropriate analogues for the powers of z, and thus polynomials, are obtained. A method called "extension" is presented which enables suitable functions to be extended as semi-discrete analytic functions into a rectangular domain of the semi-lattice.

2. **Definitions and notations.** A discussion of the basic concepts for the semi-discrete plane is given by Helmbold [1]. We will proceed in an analogous manner letting the abbreviation SD stand for semi-discrete.

A *grid-line*, 1_m, is the set of points in the xy-plane such that $y = mh$, where $h > 0$. The type I SD-plane, $L_1(h)$, is the semi-lattice

$$L_1(h) = \bigcup 1_m \qquad (m = 0, \pm 1, \pm 2, \dots); \tag{2.1}$$

the type II SD-plane, $L_2(h)$, is the semi-lattice

$$L_2(h) = \bigcup_m 1_{(1/2)m} \qquad (m = 0, \pm 1, \pm 2, \dots). \tag{2.2}$$

For $L_2(h)$, there are two associated semi-lattices; the union $G(2k)$ of the 1_m for $m = k$ called the *even* semi-lattice and the union $G(2k+1)$ of the 1_m for $m = (2k+1)/2$ called the *odd* semi-lattice. For each case, $k = 0, \pm 1, \pm 2, \dots$ and further $L_2(h) = G(2k) \cup G(2k+1)$.

Two points z_1 and z_2 of $L_2(h)$ are said to be *similar* if both belong to $G(2k)$ or both belong to $G(2k+1)$ and accordingly are called *even* or *odd.* Two grid-lines of $L_2(h)$ are *similar* if they are composed of similar points.

Whenever possible, further definitions will be stated simultaneously for both cases. Accordingly, we speak of the SD-plane $L_j(h)$ understanding that j may be either 1 or 2. Since all points of $L_1(h)$ could be called similar points, this terminology will be used for both cases.

On the continuous z-plane, to be denoted by L_c, we define the following sets

$$Q_k^{(1)} = \{(x,y)\,|\,\alpha_k \leqq x \leqq \beta_k;\ kh \leqq y \leqq kh + h\}, \tag{2.3}$$

$$Q_k^{(2)} = \{(x,y)\,|\,\alpha_k \leqq x \leqq \beta_k;\ (kh - h) \leqq 2y \leqq (kh + h)\} \tag{2.4}$$

where α_k and β_k are distinct real numbers. Let Q_j^c be a finite union of sets like $Q_k^{(j)}$. The set $Q_j^c \cap L_j(h) = Q_j$ will be called a *SD-domain* on $L_j(h)$. On $L_2(h)$, the set $Q_2^c \cap G(2k)$ is called an *even* SD-domain; the set $Q_2^c \cap G(2k+1)$ is called an *odd* SD-domain. Clearly, Q_2 is the union of an odd and an even SD-domain. If Q_j^c is simply-connected so is Q_j.

Let P_c be a polygonal path in L_c which is the connected union of a finite number of line segments; each segment being parallel to either the x or y-axis such that the horizontal segments lie on $L_j(h)$ along similar grid-lines (the lengths of the vertical segments are integral multiples of h). A *SD-path* P on $L_j(h)$ is the set

$$P = P_c \cap L_j(h). \tag{2.5}$$

The SD-path P is *closed* if P_c is closed.

Let C^c denote the boundary of Q_j^c. On $L_j(h)$ the *inner-boundary* C of Q_j is defined to be the set

$$C = C^c \cap L_j(h). \tag{2.6}$$

The *translator* E^n is defined by the relation

$$E^n(1_{am}) = 1_{a(m+n)}, \tag{2.7}$$

where $a = 1$ for $L_1(h)$ and $a = 1/2$ for $L_2(h)$. The result of applying E^n to a SD-domain Q_j is a vertical translation of nh/j.

Associated with the SD-domain Q_j is a larger domain, Q_j^*, called the *augmented* domain which is defined as follows.

$$Q_1^* = E^1(Q_1) \cup Q_1, \tag{2.8}$$

$$Q_2^* = E^1(Q_2) \cup E^{-1}(Q_2). \tag{2.9}$$

The *outer-boundary* of the SD-domain Q_j is the inner-boundary of the augmented domain Q_j^*. Clearly, both the inner and outer boundaries of the SD-domain Q_j are SD-paths on $L_j(h)$. The *total boundary*, C_T, of the SD-domain Q_j is the union of its inner and outer boundaries. The interior Q_j^0 of the SD-domain Q_j is the set $Q_j \sim C_T$. A SD-path P on Q_j is *contained* in Q_j if each point of P is also a point of Q_j^0.

A real or complex valued function of one continuous and one discrete variable is said to be a *SD-function* on a domain Q_j of $L_j(h)$ if it is defined and continuous in x (the continuous variable) for all points of Q_j and its outer-boundary. The subfamily of single-valued SD-functions defined on a SD-domain Q_1 of $L_1(h)$

which satisfies the differential difference equation (1.3) at the point $z \in Q_1$ is called the family of *h-SD analytic functions of the first kind* at the point z [abbreviated, h-SDA(I)].

On $L_2(h)$, the subfamily of single-valued SD-functions defined on a SD-domain Q_2 which satisfies the differential difference equation (1.4) at the point $z \in Q_2$ is called the family of *h-SD analytic functions of the second kind* at the point z [abbreviated, h-SDA(II)].

Equations (1.3) and (1.4) are called the *defining equations* for the respective h-SDA functions. A single-valued SD-function $f(z)$ is said to be h-SDA(I) or h-SDA(II) on a domain Q_j if $f(z)$ satisfies the appropriate defining equation for all points z of Q_j.

3. **Calculus of semi-discrete functions.** It is convenient to introduce the following operators on $L_j(h)$:

$$
\text{(3.1)} \qquad
\begin{aligned}
&\text{(a)} \quad \Delta_1 f(z) = f(z+ih) - f(z),\\
&\text{(b)} \quad \Delta_2 f(z) = f(z+ih/2) - f(z-ih/2),\\
&\text{(c)} \quad \Delta_j^{n+1} f(z) = \Delta_j[\Delta_j^n f(z)], \qquad n \geqq 1,\\
&\text{(d)} \quad \nabla_j f(z) = \frac{\partial^2 f(z)}{\partial x^2} + \Delta_j^2 f(z),\\
&\text{(e)} \quad 2S_j f(z) = \frac{\partial f(z)}{\partial x} - \frac{i}{h}\Delta_j f(z),\\
&\text{(f)} \quad 2\bar{S}_j f(z) = \frac{\partial f(z)}{\partial x} + \frac{i}{h}\Delta_j f(z), \quad \text{and}\\
&\text{(g)} \quad 2\bar{S}_B f(z) = \frac{\partial f(z)}{\partial x} + \frac{i}{h}\Delta_1 f(z-ih).
\end{aligned}
$$

Without loss of generality, we will consider in this and the following section only the case $h = 1$ and will write $L_j(1) = L_j$. The definitions and results are easily extended to $L_j(h)$. We introduce the following *basic rectangles* on L_1 and L_2 respectively

$$
\text{(3.2)} \qquad
\begin{aligned}
B_1(M,N) &= \{(x,y) \mid \alpha \leqq x \leqq \beta; \quad y = M, M+1, \dots, N\},\\
B_2(M,N) &= \{(x,y) \mid \alpha \leqq x \leqq \beta; \; 2y = M, M+1, \dots, N\},
\end{aligned}
$$

where M, N are integers, $M \leqq N$. The proofs for all theorems are established by proving the theorem for $B_j(M,N)$ and observing that the general result follows by juxtaposition. At all times D will denote a simply-connected SD-domain on $L_j(h)$, though this restriction will not always be necessary.

The *path-integral* on L_j of a SD-function $f(z)$ is defined as follows:

(a) Along the horizontal segment whose endpoints are $a = \alpha + iy$ and $b = \beta + iy$:

$$\int_a^b f(z)\delta z = \int_\alpha^\beta f(t+iy)dt. \tag{3.3a}$$

(b) Along the directed vertical segment connecting the adjacent points $a = \alpha + iy$ and $b = \alpha + i(y+1)$:

$$\int_\alpha^b f(z)\delta z \begin{array}{ll} = if(\alpha + iy) & \text{[Type I]}, \\ = if(\alpha + iy + i/2) & \text{[Type II]}. \end{array} \tag{3.3b}$$

Along the directed vertical segment connecting the adjacent points b and a the integral is defined to be the negative of the integral from a to b.

A path-integral over a closed SD-path is to be taken in the conventional positive direction. To illustrate, let P be the inner boundary of either $B_1(k, k+1)$ or $B_2(2k-1, 2k+1)$. In each case this path integral is

$$\oint_P f(z)\delta z = -\int_\alpha^\beta (\Delta_j f_k)dx + i[f_k(\beta) - f_k(\alpha)], \tag{3.4}$$

where we introduce the notational convention

$$f_k = f(x+ik) = f_k(x). \tag{3.5}$$

The following provides an analogue for Green's formula.

THEOREM 3.1. *Let $f(z)$ be a SD-function on* (I) *a SD-domain Q of L_1;* (II) *an even* (*odd*) *SD-domain Q of L_2. If P denotes the outer-boundary of Q on L_j, then*

$$2i \sum_Q \int \bar{S}_j(f)dx = \oint_P f(z)\delta z. \tag{3.6}$$

The proofs for this theorem and the remaining theorems in this section are straightforward and accordingly are omitted. Note that Theorem 3.1 is applicable to the usual domain on L_2 if the path P denotes the total-boundary.

To obtain an analogue for the Cauchy integral formula, a modified formula is required. Let f, g be a pair of SD-functions on the basic-rectangles $B_1(N, M)$ and $E^{-1}(B_1)$ respectively and let the outer-boundary of B_1 be denoted by P. We define the *modified path-integral*

$$\oint_P [f(z);\ g(z)]\delta z = \int_\alpha^\beta [f_k g_{k-1}]_{k=M+1}^N dx + i\sum_{k=N}^M [f_k g_k]_{x=\alpha}^\beta \quad \text{[Type I]}. \tag{3.7}$$

For the special case $g(z) = 1$, (3.7) reduces to the previous definition for the path-integral of $f(z)$ over P.

Sum-integration over $B_1(N, M)$ of the identity

$$\frac{\partial}{\partial x}(f_k g_k) + i\Delta_1(f_k g_{k-1}) = 2[f_k \bar{S}_B(g_k) + g_k \bar{S}_1(f_k)]$$

establishes a further analogue of Green's theorem on L_1.

THEOREM 3.2. *Let $f(z)$ be a SD-function on the SD-domain D of L_1 and let $g(z)$ be a SD-function on $E^{-1}(D)$. If P is the outer-boundary of D, then*

$$2i\sum_D \int [g\bar{S}_1(f) + f\bar{S}_B(g)]dx = \oint_P [f;g]\delta z. \tag{3.8}$$

To develop the Green's theorem for a pair of SD-functions f, g of the second kind, we let P denote the total-boundary of $B_2(M, N)$, and define

$$\begin{aligned}\oint_P [f;g]\delta z = \int_\alpha^\beta &(f_M g_{M-1/2} + f_{M-1/2}\, g_M - f_{N+1/2}\, g_N - f_N g_{N+1/2})dx \\ &+ i\sum_{k=M}^{N-1} [f_{k+1/2}\, g_{k+1/2}]_\alpha^\beta + i\sum_{k=M}^{N} [f_k g_k]_\alpha^\beta \qquad \text{[Type II]}.\end{aligned} \tag{3.9}$$

This definition reduces to the definition for path-integration of a SD(II)-function about the total-boundary of $B_2(M,N)$ if either $f = 1$ or $g = 1$. The analogue of Green's theorem on L_2 is a consequence of (3.9).

THEOREM 3.3. *If $f(z)$ and $g(z)$ are SD-functions of the second kind on a SD-domain D of L_2 and if P denotes the total-boundary of D, then*

$$2i\sum_D \int [f\bar{S}_2(g) + g\bar{S}_2(f)]dx = \oint_P [f;g]\delta z. \tag{3.10}$$

4. **Basic properties of semi-discrete analytic functions.** Functions which are SDA (semi-discrete analytic) on a domain D of L_j satisfy a pair of equations analogous to the Cauchy-Riemann equations of classical function theory. Letting $f(z) = u(x, y) + iv(x, y)$ be SDA on D, where u and v are real-valued SD-functions, equating the real and imaginary parts of (1.3) and (1.4) respectively yields the *SD Cauchy-Riemann equations.*

$$\begin{aligned}&\text{Type (I)} && \frac{\partial u(x,y)}{\partial x} = v(x, y+1) - v(x, y), \text{ and} \\ & && \frac{\partial v(x,y)}{\partial x} = u(x, y) - u(x, y+1). \\ &\text{Type (II)} && \frac{\partial u(x,y)}{\partial x} = v(x, y+\tfrac{1}{2}) - v(x, y-\tfrac{1}{2}), \text{ and} \\ & && \frac{\partial v(x,y)}{\partial x} = u(x, y-\tfrac{1}{2}) - u(x, y+\tfrac{1}{2}).\end{aligned} \tag{4.1}$$

Since

$$4S_j[\bar{S}_j(f)] = 4\bar{S}_j[S_j(f)] = \nabla_j(f), \tag{4.2}$$

if $f(z)$ is SDA on D, then $\nabla_j(f) = 0$ for all $z \in D^0$ and consequently

$$\nabla_j(u) = \nabla_j(v) = 0.$$

SD-functions g such that $\nabla_j(g) = 0$ are called *semi-discrete harmonic* functions of the first or second kind (SDH). The SDH functions of the second kind are the semi-discrete harmonic functions considered by Helmbold [1], see equation (1.5), who called such functions $1/2$-*harmonic*.

The common value of the defining equation is called the *derivative* of $f(z)$, denoted by $f'(z)$; i.e.,

$$f'(z) = \frac{\partial f(z)}{\partial x} = -i\Delta_j f(z). \tag{4.3}$$

It should also be noted that $f'(z)$ is given by

$$f'(z) = S_j(f). \tag{4.4}$$

THEOREM 4.1. *If $f(z)$ is SDA on the augmented SD-domain D^* of L_j, its derivative $f'(z)$ is SDA on D.*

This theorem and many of the others in this section are either direct consequences of definitions or result from classic arguments. Hence their proofs will not be included.

THEOREM 4.2. *If $f(z)$ is SDA on a simply-connected domain D of L_j and if C is a closed SD-path contained in D, then*

$$\oint_C f(z)\delta z = 0. \tag{4.5}$$

As in classic function theory, D need not be simply connected; only the subdomain of D whose inner-boundary is the SD-path C must be simply-connected.

As a consequence of (4.5), the value of the path-integral of f taken along a SD-path joining two similar points of D is independent of the SD-path chosen in D provided that f is SDA on D. Accordingly, for each SDA function $f(z)$ on D we may introduce the function

$$F(z) = \int_a^z f(z)\delta z, \tag{4.6}$$

where a and z are similar points of D and the path of integration is contained in D. The function $F(z)$ given by (4.6) is called the *primitive* of $f(z)$.

THEOREM 4.3. *If $f(z)$ is SDA on a SD-domain D of L_j, its primitive $F(z)$ is also SDA on D and further $F'(z) = f(z)$.*

Proof. It is sufficient to consider the following SD-path C joining the similar points $a = \alpha + i\beta$ and $z = x + iy$.

$$C: \left\{ \begin{array}{l} \text{Along the horizontal segment } y = \beta \text{ from } \alpha \text{ to } x, \\ \text{then along the vertical segment at } x \text{ from } \beta \text{ to } y. \end{array} \right\}$$

The stated result is then a consequence of definitions. QED.

Evaluation along C of (4.6) with $f = g'$ establishes the following.

THEOREM 4.4. *Let $g(z)$ be SDA on the augmented SD-domain D^* of L_j; let a and z be similar points of D and let C be a SD-path contained in D which joins a and z. Then,*

$$\int_a^z g'(z)\delta z = g(z) - g(a). \tag{4.7}$$

Following the terminology of Duffin [3], a SD-function on $L_2(h)$ which is constant on both $G(2k)$ and $G(2k+1)$, these values not necessarily identical, is said to be a *bi-constant*. A SD-function which is constant on $L_1(h)$ will also be called a bi-constant. Hence, by Theorem 4.4, a SDA function is, up to an additive bi-constant, determined uniquely by the primitive of its derivative.

THEOREM 4.5. *Let $f(z)$ be SDA on the augmented domain D^* of L_j such that $f'(z) = 0$ for all points z of D. Then $f(z)$ is identically a bi-constant on D^*.*

The following is a converse to Theorem 4.2 and as such is analogous to Morera's theorem:

THEOREM 4.6. *Let $f(z)$ be a SD-function defined on the augmented domain D^* of L_j. If the path-integral of $f(z)$ about all closed SD-paths contained in D^* is zero, then $f(z)$ is SDA on D.*

THEOREM 4.7. *If $f(z)$ is h-SDA at the point z of $L_j(h)$ and λ is a real constant, then $f(\lambda z)$ is λh-SDA at the point λz.*

A direct consequence of (1.3) and (1.4) is the following "uniqueness" theorem.

THEOREM 4.8. *Let $f(z)$ be SDA on a SD-domain of L_j such that* (I) $|f(z)| = 0$ *for all points z of P, the outer-boundary of D,* (II) $|f(z)| = 0$ *for all points z of P, the total-boundary of D. Then $f(z)$ is identically zero on D.*

By the above, a SDA function is determined up to a bi-constant on D by its values on (I) the outer-boundary P of D, (II) the total-boundary P of D.

Application of Theorems 3.2 and 3.3 results in a further analogue of Cauchy's theorem.

THEOREM 4.9. (I) *Let $f(z)$ and $g(z)$ be SD-functions on a domain D of L_1 such that $\bar{S}_1(f) = 0$ and $\bar{S}_B(g) = 0$ for all z of D. If C is a closed SD-path contained in D, then*

$$\oint_C [f(z); g(z)]\delta z = 0. \tag{4.8}$$

(II) *Let* $f(z)$ *and* $g(z)$ *be* SDA(II) *on a SD-domain* D *of* L_2. *If* P *is the total-boundary of a subdomain* D' *of* D, *then*

$$\oint_P [f(z); g(z)]\delta z = 0. \tag{4.8'}$$

DEFINITION. The SD-path of Theorem 4.9(II) is called a *closed total-path*.

To develop an analogue for the Cauchy integral formula, a *singularity function*, $G_j(z\,;\zeta)$, having the following properties is required:

(A). $G_j(z\,;\zeta)$ is continuous for all z of L_j except the point ζ where for real $\varepsilon > 0$ it has the following jump-discontinuity:

$$\lim_{\varepsilon\to 0}\ [G_j(\zeta+\varepsilon\,;\zeta) - G_j(\zeta-\varepsilon\,;\zeta)] = 1.$$

(BI). $G_1(z\,;\zeta)$ satisfies the differential-difference equation $S_B(G) = 0$ for all z of L_1 except the point $z = \zeta$.

(BII). $G_2(z\,;\zeta)$ is SDA(II) for all z of L_2 except the point $z = \zeta$.

(C). As $|y| \to \infty$, $G_j(z\,;\zeta) = 0(|y|^{-1})$; as $|x| \to \infty$, $G_j(z\,;\zeta) = 0(|x|^{-1})$.

The explicit expression for the respective singularity function may be obtained formally by applying the operational calculus described by Helmbold [1]. Accordingly, letting $\zeta = 0$ and defining

$$\operatorname{sgn}(t) = \begin{matrix} 1, & t > 0, \\ -1, & t < 0, \end{matrix} \tag{4.9}$$

we obtain the following functions:

$$G_1(z\,;0) = \frac{\operatorname{sgn}(t)}{2\pi}\int_0^\pi F(z\,;y)dy, \qquad t \neq 0, \qquad \text{[Type I]} \tag{4.10}$$

where $z = t + im$, $m = 0, \pm 1, \pm 2, \ldots$ and

$$F(z, y) = \exp\left[-t\left(\operatorname{sgn}(t)\sin y + 2i\sin^2\frac{y}{2}\right) - imy\operatorname{sgn}(t)\right].$$

$$\begin{aligned} G_2(z\,;0) = {} & \frac{\operatorname{sgn}(t)}{2\pi}\int_0^\pi e^{-2|t|\sin u}\cos(mu)\,du \\ & - \frac{i}{2\pi}\int_0^\pi e^{-2|t|\sin u}\sin(mu)\,du, \end{aligned} \qquad \text{[Type II]} \tag{4.11}$$

where $z = t + im/2$, $m = 0, \pm 1, \pm 2, \ldots$. The singularity function whose singularity occurs at the point (I) $\zeta = c + in$,(II) $\zeta = c + in/2$ is obtained from (4.10), (4.11) with the substitution $t = (x - c)$and $m = (k - n)$.

It is easily established that each of the above functions possesses the properties (A), (C), and (BI) or (BII). To develop the SD analogue for Cauchy's integral formula, we consider each case separately.

If $f(z)$ is SDA(I) on a SD-domain D of L_1 and $g(z)$ is a SD-function on $E^{-1}(D)$, (3.8) shows that for every closed SD-path P in D

$$\oint_P [f;g]\delta z = 2i \sum_D \int f S_B(g)dx. \tag{4.12}$$

In (4.12), let g be the singularity function $G_1(z;\zeta)$. By property (BI), the value of this path-integral for each closed SD-path P not having $z = \zeta$ (ζ not a point of P) as an interior point is zero. Thus, we need consider only the SD-path P which is the outer-boundary of the basic-rectangle $B_1(M,N)$ containing ζ as an interior point. Let P_1 denote the outer-boundary of the rectangle $B_1'(M,N)$ which for real $\varepsilon > 0$ is defined to be the rectangle obtained on replacing β with $c-\varepsilon$ in (3.2) and let P_2 denote the outer-boundary of the rectangle $B_1''(M,N)$ obtained on replacing α with $c+\varepsilon$ in (3.2). Clearly,

$$B_1(M,N) = \lim_{\varepsilon\to 0} (B_1' \cup B_1'').$$

Consideration of (4.12) for $B_1' \cup B_1''$ as $\varepsilon \to 0$, using properties (A) and (BI), leads to the following analogue of the Cauchy integral formula on L_1.

THEOREM 4.10. *Let $f(z)$ be SDA*(I) *on a SD-domain D of L_1. For an interior point $\zeta = c + in$ of D, let $G_1(z;\zeta)$ be the singularity function. If P is the outer-boundary of a subdomain R of D which contains the point ζ, then*

$$\oint_P [f(z);G_1(z;\zeta)]\,\delta z = if(\zeta). \tag{4.13}$$

Otherwise (provided ζ is not a point of P,) the value of this path integral on D is zero.

A further integral formula involving $G_1(z;\zeta)$, proven with a similar argument, is the following:

THEOREM 4.11. *Let $f(z)$ be SDA*(I) *on an augmented SD-domain D^* of L_1 and let P be a closed path in D such that the point ζ is an interior point. Then*

$$\oint_P [f(z);G_1'(z;\zeta)]\,\delta z = -if'(\zeta), \qquad \textit{where} \tag{4.14}$$
$$G_1'(z;\zeta) = \frac{\partial G_1(z;\zeta)}{\partial x} = -i\Delta_1 G_1(z-i;\zeta).$$

The analogue of the Cauchy integral formula on L_2 is established similarly. For real $\varepsilon > 0$, the rectangles $B_2'(M,N)$ and $B_2''(M,N)$ are defined as before so that $B_2(M, N) = \lim_{\varepsilon\to 0} B_2' \cup B_2''$. Applying Theorem 3.3, using properties (A) and (BII) as $\varepsilon \to 0$, leads to the integral formula for SDA(II) functions.

THEOREM 4.12. *Let $f(z)$ be SDA(II) on a SD-domain D of L_2. If $G_2(z;\zeta)$ is the singularity function whose singularity occurs at the point $\zeta = c + in/2$ and if P is a closed total-path contained in D having ζ as an interior point, then*

$$\oint_P [f(z); G_2(z;\zeta)]\,\delta z = if(\zeta). \tag{4.15}$$

A further analogue of the integral formula is the following:

THEOREM 4.13. *Let $f(z)$ be SDA(II) on a SD-domain D of L_2 and let P be a closed total-path contained in D having ζ as an interior point. Then,*

$$\oint_P [f(z); G_2'(z;\zeta)]\,\delta z = -if'(\zeta). \tag{4.16}$$

As noted previously, the real and imaginary part of an SDA(II) function on D are 1/2-harmonic on both D_0 and D_E, the odd and even domains which comprise D. Helmbold [1] proves that a 1/2-harmonic function on $D_0(D_E)$ cannot attain its maximum or minimum value on $D_0^0(D_E^0)$ unless it is identically constant on $D_0(D_E)$, a result which is used in the proof for the following "maximum" theorem:

THEOREM 4.14. *Let $f(z)$ be SDA(II) on a SD-domain of L_2 and let P denote the total-boundary of this domain D. Then $|f(z)|$ cannot attain its maximum value for a point z of the total-interior of D, unless $f(z)$ is identically a bi-constant on D.*

Proof. Let z_1 be a fixed, arbitrary point of the total interior of D and let $f = u + iv$ be SDA(II) on D. If $|f(z)| = 0$ the theorem is trivially true. If $|f(z)| \neq 0$, application of the maximum principle of Helmbold [1] to the real-valued 1/2-harmonic SD-function

$$\Phi(z) = u(z_1)\,u(z) + v(z_1)\,v(z)$$

leads to the stated result. QED.

Since $|f|$ is continuous on each closed line segment of D and since D contains only a finite number of segments, by the theorem of Weierstrass, $|f|$ must take its maximum value on D. By the above, this extremal must occur on the total-boundary of D.

Such an analogue for the maximum theorem has not been established for SDA(I) functions. The following result concerning the uniqueness of the singularity function of the second kind is a consequence of the above maximum theorem:

THEOREM 4.15. *The type* II *singularity function* $G_2(z;\zeta)$ *having the defining properties* (A), (BII), *and* (C) *is unique up to additive SDA*(II) *functions.*

Proof. Let $g(z)$ be another SD-function having the defining properties (A), (BII), and (C) and let $\zeta = 0$. Defining the function $h(z) = G_2(z;0) - g(z)$ and integrating $\bar{S}_2(h)$ along the line $(x,0)$ of L_2 shows that

$$0 = h_0(t) + i\int_\alpha^t \Delta_2(h_0)dx + C,$$

where C is a constant. Consequently, $h_0(t)$ is continuous for all t and, in addition, satisfies the defining equation (1.4). Thus $h(z)$ is SDA(II) on L_2.

Given a real $\varepsilon > 0$, choose a total-boundary P such that $|G| \leqq \varepsilon/2$ and $|f| \leqq \varepsilon/2$ for all z_p of P. Then $|h(z_p)| \leqq \varepsilon$. Hence by Theorem 4.14, if z is an arbitrary point of the region interior to P, $0 \leqq |h(z)| \leqq \varepsilon$. QED.

5. **Analytic extension and multiplication.** A direct solution of the defining equation (1.3) can be obtained by applying Boole's symbolic method [9]. This solution is

$$F(z) = \sum_{k=0}^{y/h} C_k^y(h) i^k \frac{d^k f(x)}{dx^k}, \qquad y = Nh, \tag{5.1}$$

where $f(x)$ is an arbitrary function of x having at least N continuous derivatives and $C_k^y(h)$ is the factorial polynomial for $y \geqq 0$ defined by

$$\begin{aligned} &\text{(a)} \quad k!\,C_k^y(h) = y(y-h)\dots(y-[k-1]h), \\ &\text{(b)} \quad C_0^y(h) = 1, \text{ and } C_k^y(h) = 0 \text{ whenever } k > y. \end{aligned} \tag{5.2}$$

LEMMA 1. *If* $C_k^y(h)$ *is the factorial polynomial of* y *on* $L_1(h)$, *then*

$$\Delta_1 C_0^y(h) = 0 \text{ and } \Delta_1 C_k^y(h) = hC_{k-1}^y(h);\ k \geqq 1. \tag{5.3}$$

We will call the h-SDA(I) function $F(z)$ defined by (5.1) the *extension* of $f(x)$. By (1.3), $F(z)$, is uniquely determined by $f(x)$.

THEOREM 5.1. *Let* $f(x)$ *be continuous with* M *continuous derivatives on the closed interval* $[\alpha,\beta]$. *The extension* $F(z)$ *of* $f(x)$ *is h-SDA*(I) *on the rectangle* $B_1(0,M)$ *and further, this extension is unique.*

As an example, let $f(x) = x^n$, where n is a positive integer. From (5.1), the following function, denoted by $(z;h)_1^{(n)}$, is the extension of x^n on $L_1(h)$:

$$(z;h)_1^{(n)} = \sum_{p=0}^{n} C_p^y(h) i^p \frac{n!}{(n-p)!} x^{n-p}. \tag{5.4}$$

These functions will be called the *pseudo-powers* of z on $L_1(h)$.

It is easily seen that the usual product of two SDA(I) functions is not SDA(I). Accordingly, the method of extension is applied to investigate "multiplication" on $L_1(h)$. Let $F(z)$, $G(z)$, and $H(z)$ be the respective extensions of $f(x)$, $g(x)$, and $h(x)$ on $B_1(0,M)$. Using Leibnitz' rule for the nth-derivative of the product $h(x) = f(x)g(x)$,

$$\text{(a)} \qquad H(z) = \sum_{k=0}^{y} \sum_{p=0}^{k} C_k^y \binom{k}{p} i^k \frac{d^p f(x)}{dx^p} \frac{d^{k-p} g(x)}{dx^{k-p}},$$

$$\text{where } \binom{k}{p} = \frac{k!}{p!(k-p)!} \text{ and } C_k^y = C_k^y(1).$$

We call $H(z)$ the *generalized dot-product* of $F(z)$ with $G(z)$, to be denoted by $F(z) \cdot G(z)$. Algebraic manipulation enables us to rewrite (a) in the form

$$\text{(5.5)} \qquad F(z) \cdot G(z) = \sum_{k=0}^{y} i^k C_k^y G(z - ik) \frac{d^k f(x)}{dx^k},$$

which provides a simple definition for the generalized dot-product on L_1. By construction, this "multiplication" is commutative and associative.

As an application, let $f(x) = x$. By (5.4), the extension of x is $F(z) = (z\,;1)_1^{(1)}$ and (5.5) yields

$$\text{(5.6)} \qquad (z\,;1)_1^{(1)} \cdot G(z) = \sum_{k=0}^{1} i^k C_k^y G(z - ik) \frac{d^k[x]}{dx^k} = xG(z) + iyG(z - i).$$

Note also that if $I(z)$ is the extension of x^{-1} on L_1, then $(z\,;1)_1^{(1)} \cdot I(z) = 1$. Another consequence of the definition for the generalized dot-product is the following.

THEOREM 5.4. *Let $H(z) = F(z) \cdot G(z)$. The derivative of $H(z)$ is obtained by the usual rule for product differentiation; that is,*

$$\text{(5.7)} \qquad H'(z) = F(z) \cdot G'(z) + F'(z) \cdot G(z).$$

We next consider extension on $L_2(h)$. Let $T_n^y(h)$ denote the symmetric factorial polynomials of y and $n \geqq 0$ which were discussed by Isaacs [4].

Odd n

$$n!\,T_n^y(h) = \left[y + \frac{n-2}{2}h\right] \cdots \left[y + \frac{3}{2}h\right] \left[y + \frac{1}{2}h\right] y \left[y - \frac{1}{2}h\right] \cdots \left[y - \frac{n-2}{2}h\right],$$

(5.8)

Even n

$$n!\,T_n^y(h) = \left[y + \frac{n-2}{2}h\right] \cdots [y+h] y^2 [y-h] \cdots \left[y - \frac{n-2}{2}h\right],$$

where $T_0^y(h) = 1$, $T_1^y(h) = y$, and $T_k^y(h) = 0$ whenever $k \geqq 2(|y| + h)/h$. The proof of the following lemma is given by Isaacs [4] for a unit gap, but is easily extended to the general case.

LEMMA 2. *If $T_n^y(y)$ is the factorial polynomial of y on $L_2(h)$, then*

$$\Delta_2 T_0^y(h) = 0 \text{ and } \Delta_2 T_n^y(h) = hT_{n-1}^y(h),\ n \geqq 1. \tag{5.9}$$

Following the discussion concerning functions of the first kind, we define a function $F(z)$ on $L_2(h)$ called the extension of $f(x)$ into $L_2(h)$:

$$F(z) = \sum_{k=0}^{\infty} i^k T_k^y(h) \frac{d^k f(x)}{dx^k}. \tag{5.10}$$

Although F is written as an infinite series, by definition there are only a finite number of nonzero terms for finite values of y. Thus, the convergence of (5.10) is trivial.

THEOREM 5.3. *Let $f(x)$ have $M+1$ continuous derivatives on the closed interval $[\alpha, \beta]$. Then the extension $F(z)$ of $f(x)$ is h-SDA(II) on $B_2(0,M)$.*

For functions of the first kind, (5.1) represents the only way that $f(x)$ can be extended into the upper-half of $L_1(h)$ as an h-SDA(I) function. This is not true for the analogous expression (5.10) defining the extension of $f(x)$ as an h-SDA(II) function since (1.4) shows that, for a unique extension, $f(x - ih/2)$ must also be given. In fact, summation of (1.4) gives the following

$$f_n = f_{-1/2} + i \sum_{k=0}^{2n-1} \frac{\partial f_{(1/2)k}}{\partial x} \tag{5.11}$$

which provides a recursive method for extension into $L_2(h)$ that is unique if f_0 and $f_{-1/2}$ are specified. Accordingly, this represents the general method for extension on $L_2(h)$. However, if $F(x + ih/2)$, as given by (5.10), and $f(x + ih/2)$, as specified in advance, agree, the extension (5.10) of $f(x)$ will be unique. For the present purpose, we confine our attention to specific applications of (5.10) for considering a "multiplication" on $L_2(h)$.

The nth *pseudo-power* of z on $L_2(h)$ is defined to be the extension of x^n on $L_2(h)$; that is,

$$(z\,; h)_2^{(n)} = \sum_{k=0}^{n} i^k T_k^y(h) \frac{n!}{(n-k)!} x^{n-k}; \quad n = 0,1\ 2, \ldots. \tag{5.12}$$

Letting $F(z)$ and $G(z)$ be the respective extensions of $f(x)$ and $g(x)$ on $B_2(0,M)$, the generalized dot-product of F and G is defined to be the extension of fg into $B_2(0,M)$; that is,

$$F(z) \cdot G(z) = \sum_{k=0}^{\infty} i^k T_k^y(h) \frac{d^k}{dx^k} [fg]. \tag{5.13}$$

By Theorem 5.3, this dot-product results in an h-SDA(II) function on $B_2(0,M)$. The derivative is computed by the usual rule for the differentiation of products

(5.7). In particular, if $G(z)$ represents the extension of $g(x)$ into $B_2(0, M)$, we define the dot-product of G with $(z\,; h)_2^{(1)}$ to be the extension of xg into $B_2(0, M)$. It is unfortunate, however, that it is not possible to obtain a simple expression for $(z\,; h)_2^{(1)} \cdot G$ to correspond to (5.6).

The following establishes that the respective pseudo-powers of z on $L_j(h)$ behave in a manner analogous to the powers of z on L_c.

THEOREM 5.4. *If $P_n(z, h) = (z\,; h)_j^{(n)}$, where n is an integer, $n \geqq 0$, then for all z of $L_j(h)$*

$$P_0'(z, h) = 0 \text{ and } P_n'(z, h) = nP_{n-1}(z, h), \quad n \geqq 1; \tag{5.14}$$

$$n \int_0^z P_{n-1}(z, h)\delta z = P_n(z, h), \quad n \geqq 1. \tag{5.15}$$

Inspection of the explicit formulae (5.4) and (5.12) for the respective kth pseudo-power of z on $L_j(h)$ shows that as the gap of $L_j(h)$ approaches zero, the pseudo-power $(z\,; h)_j^{(k)}$ converges to z^k.

THEOREM 5.5. *Let R be the rectangle $R = \{(x, y) : |x| \leqq A, |y| \leqq B\}$ of $L_j(h)$. Then, for all z of R*

$$\lim_{h \to 0} (z\,; h)_j^{(k)} = z^k, \tag{5.16}$$

where the convergence is uniform on R.

6. **Semi-discrete analytic polynomials.** A SD-polynomial is a SD-function of the form

$$p(z) = \sum_{m=0}^{M} \sum_{n=0}^{N} C_{mn}(h) x^m y^n, \tag{6.1}$$

where the coefficients $C_{mn}(h)$ may be polynomials in h with real or complex coefficients. Since $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/2i$, $p(z)$ can be considered as a function of z and $\bar{z}$ and written in the conjugate form $P(z, \bar{z})$. The *degree* of the SD-polynomial $p(z)$, denoted by $\deg(p)$, is the total degree of the conjugate form $P(z, \bar{z})$ in z and $\bar{z}$.

A SD-polynomial which is h-SDA on $L_j(h)$ is said to be a *SDA polynomial.* By the expressions (5.4) and (5.12) for the respective pseudo-powers of z, these functions are SDA polynomials, as are finite linear combinations of these powers. Following Isaacs [4], we will show that essentially the only SDA polynomials are the SD-polynomials which are linear combinations of the pseudo-powers of z.

THEOREM 6.1. *If $p(z)$ is a SDA polynomial of degree n, it is of the form*

$$p(z) = k(h)z^n + G(z, \bar{z}), \tag{6.2}$$

where $\deg(G) < n$, and $k \not\equiv 0$.

Proof. By hypothesis, the conjugate form $P(z,\bar{z})$ is a SDA polynomial and thus the polynomial $\lambda^{-n}P(\lambda z,\lambda\bar{z})$ is λh-SDA. Clearly, the limit

$$\lim_{\lambda\to 0} \lambda^{-n}P(\lambda z,\lambda\bar{z}) = Q(x,y)$$

exists and is a polynomial. If $Q(x,y)$ does not satisfy the Cauchy-Riemann equations, then for some λ sufficiently small $\lambda^{-n}P(\lambda z,\lambda\bar{z})$ would not satisfy the defining equation (1.3) or (1.4). Hence, $Q(x,y)$ must be analytic. Further, since $\deg(P) = n$, P has at least one term of degree n with a nonzero coefficient $k(h)$. If this term were to involve $\bar{z}$, then so must Q; in which case, Q would not be analytic. QED.

An immediate consequence of Theorems 6.1. and 4.1 is the following.

THEOREM 6.2. *If $p(z)$ is a SDA polynomial of degree n, its derivative is a SDA polynomial of degree* $(n-1)$.

If $f(z)$ is SDA, $f'(z)$ is given by $S_j(f)$. Letting $f^{[n]}(z)$ denote the nth derivative of $f(z)$, by recursion

$$S_j^n(f) = f^{[n]}(z). \tag{6.3}$$

The following notation is used in subsequent discussions:

$$f^{[n]}(z)\,\big|_{z=a} = S_j^n f(a). \tag{6.4}$$

THEOREM 6.3. *If $p(z)$ is a SDA polynomial,* $\deg(p) = n$, *then*

$$S_j^n(p) = kn!\ , \textit{ and } S_j^{n+1}(p) = 0. \tag{6.5}$$

Proof. Using Theorem 6.1, $p(z)$ is of the form (6.2). Repeated operation on $p(z)$ with the operator S_j and the use of Theorem 6.2 after each such operation yields (6.5). QED.

THEOREM 6.4. *Let $p(z)$ be a SDA polynomial such that* $\deg(p) \leqq n$. *If* $p(0) = 0$, $S_jp(0) = 0, \ldots, S_j^np(0) = 0$, *then* $p(z) \equiv 0$.

Proof. By Theorem 6.1, $p(z)$ is possibly of the form (6.2) with $k \neq 0$. However, Theorem 6.3, shows that k must be zero. Thus, $\deg(p) \leqq (n-1)$. By repetition of this argument $p(z)$ must be identically zero. QED.

The following is the SD-analogue of the Taylor expansion for SDA polynomials:

THEOREM 6.5. *If $p(z)$ is a SDA polynomial such that* $\deg(p) = n$, *it may be written in the form*

$$p(z) = \sum_{k=0}^{n} a_k(z\,;h)_j^{(k)}, \textit{ where } a_k = \frac{1}{k!}S_j^kp(0). \tag{6.6}$$

Proof. Application of Theorem 6.4 to the SDA polynomial

$$g(z) = p(z) - \sum_{k=0}^{n} \left[\frac{1}{k!} S_j^k p(0)\right] (z\,;h)_j^{(k)}$$

shows that $g(z) \equiv 0$. QED.

Consider the h-SDA polynomial $P(z,h)$ defined by

$$P(z,h) = \sum_{m=0}^{n} a_m h^{n-m} (z\,;h)_j^{(m)}. \tag{6.7}$$

It is easily verified that each term of this polynomial is of total degree n in the three variables x, y, and h. Consequently, the polynomial $P(z,h)$ of (6.7) is the homogeneous polynomial of degree n in x, y, and h corresponding to the SDA polynomial

$$p(z) = \sum_{m=0}^{n} a_m z_j^{(m)}, \text{ where } z_j^{(m)} \equiv (z\,;1)_j^{(m)}. \tag{6.8}$$

THEOREM 6.6. *The real and imaginary parts of a SDA polynomial are relatively prime.*

Proof. Consider the SDA polynomial of (6.8) and let

$$F(x,y) = \operatorname{Re}[p(z)], \quad G(x,y) = \operatorname{Im}[p(z)].$$

Assume that F and G have a nonconstant, common factor. Then, we can write

$$F(x,y) = [s + iq]A(x,y) \text{ and } G(x,y) = [s + iq]B(x,y).$$

Since the left side of each of these expressions is real, either (1) $q \equiv 0$, or (2) $[s - iq]$ is a common factor of A and B, and thus also of F and G such that $A/[s - iq]$ or $B/[s - iq]$ is nonconstant, or (3) both A/C_1 and B/C_2 equal $[s - iq]$ where C_1 and C_2 are constants. In the event of either (1) or (2), the common factor of F and G is real. Case (3) is not possible for nonconstant polynomials which are SDA, since $p(z) = [C_1 + C_2]g(x,y)$, where g is a real polynomial, does not satisfy the defining equations. Thus, the nonconstant common factor $r(x,y)$ of F and G is real and $p(z)$ can be written $p(z) = r(x,y)\,t(x,y)$. We introduce the SDA homogeneous polynomial of degree n given by (6.7) which corresponds to the SDA polynomial (6.8) under consideration. By Bôcher [6], since $p(z)$ is reducible, so is $P(z,h)$ and the factors correspond. Hence, $P(z,h)$ can be factored.

$$P(z,h) = R(x,y,h)\,T(x,y,h), \tag{a}$$

where R corresponds to r and T corresponds to t. The factor $R(x,y,h)$ is obtained from $r(x,y)$ on multiplication of the terms of r by suitable powers of h (real); hence, $R(x,y,h)$ is real and nonconstant. By construction, some variable term of R and also of T must be free of h. By Theorem 5.5, and (a)

(b) $$\lim_{h \to 0} P(z,h) = a_n z^n = R(x,y,0)\, T(x,y,0),$$

where by previous remarks, $R(x,y,0)$ is real and nonconstant. By unique factorization, this is impossible and thus F and G must be relatively prime. QED.

The result of the above theorem and a well-known property concerning relatively prime polynomials gives the following:

THEOREM 6.7. *If $p(z)$ is a SDA polynomial,* $\deg(p) = n$, *there exists only a finite number of values $z = \alpha_k$, $k = 1, 2, \ldots, N$ such that $p(\alpha_k) = 0$.*

Proof. As above, let $F(x,y)$ and $G(x,y)$ denote the real and imaginary parts of $p(z)$ respectively. Both F and G are real polynomials of the real variables x and y and $\deg(F) \leqq n$, $\deg(G) \leqq n$. By Theorem 6.6, F and G have no common, nonconstant factor. Therefore, by Bôcher [6], there exist only a finite number of points for which both F and G vanish. QED.

In the continuous case, a polynomial of degree n has exactly n roots. This statement is not true for the analogous SDA polynomial of degree n. For example, the equation $z_1^{(2)} = 0$ has three zeros, a double zero for $z = 0$ and a simple zero for $z = i$. The discrete "monodiffric polynomials" considered by Isaacs in [4] and [5] also exhibited this property. Terracini [10] shows that the number of zeros of an nth degree monodiffric polynomial is at least n and at most n^2.

The author would like to thank Professor R. J. Duffin for his invaluable advice and guidance during the course of this work.

REFERENCES

1. R. L. Helmbold, *Semi-discrete potential theory*, Carnegie Institute of Technology, Technical Report No. 34, Contract No. DA-36-061-ORD-490, Office of Ordnance Research, U.S. Army.

2. R. J. Duffin, *Discrete potential theory*, Duke Math. J. **20** (1953), 233–251.

3. ———, *Basic properties of discrete analytic functions*, Duke Math. J. **23** (1956), 335–363.

4. R. P. Isaacs, *A finite difference function theory*, Univ. Nac. Tucumán Rev. **2**(1941), 177–201.

5. ———, *Monodiffric functions*, Nat. Bur. Standards Appl. Math. Ser. **18** (1952), 257–266,

6. M. Bôcher, *Introducion to higher algebra*, Macmillan Company, New York, (1907).

7. J. Ferrand, *Fonctions préharmoniques et fonctions préholomorphes*, Bull. Sci. Math. **68** (1944), 152–180.

8. ———, *Représentation conforme et transformations à integrale de Dirichlet bornée*, Cahiers Scientifiques Fascicule XXII, Chapter 5, Gauthier-Villars, Paris, 1955.

9. C. Jordan, *Calculus of finite differences*, 2nd ed., Chelsea, New York, 1950.

10. A. Terracini, *A first contribution to the geometry of monodiffric polynomials*, Actas Acad. Ci. Lima **8** (1945), 217–250.

CARNEGIE INSTITUTE OF TECHNOLOGY,
PITTSBURGH, PENNSYLVANIA
DUKE UNIVERSITY,
DURHAM, NORTH CAROLINA