HEREDITARY ORDERS

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Introduction. Let R be an integral noetherian domain with field of quotients K. Let Σ be a semi-simple K-algebra with finite dimension over K. By an order over R we mean a subring Λ in Σ such that Λ is a finitely generated R-module which spans Σ over K, and that Λ contains the identity element in Σ.

A ring Λ will be called hereditary if every left and right ideal in Λ is Λ-projective.

In the paper of Auslander and Goldman [3], they have obtained the following fact: Let R be a discrete, rank one valuation ring, and Σ a central simple K-algebra; then Λ is maximal if and only if Λ is hereditary and the radical of Λ is a unique maximal ideal in Λ (see Corollary 3.5). Furthermore, they have given a nonmaximal hereditary order in which there are two maximal two-sided ideals and over which there are two maximal orders (cf. [9]).

This fact suggests that there are some relations between maximal orders containing Λ and maximal two-sided ideals.

The purpose of this paper is to investigate such a relationship and to give analogous properties of hereditary orders over a Dedekind domain to classical properties of maximal orders.

In §1, we give a fundamental theorem: There is a one-to-one correspondence between orders containing an hereditary order Λ and idempotent ideals in Λ. Using this fact, we shall reduce, in §2, problems to the case where R is a Dedekind domain and Σ is a central simple K-algebra.

In §§3, 4, 5, and 6, we study hereditary orders over a discrete, rank one valuation ring. We shall give a complete description of orders containing an hereditary order Λ, and some relations of orders containing Λ. Furthermore, we see that the associated division rings of simple components of Λ/N do depend only on Σ, not on Λ, where N is the radical of Λ. We shall consider in §5 some criteria of hereditary order.

In §6, we consider a group structure of two-sided ideals with respect to Λ and in §7, generalizing the above results to the case of a Dedekind ring, we obtain

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that the set of inversible two-sided ideals with respect to Λ is an abelian group which is a direct product of cyclic group.

Some of our results are already given in [1; 2; 3; 10].

1. **Fundamental theorem in an hereditary order.** Throughout this section, R stands for a commutative noetherian domain, and K its quotient field, unless stated to the contrary.

Let Σ be a semi-simple K-algebra with finite dimension over K. By an order over R we mean a subring Λ such that Λ is a finitely generated R-module containing the identity element in Σ, which spans Σ over K. Hence, Λ contains a K-basis of Σ and Λ is a left and right noetherian ring.

Let Λ_1 and Λ_2 be orders in Σ. Then subset $C_{\Lambda_2}(\Lambda_1)$ in Σ consisting of all elements x in Σ such that $\Lambda_1 x \subseteq \Lambda_2$ is a left Λ_1- and right Λ_2-module. We call $C_{\Lambda_2}(\Lambda_1)$ the (right) conductor of Λ_1 with respect to Λ_2. Similarly, we can define the left conductor and we shall denote it by $D_{\Lambda_2}(\Lambda_1)$. If we fix Λ_2, then we denote briefly $C_{\Lambda_2}(\Lambda_1)$ by $C(\Lambda_1)$. We shall use frequently the following well-known result (Lemma 1.1) in this paper and so we recall the definition of trace ideal.

Let S be any ring and E a finitely generated left S-module. By the trace mapping τ of E we mean the two-sided S-homomorphism of $E \otimes_T \mathrm{Hom}_S(E,S)$(2) to S by setting $\tau(e \otimes f) = f(e)$, where $T = \mathrm{Hom}_S(E,E)$, $e \in E$ and $f \in \mathrm{Hom}_S(E,S)$. Then the image $\tau_S(E)$ is the two-sided ideal generated by the image of f where f runs through all elements in $\mathrm{Hom}_S(E,S)$. Therefore, we obtain that $\mathrm{Hom}_S(E,S) = \mathrm{Hom}_S(E, \tau_S(E))$. We call $\tau_S(E)$ the trace ideal of E.

The following lemma is given in [3, Appendix]:

LEMMA 1.1. *Let S be a ring and E a finitely generated left S-module. Let* $T = \mathrm{Hom}_S(E,E)$. *Then* (1) *If $\tau_S(E) = S$, E is a finitely generated projective T-module and $S = \mathrm{Hom}_T(E,E)$.* (2) *If E is a finitely generated projective S-module, then $\tau_S(E)E = E$ and $\tau_T(E) = T$.*

LEMMA 1.2. *Let $\Lambda_1 \supseteq \Lambda_2$ be orders in Σ and let E_1 and E_2 be left Λ_1-modules such that E_2 is R-torsion free. Then we have* $\mathrm{Hom}_{\Lambda_1}(E_1, E_2) = \mathrm{Hom}_{\Lambda_2}(E_1, E_2)$.

Proof. It is clear that $\mathrm{Hom}_{\Lambda_1}(E_1, E_2) \subseteq \mathrm{Hom}_{\Lambda_2}(E_1, E_2)$. By the definition of an order, we can find an element $r \neq 0$ in R such that $r\Lambda_1 \subseteq \Lambda_2$. For $f \in \mathrm{Hom}_{\Lambda_2}(E_1, E_2)$, $e_1 \in E_1$ and $\lambda_1 \in \Lambda_1$, we have that $f(r\lambda_1 e_1) = rf(\lambda_1 e_1)$, and $f(r\lambda_1 e_1) = r\lambda_1 f(e_1)$. Since E_2 is R-torsion free, $f(\lambda_1 e_1) = \lambda_1 f(e_1)$.

LEMMA 1.3. *Let $\Lambda \subset \Gamma$ be orders and E a finitely generated left Γ-module and R-torsion free. If E is Λ-projective, then E is Γ-projective.*

(2) For a left (right) S-module E, every element of a ring of endomorphism of E as a left (right) S-module operates on E from the right (left) side.

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}^l_\Lambda(E,\Lambda) \underset{\Lambda}{\otimes} E & \xrightarrow{\psi_\Lambda} & \mathrm{Hom}^l_\Lambda(E,E) \\ \Big\downarrow{\scriptstyle i\otimes I} & & \Big\downarrow{\scriptstyle i'} \\ \mathrm{Hom}^l_\Gamma(E,\Gamma) \underset{\Gamma}{\otimes} E & \xrightarrow{\psi_\Gamma} & \mathrm{Hom}^l_\Gamma(E,E) \end{array}$$

where i' is the identity mapping by Lemma 1.2 and $i:\mathrm{Hom}^l_\Lambda(E,\Lambda)\to\mathrm{Hom}^l_\Lambda(E,\Lambda) = \mathrm{Hom}^l_\Gamma(E,\Gamma)$, and $\psi_\Lambda(f\otimes e)(e') = f(e')e$, $f\in\mathrm{Hom}^l_\Lambda(E,\Lambda)$ and e, $e'\in E$. By assumption and [5, p. 123, Proposition 3.1], ψ_Λ is epimorphic, and hence ψ_Γ is epimorphic, which implies E is Γ-projective.

COROLLARY 1.4. *Let Λ be an hereditary order in Σ. Then every order containing Λ is also hereditary.*

Proof. Since there exists an element $r\neq 0$ in R such that $\Gamma r\subseteq\Lambda$, every left (right) ideal in Γ is Λ-isomorphic to a left (right) ideal in Λ. Hence, Γ is hereditary.

LEMMA 1.5. *Let S be any ring and A a two-sided ideal in S such that A is left Λ-projective. Then*(3) *$\tau^l_S(A) = A$ if and only if A is idempotent, i.e., $A^2 = A$.*

Proof. If $\tau^l_S(A) = A$, then by assumption, we have $A = \tau^l_S(A)A = A^2$ by Lemma 1.1. Conversely, if $A = A^2$, then for any element f in $\mathrm{Hom}^l_S(A,S)$, we have $f(A) = f(AA) = Af(A)\subseteq A$, which means $\tau^l_S(A)\subseteq A$. It is clear for any ideal A that $\tau^l_S(A)\supseteq A$.

From now on, when we fix an order Λ, then an ideal A means a fractional two-sided ideal with respect to Λ, namely A is a two-sided Λ-module in Σ such that $AK = \Sigma$ and $Ar\subseteq\Lambda$ for some $r\neq 0$ in R.

Let A be an ideal in an order Λ in Σ. Then $\mathrm{Hom}^r_\Lambda(A,A) = \{x\,|\in\Sigma,\ xA\subseteq A\}$ and we shall denote it by $\mathrm{End}^r_\Lambda(A)$. Since A is a faithful Λ-module, $\mathrm{End}^r_\Lambda(A)$ is an order containing Λ.

PROPOSITION 1.6. *Let Λ be an order over R in the semi-simple K-algebra Σ, and A an ideal in Λ. Then,* (1) $C(\mathrm{End}^r_\Lambda(A))\supseteq\tau^l_\Lambda(A)$. *If A is right Λ-projective, then* $C(\mathrm{End}^r_\Lambda(A)) = \tau^l_\Lambda(A)\cdot$(2) $C(\Gamma) = \tau^l_\Lambda(C(\Gamma))$ *for any order Γ containing Λ. If Γ is left Λ-projective, then $C(\Gamma)$ is right Λ-projective. Furthermore, if $C(\Gamma)$ is idempotent, then* $\mathrm{End}^r(C(\Gamma)) = \Gamma$.

Proof. (1) We can easily see by the same method as in the proof of Lemma 1.2 that $C(\mathrm{End}^r_\Lambda(A))\supset\tau^l_\Lambda(A)$. If A is right Λ-projective, then we have an isomorphism $\psi: A\otimes_\Lambda\mathrm{Hom}^l_\Lambda(A,\Lambda)\to\mathrm{Hom}^l_\Lambda(\mathrm{Hom}^r_\Lambda(A,A),\Lambda)$ by setting $\psi(a\otimes f)g = f(g(a))$, where $f\in\mathrm{Hom}^l_\Lambda(A,\Lambda)$, $g\in\mathrm{Hom}^r_\Lambda(A,A)$, and $a\in A$. However, the right side is

(3) Let F be functor of a category of left (right) Λ-modules to a category. We denote $F(\ )$ by $F^l(\ )$ $(F^r(\ ))$ if there is ambiguity.

equal to $C(\mathrm{End}^r_\Lambda(A))$, and for $\sigma \in C(\mathrm{End}^r_\Lambda(A))$, $\psi^{-1}(\sigma) = \Sigma a_i \otimes f_i$ and $\sigma = I\sigma = \psi(\Sigma a_i \otimes f_i)(I) = \Sigma f_i(a_i) \in \tau^l_\Lambda(A)$, where I is the identity in $\mathrm{End}^r_\Lambda(A) \subseteq \Sigma$.

(2) Since $C(\Gamma)$ is a left Γ-module, we have $\mathrm{End}^r_\Lambda(C(\Gamma)) \supseteq \Gamma$, and hence, $C(\Gamma) \subseteq \tau^l_\Lambda(C(\Gamma) \subseteq C(\mathrm{End}^r_\Lambda(C(\Gamma))) \subseteq C(\Gamma)$. It is clear that $C(\Gamma)$ is isomorphic to $\mathrm{Hom}^l_\Lambda(\Gamma, \Lambda)$ as a two-sided Λ-module. Hence, $C(\Gamma)$ is a right projective Λ-module if Γ is left Λ-projective. Furthermore, we assume that $C(\Gamma)$ is idempotent. Then $\mathrm{Hom}^l_\Lambda(C(\Gamma), C(\Gamma)) = \mathrm{Hom}^l_\Lambda(C(\Gamma), \Lambda) = \mathrm{Hom}^r_\Lambda(\mathrm{Hom}^l_\Lambda(\Gamma, \Lambda), \Lambda)$. Since Γ is left Λ-projective, $\mathrm{Hom}^r_\Lambda(\mathrm{Hom}^l_\Lambda(\Gamma, \Lambda), \Lambda) \approx \mathrm{Hom}^r_\Lambda(\Lambda, \Lambda) \otimes_\Lambda \Gamma \approx \Gamma$. Hence, we have $\Gamma = \mathrm{End}^r_\Lambda(C(\Gamma))$ as above.

We shall call briefly an hereditary order an h-order.

Summarizing the above results, we have

THEOREM 1.7. *Let R be a commutative noetherian domain with field of quotients K. Let Λ be an h-order over R in the semi-simple K-algebra Σ. Then every order containing Λ is also an h-order, and there is a one-to-one correspondence between two-sided idenpotent ideals A in Λ and orders Γ containing Λ as follows*:

$$\Gamma = \mathrm{End}^r_\Lambda(A), \quad A = C(\Gamma),$$
$$\Gamma_1 \supseteq \Gamma_2 \quad \textit{if and only if } C(\Gamma_1) \subseteq C(\Gamma_2), \textit{ and}$$
$$A_1 \supseteq A_2 \quad \textit{if and only if } \mathrm{End}^r_\Lambda(A_1) \subseteq \mathrm{End}^r_\Lambda(A_2).$$

We close this section with the following proposition:

PROPOSITION 1.8. *Let Λ be an order in Σ, and A an ideal in Λ such that A is left Λ-projective. If $\tau^l_\Omega(A) = \Omega$ for $\Omega = \mathrm{End}^r_\Lambda(A)$, then we have $\Omega\tau^l_\Lambda(A)\Omega = \Omega$.*

Proof. Since A is a finitely generated projective left Λ-module, we have an isomorphism $\phi : \mathrm{Hom}^l_\Lambda(A,\Lambda) \otimes_\Lambda \Omega \to \mathrm{Hom}^l_\Omega(\Omega \otimes_\Lambda A, \Omega)$ by setting $\phi(f \otimes \omega)(\omega' \otimes a) = \omega' f(a)\omega$, where $f \in \mathrm{Hom}^l_\Lambda(A,\Lambda)$; $\omega, \omega' \in \Omega$, (since for $\lambda \in \Lambda$, $\phi(f\lambda \otimes \omega)(\omega' \otimes a) = \omega'(f\lambda)(a)\omega = \omega' f(a)\lambda\omega = \phi(f \otimes \lambda\omega)(\omega' \otimes a)$). Hence, by the definition of trace ideal, we have $\tau^l_\Omega(\Omega \otimes_\Lambda A) = \Omega\tau^l_\Lambda(A)\Omega$. Furthermore, from the natural epimorphism: $\Omega \otimes_\Lambda A \to A \to 0$, we obtain that $\Omega = \tau^l_\Omega(A) \subseteq \tau^l_\Omega(\Omega \otimes_\Lambda A) = \Omega\tau^l_\Lambda(A)\Omega \subseteq \Omega$.

COROLLARY 1.9. *Let Λ be an order and A an idempotent ideal in Λ which is left and right Λ-projective. If $A \neq \Lambda$, then $\mathrm{End}^r_\Lambda(A)$ ($\mathrm{End}^l_\Lambda(A)$) does not contain $\mathrm{End}^l_\Lambda(A)$ ($\mathrm{End}^r_\Lambda(A)$).*

Proof. Let $\Gamma_1 = \mathrm{End}^r_\Lambda(A)$, and $\Gamma_2 = \mathrm{End}^l_\Lambda(A)$. We assume $\Gamma_1 \subseteq \Gamma_2$. Then we have, by Lemmas 1.1 and 1.5 and Proposition 1.8, we have $\Gamma_1 = \Gamma_1\tau^l_\Lambda(A)\Gamma_1 = A\Gamma_1 \subseteq A\Gamma_2 = A \subseteq \Lambda$. Hence, $A = \Lambda$.

2. **The center of an h-order.** The purpose of this section is to show that an h-order in a semi-simple algebra Σ is the direct sum of h-orders in simple components of Σ, whose centers are Dedekind domains.

Let R be a commutative noetherian domain with field of quotients K, and let Λ be an h-order in the semi-simple K-algebra Σ. As a preliminary to the main theorem in this section, we make the following observation.

Let A be an ideal in Λ and let $\Lambda_1 = \mathrm{End}^r_\Lambda(A)$, and $\Lambda_2 = \mathrm{End}^l_\Lambda(A)$. Then we have, by Lemma 1.1, $\tau_{\Lambda_1}(A) = \Lambda_1$, and $\tau_{\Lambda_2}(A) = \Lambda_2$. By A^{-1} we mean the subset $= \{x \mid \in \Sigma,\ xA \subseteq \Lambda_2\} = \{x \mid \in \Sigma,\ AxA \subseteq A\} = \{x \mid \in \Sigma,\ Ax \subseteq \Lambda_1\}$. It is clear that A^{-1} is left Λ_2- and right Λ_1-ideal in Σ. Then by the definition of trace ideal, we have that $\Lambda_1 = \tau^l_{\Lambda_1}(A) = AA^{-1}$ and $\Lambda_2 = \tau^r_{\Lambda_2}(A) = A^{-1}A$. Consequently, we have $\Lambda_2 = (A^{-1}A)(A^{-1}A) = A^{-1}\Lambda_1 A$.

Lemma 2.1. *Let Λ be an h-order in Σ, and a central element in Σ. If the ring $\Lambda[a]$ generated by Λ and a is an order, then a is contained in Λ.*

Proof. Let $C = C(\Lambda[a])$. Then we have, by Theorem 1.7, $\Lambda[a] = \mathrm{End}^r_\Lambda(C)$. Put $\Gamma = \mathrm{End}^l_\Lambda(C)$. By the above observation, $\Gamma = C^{-1}\Lambda[a]C$. However, since a is central, $C^{-1}\Lambda[a]C \ni a$, which implies that $\Gamma \supseteq \Lambda[a]$. Hence, by Corollary 1.9, we have $C = \Lambda$, and hence, $\Lambda[a] = \Lambda$.

Proposition 2.2. *Let $\Sigma = \Sigma e_1 \oplus \Sigma e_2 \oplus \cdots \oplus \Sigma e_n$ be the simple decomposition of Σ. Then for any h-order Λ, we obtain that $\Lambda = \Lambda e_1 \oplus \cdots \oplus \Lambda e_n$, and that Λe_i is an h-order in Σe_i and the center of Λe_i is integrally closed over Re_i.*

Proof. It is clear that $\Lambda[e_i]$ is a finitely generated R-module, and hence, $e_i \in \Lambda$. Therefore, we have $\Lambda = \Lambda e_i \oplus \cdots \oplus \Lambda e_n$, and Λe_i is an h-order over Re_i in Σe_i. The second half is also clear.

By virtue of this proposition, we may assume that Σ is a central simple K-algebra. Thus, from now on we always assume that Σ is central simple.

The essential part of the following lemma is known (cf. [2, Theorem 6.34]), but we give the proof for the sake of completeness.

Lemma 2.3. *Let Λ be an hereditary, maximal order in Σ. Then the center R of Λ is a Dedekind domain.*

Proof. First, we shall show that every nonzero prime ideal P in Λ is maximal. Since Λ is maximal, we have, by Corollary 1.9, $\tau^l_\Lambda(A) = \tau^r_\Lambda(A) = \Lambda$ and $\mathrm{End}^r_\Lambda(A) = \mathrm{End}^l_\Lambda(A) = \Lambda$ for every ideal A in Λ. Hence, $\Lambda = AA^{-1}$ by the above observation. If $P \subsetneqq A \subset \Lambda$, then $B = A^{-1}P$ is an ideal in Λ, and $P = AB$. Since P is prime and $A \subsetneqq P$, $B = A^{-1}P \subseteq P$ which implies $A^{-1} \subseteq \Lambda$ and $\Lambda = AA^{-1} \subseteq A\Lambda = A$. Let p be a prime ideal in R, and let $\Lambda_p = \Lambda \otimes R_p$. Let M' be a maximal ideal in Λ_p, and $M = M' \cap \Lambda \neq (0)$. Then M is prime in Λ, and hence M is maximal. On the other hand, M' contains $p\Lambda_p$. Hence, $M \cap R = M' \cap R_p \cap R = p$. Let q be a prime ideal containing p in R. Since M is maximal in Λ and $M \cap (R - q) = \phi$, $M \otimes R_q$ is maximal in Λ_q. Hence, we have $p = M \cap R = (M \otimes R_q) \cap \Lambda \cap R = qR_q \cap R = q$ as above. Therefore, the rank of R does not exceed one. By Proposition 2.2, R is integrally closed and hence, R is a Dedekind domain.

LEMMA 2.4. *Let R be a semi-local noetherian ring. Let S be an R-algebra which is finitely generated as an R-module. Then for any two-sided idempotent ideal A in S, we have $A = \bigcap_n (A+N)^n$, where N is the radical of S. Consequently, for two-sided idempotent ideals A and B, we have that $A \subseteq B$ if and only if $A + N \subseteq B + N$.*

Proof. Let m be the radical of R. Since S is R-finitely generated, $N \supseteq mS$ and $N^t \subseteq mS$ *for some integer* t. Let $C = A + N$. Then we can easily see that $C^n = A + N^n$. Hence, $\bigcap_n C^n = \bigcap_n (A + N^n) = \bigcap_n (A + (mS)^n)$. Therefore, we have, by Artin-Rees theorem (see [11, p. 262, Theorem 9]), $A = \bigcap_n (A + (mS)^n)$.

COROLLARY 2.5. *Let Λ be an h-order in the central simple K-algebra Σ. If the center of Λ is semi-local, then there is only a finite number of orders containing Λ in Σ. Consequently, there exists a maximal order containing Λ in Σ.*

Proof. Since Λ/N is semi-simple, it is clear by Theorem 1.7 and Lemma 2.4.

THEOREM 2.6. *Let Λ be an h-order in the central simple K-algebra Σ. Then the center R of Λ is a Dedekind domain.*

Proof. First, we assume that R is local. Then there exists, by Corollary 2.5, a maximal order Λ containing Γ. Then $\Gamma \cap K$ contains R and it is a finitely generated R-module. However, R is integrally closed by Proposition 2.2. Hence, R is a Dedekind domain by Corollary 1.4 and Lemma 2.3. By the usual localization process, we have proved the theorem.

COROLLARY 2.7. *Let Λ be an h-order over a noetherian domain R in Σ. Then Λ is R-projective if and only if R is integrally closed in K.*

Proof. If R is integrally closed, then R is a Dedekind domain. Hence, Λ is R-projective by [5, p. 133, Proposition 4.2]. Conversely, let Λ be R-projective. Since Λ is a projective module over the center Z of Λ, Z is a direct summand of Λ as Z-module by [4, p. 371]. Hence, Z is R-projective. Since Z and R have the same quotient field K, $Z = R$.

3. **Orders containing an *h*-order over a valuation ring.** By virtue of Theorem 2.6, we may assume that the base ring R of an h-order in the central simple K-algebra Σ is a Dedekind domain. Thus, throughout the rest of the paper, we consider h-orders over a Dedekind domain R, unless otherwise stated.

The main purpose of this section is to give the complete description of orders containing a fixed h-order over a discrete rank one valuation ring.

PROPOSITION 3.1. *Let Λ be an h-order over a Dedekind domain R in Σ. Then we have:*

(1) *Let* $\Lambda_2 \supseteq \Lambda_1$ *be orders containing* Λ *in* Σ, *then* $C(\Lambda_1)\Lambda_1 = \Lambda_1$ *and* $C_{\Lambda_1}(\Lambda_2) = C(\Lambda_2)\Lambda_1$.

(2) *For any idempotent ideals* A *and* B *in* Λ, *we have* $\mathrm{End}^r_\Lambda(A) \cap \mathrm{End}^r_\Lambda(B) = \mathrm{End}^r_\Lambda(A+B)$. *Furthermore, the ring* $\mathrm{End}^r_\Lambda(A) \cup \mathrm{End}^r_\Lambda(B)$ *generated by* $\mathrm{End}^r_\Lambda(A)$ *and* $\mathrm{End}^r_\Lambda(B)$ *in* Λ *is an order if and only if* $(AB)^n$ *is idempotent for some integer* n. *In this case, we have* $\mathrm{End}^r_\Lambda(A) \cup \mathrm{End}^r_\Lambda(B) = \mathrm{End}^r_\Lambda((AB)^n)$.

Proof. (1) By Propositions 1.6 and 1.8, we have $\Lambda_1 = \Lambda_1 C(\Lambda_1)\Lambda_1 = C(\Lambda_1)\Lambda_1$. It is clear that $C(\Lambda_2)\Lambda_1 \subseteq C_{\Lambda_1}(\Lambda_2)$ and that $C_{\Lambda_1}(\Lambda_2)C(\Lambda_1) \subseteq C(\Lambda_2)$. Hence, $C_{\Lambda_1}(\Lambda_2) = C_{\Lambda_1}(\Lambda_2)C(\Lambda_1)\Lambda_1 \subseteq C(\Lambda_2)\Lambda_1$.

(2) Let A and B be idempotent in Λ. Since $(A+B)^2 = A+B$, $\mathrm{End}(A) \cap \mathrm{End}(B) \supseteq \mathrm{End}(A+B)$ by Theorem 1.7. However, it is clear that $\mathrm{End}(A) \cap \mathrm{End}(B) \supseteq \mathrm{End}(A+B)$. Consequently, $\mathrm{End}(A) \cap \mathrm{End}(B) = \mathrm{End}(A+B)$. Let Γ be an order containing $\mathrm{End}(A)$ and $\mathrm{End}(B)$. Then $C(\Gamma)$ is contained in $A \cap B$. Furthermore, since $C(\Gamma)$ is idempotent, $C(\Gamma)$ is contained in $(AB)^n$ for any n. Therefore, $(AB)^t$ is idempotent for some t, since $\Lambda/C(\Gamma)$ satisfies the minimal condition. Thus, we have $\Gamma \supseteq \mathrm{End}((AB)^t) \supseteq \mathrm{End}(A) \cup \mathrm{End}(B)$, which implies that $\mathrm{End}(A) \cup \mathrm{End}(B) = \mathrm{End}((AB)^t)$. The converse is clear.

We shall reduce, in §7, the problems to the case where R is a semi-local ring, and so we first study h-orders over a discrete valuation ring.

In the rest of this section, we always assume that R is a discrete rank one valuation ring.

LEMMA 3.2. *Let* Λ *be an order in* Σ. *Then for every ideal* A *properly containing the radical* N *of* Λ, *there exists a unique idempotent ideal* $I(A)$ *such that* $A = I(A) + N$. *Furthermore, if* $A \supseteq B$ *for ideals properly containing* N, *then* $I(A) \supseteq I(B)$.

Proof. Let $\hat{R}$ and $\hat{\Lambda}$ be completions of R and Λ with respect to the maximal ideal in R, respectively. Since R is a discrete, rank one valuation ring, $\hat{R}$ is a local domain of rank one. Let $\hat{K}$ be the quotient field of $\hat{R}$. Then $\hat{\Sigma} = \Sigma \otimes_K \hat{K} \supseteq \hat{\Lambda} = \Lambda \otimes_R \hat{R}$, and hence, $\hat{\Lambda}$ is an order in the central simple $\hat{K}$-algebra $\hat{\Sigma}$. Furthermore, we have that $\hat{N} = N \otimes \hat{R}$ is the radical of $\hat{\Lambda}$ and that $\hat{\Lambda}/\hat{N} \approx \Lambda/N$. Let A be an ideal properly containing N in Λ; then $\hat{A} = A \otimes \hat{R}$ contains properly $\hat{N}$. Since $\hat{\Lambda}/\hat{N}$ is a semi-simple ring with the minimal condition, $\hat{A}$ has an element a such that $a \not\equiv 0 \bmod \hat{N}$ and $a^2 \equiv a \bmod \hat{N}$. However, since $\hat{\Lambda}$ is a completion with respect to $\hat{N}$, we can find an idempotent element e in $\hat{\Lambda}$ such that $a \equiv e \bmod \hat{N}$ by [8, Theorem A]. Since $\hat{\Lambda}e\hat{\Lambda}$ is a nonzero ideal in $\hat{\Lambda}$, $\hat{\Lambda}/\hat{\Lambda}e\hat{\Lambda}$ satisfies the minimal condition. Hence, $\hat{A}^t$ is idempotent for some integer t, since $\hat{A}$ contains the idempotent ideal $\hat{\Lambda}e\hat{\Lambda}$. Therefore, A^t is idempotent by the property of completion. It is clear that $A = A^t + N$. The second half is an immediate consequence of Lemma 3.2.

From Theorem 1.7, Proposition 3.1 and Lemma 3.2, we have

THEOREM 3.3. *Let R be a discrete, rank one valuation ring with field of quotients K. Let Λ be an h-order in the central simple K-algebra Σ. Let n be the number of maximal two-sided ideals in Λ. Then we have*:

(1) *There exist precisely n maximal orders Λ_i containing Λ and n minimal*(4) *orders Γ_i containing Λ.*

(2) *Every order Ω properly containing Λ is uniquely written by the form $\Omega = \bigcap_{j=1}^{r}\Lambda_{i_j} = \bigcup_{k=1}^{n-r}\Gamma_{j_k}$, and $\Lambda = \bigcap_{i=1}^{n}\Lambda_i$. Consequently, the number of orders containing Λ is equal to $2^n - 1$, and the number of maximal two-sided ideals in $\Omega = \bigcap_{j=1}^{r}\Lambda_{i_j}$ is equal to r.*

COROLLARY 3.4. *Maximal lengths of chain for h-orders in Σ do not exceed the dimension of Σ over K.*

COROLLARY 3.5. *Let Λ be an order as above. Then Λ is maximal if and only if the radical of Λ is a unique maximal two-sided ideal in Λ, and Λ is an h-order* (*cf.* [3, *Theorem* 2.3]).

If Λ is maximal, then the radical N is inversible (see [6, p. 74, Satz 9]) and hence, N is Λ-projective. Thus, the corollary is true by the following result [3, p. 4, Theorem 2.2].

LEMMA 3.6. *Let R be a local noetherian ring and Λ an R-algebra such that Λ is a finitely generated R-module. If the radical N of Λ is a projective left Λ-module, then Λ is hereditary. Furthermore, if the completion of R with respect to the maximal ideal is an integral domain and Λ is R-torsion free, then for any finitely generated left Λ-module E, we have that E is Λ-projective if and only if E is R-torsion free.*

Proof. Let $\hat{\Lambda}$, $\hat{R}$ be completions of Λ and R with respect to the maximal ideal in R. Then by the usual argument (cf. [5, p. 129, Exercise 11]), we have $\mathrm{gl.dim}\,\Lambda \leqq \mathrm{gl.dim.}\,\hat{\Lambda}$. By [7, Theorem 11], $\mathrm{gl.dim}\,\hat{\Lambda} = \mathrm{gl.dim}\,\hat{N} + 1 = 1$, since $\hat{N} = N \otimes_R \hat{R}$ is $\hat{\Lambda}$-projective. Hence, Λ is hereditary. Next, we assume that $\hat{R}$ is an integral domain, and that E is R-torsion free. In order to prove that E is Λ-projective, it is sufficient to show that $\hat{E} = E \otimes_R \hat{R}$ is $\hat{\Lambda}$-projective. Hence, we may assume that R is complete. Let $0 \to K \to P \to E \to 0$ be a minimal resolution. Then $\mathrm{Tor}_1^{\Lambda}(\Lambda/N,E) \approx \Lambda/N \otimes_{\Lambda} K = K/NK$. Since Λ is hereditary, $\mathrm{Tor}_1^{\Lambda}(\Lambda/N,E)$ is R-torsion free. Therefore, $K/NK = 0$, which implies that $K = (0)$.

If Λ is maximal, then for every finitely generated projective Λ-module E, we have $\tau_{\Lambda}(E) = \Lambda$ by [3, Proposition 3.10]. Conversely:

PROPOSITION 3.7. *Let R be a complete, discrete rank one valuation ring with field of quotients K. Let Λ be an h-order in Σ. If there exists an indecomposable projective Λ-module E such that $\tau_{\Lambda}(E) = \Lambda$, then Λ is maximal.*

(4) By a minimal order we mean an order Γ containing Λ such that if $\Gamma \supseteq \Omega \supsetneqq \Lambda$ for an order Ω, then $\Gamma = \Omega$.

Proof. Since E is indecomposable, $E \otimes_R K$ is, by [3, Proposition 2.8], isomorphic to a simple left ideal in $\Sigma = \Delta_n$, where Δ is the associated division ring of Σ. Hence, $\Omega = \mathrm{Hom}_\Lambda(E,E)$ is an order over R in Δ. By Lemma 3.6 and [3, Theorem A.5], Ω is hereditary. If Ω is not maximal, then there exist at least two maximal orders in Δ by Theorem 3.3, which contradicts [3, p. 14, Corollary]. Hence, Λ is maximal by [3, Theorem 3.6].

REMARK 1. Proposition 3.7 is not true in general unless R is complete; cf. [9].

REMARK 2. By virtue of Theorem 3.3, every h-order is written by the intersection of a finite number of maximal orders. However, the intersection of two maximal orders, in general, is not an h-order. For example, let R be integers and K rationals. Let Σ be a matrix ring over K with degree two. A subring Λ in Σ which is a matrix ring over R_2 with degree two is a maximal order in Σ. We take a regular element $t_n = 1/2^n e_{1,1} + e_{2,1} + 2^n e_{2,2}$, where the $e^1_{i,j}$'s are matrix units. Then $\Omega \cap t_n\Omega t_n^{-1} = R_2 f_{1,1} + 2^{2n}R_2 f_{1,2} + R f_{2,1} + R f_{2,2}$, where $f_{i,j} = t_n e_{i,j} t_n^{-1}$. There exists an order $R_2 f_{1,1} + 2R_2 f_{1,2} + R_2 f_{2,1} + R_2 f_{2,2}$ between $t_n\Omega t_n^{-1}$ and $\Omega \cap t_n\Omega t_n^{-1}$, and hence, $\Omega \cap t_n\Omega t_n^{-1}$ is not hereditary for any $n \geqq 1$.

4. **Relations between h-orders over a valuation ring.** Let R be a Dedekind domain with field of quotients K and Σ the central simple K-algebra. For two orders Λ_1 and Λ_2 in Σ, we say that Λ_1 and Λ_2 belong to the same type through C if there exists a left Λ_1- and right Λ_2-ideal C in Σ such that $\mathrm{End}^l_{\Lambda_1}(C) = \Lambda_2$ and $\mathrm{End}^r_{\Lambda_2}(C) = \Lambda_1$ (notation $(\Lambda_1, C, \Lambda_2)$); cf. [1]. It is clear that two maximal orders belong to the same type through the conductor.

Furthermore, if Λ_1 and Λ_2 are h-orders, then we have by Lemma 1.1 that $\Lambda_1 = \tau^l_{\Lambda_1}(C) = CC^{-1}$ and $\Lambda_2 = \tau^r_{\Lambda_2}(C) = C^{-1}C$ and hence, $\Lambda_2 = C^{-1}\Lambda_1 C$.

It is clear that in the category of h-orders the relation of the same type is reflexible and transitive.

PROPOSITION 4.1. *Let R be a Dedekind domain and let Λ_1 and Λ_2 belong to the same type. If Λ_1 is maximal, then Λ_2 is maximal. If Λ_1 and Λ_2 are h-orders, then there is a one-to-one correspondence between ideals A_1 in Λ_1 and ideals A_2 in Λ_2 by the mapping $A_1 \to C^{-1}A_1C$ and $A_2 \to CA_2C^{-1}$ which preserves inclusion and multiplication of ideals (cf.* [3, *Theorem* A.5; 6, p. 75, *Satz* 12]).

Proof. The second half is clear from the above observation. By [3, p. 2, Corollary and Lemma 2.4], we may restrict ourselves to the case where R is a local ring. Since Λ_1 is maximal, $\tau_{\Lambda_1}(C) = \Lambda_1$ by [3, Proposition 3.10]. Hence, Λ_2 is maximal by Corollary 3.5 and [3, Theorem A.5].

PROPOSITION 4.2. *Let Λ be an h-order over a Dedekind domain R in Σ. Let Ω_i be orders containing Λ $(i = 1,2,\cdots,t)$. If $(\Omega_1, C_1, \Omega_2)$ $(\Omega_2, C_2, \Omega_3), \cdots, (\Omega_t, C_t, \Omega_1)$ belong to the same type, then there is no order in Σ containing all the Ω_i's, where $C_i = C_\Lambda(\Omega_i)$.*

Proof. We assume that there exists an order Γ containing all the Ω_i's. Then $C(\Gamma) \subseteq \bigcap_i C_i$. Since $C(\Gamma)$ is idempotent, there exists an integer r such that $B=(C_1C_2\cdots C_t)^r$ is idempotent. Then we have, by Proposition 3.1, $\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_t = \mathrm{End}^r_\Lambda(B)$. On the other hand, by interchanging left and right, we obtain by Proposition 1.6 that $\Omega_2 \cup \Omega_3 \cup \cdots \cup \Omega_t \cup \Omega_1 = \mathrm{End}^l_\Lambda(B)$ as above, which is a contradiction to Corollary 1.9.

THEOREM 4.3. *Let R be a discrete, rank one valuation ring with field of quotients K. Let Λ be an h-order in Σ. Then two orders Ω_1 and Ω_2 containing Λ belong to the same type if and only if Ω_1 and Ω_2 have the same number of maximal ideals.*

Proof. If Ω_1 and Ω_2 belong to the same type, then they have the same number of maximal ideals by Proposition 4.1. Conversely, if Ω_1 and Ω_2 have the same number of maximal ideals, then $\Omega_1=\bigcup_{j=1}^r\Gamma_{i_j}$ and $\Omega_2=\bigcup_{l=1}^r\Gamma_{k_l}$ by Theorem 3.3, where the Γ's are minimal orders containing Λ. First we shall show that every minimal order belongs to the same type. Let $\{\Gamma_1,\cdots,\Gamma_n\}$ be the set of minimal orders containing Λ. Let $C_1 = C(\Gamma_1)$; then we have, by Theorem 1.7, that $\Gamma_1=\mathrm{End}^r_\Lambda(C_1)$. Furthermore, $\mathrm{End}^l_\Lambda(C_1)$ is a minimal order by Proposition 4.1, say $\Gamma_2=\mathrm{End}^l_\Lambda(C_1)$. Then Γ_1 and Γ_2 belong to the same type. Again from the conductor $C_2=C(\Gamma_2)$, we obtain the same type (Γ_2,C_2,Γ_3). Repeating this argument, we may have a set of minimal orders Γ_i such that (Γ_1,C_1,Γ_2), $(\Gamma_2,C_2,\Gamma_3),\cdots,(\Gamma_t,C_t,\Gamma_1)$. If $t<n$, then $\bigcup_{i=1}^t\Gamma_i$ is an order in Σ by Theorem 3.3. Therefore, $t=n$ by Proposition 4.2. Thus we have proved the theorem for $r=1$. We assume that $r\geqq 2$, and $\Gamma=\bigcup_{j=1}^t\Gamma_{i_j}=\bigcup_{l=1}^t\Gamma_{k_l}$, $t<r$ and that $\Gamma_{i_p}\neq\Gamma_{k_q}$ if $p,q>t$. We consider $\Omega=\Gamma\cup\Gamma_{k_{t+1}}\cup\Gamma_{i_{t+2}}\cup\cdots\cup\Gamma_{i_r}$. Then Ω_1 and Ω are minimal orders containing $\Gamma\cup\Gamma_{i_{t+2}}\cup\cdots\cup\Gamma_{i_r}$. Hence, they belong to the same type of the above argument. Therefore, by using the induction on t, we have proved that Ω_1 and Ω_2 belong to the same type.

In the rest of this section, we shall consider associated division ring of simple components of Λ/N, where N is the radical of Λ.

PROPOSITION 4.4. *Let Ω_1 and Ω_2 be h-orders with radicals N_1 and N_2 respectively, which belong to the same type through A. Then we have an isomorphism Ω_1/N_1 to $\mathrm{Hom}^r_{\Omega_2/N_2}(A/AN_2,A/AN_2)$ by the natural mapping. The associated division rings of Ω_1/N_1 and Ω_2/N_2 are isomorphic.*

Proof. Let m be the maximal ideal in R. Since A is Ω_2-projective, we have $\Omega_1/m\Omega_1=\mathrm{Hom}^r_{\Omega_2}(A,A)\otimes_R R/m=\mathrm{Hom}^r_{\Omega_2/m\Omega_2}(A/mA,A/mA)$. $\bar{\Omega}_i=\Omega_i/m\Omega_i\,(i=1,2)$ are semi-primary rings with radical $N_i/m\Omega_i$ and $\bar{A}=A/mA$ is $\bar{\Omega}_2$-projective. Let $\bar{\Omega}_2=\sum_{i=1}^n\sum_{j=1}^{p(i)}e_{i,j}\bar{\Omega}_2$ be a decomposition of indecomposable components. Then by [7, Corollary 4], $A=\sum_{i=1}^m\sum_{j=1}^{n(i)}e_{i,j}\bar{\Omega}_2$. It is clear that the images of elements in $\mathrm{Hom}^r_{\bar{\Omega}_2}(A,\bar{\Omega}_2)$ are contained in $\sum_{i=1}^m e_{i,j}\bar{\Omega}_2+\sum_{l=m+1}^n e_{l,k}\bar{N}_2$. However, $\tau^r_{\Omega_2}(A)=\Omega_2$ implies $\tau^r_{\bar{\Omega}_2}(\bar{A})=\bar{\Omega}_2$. Hence, we have $n=m$. Furthermore, we have a natural homomorphism ϕ of $\mathrm{Hom}^r_{\bar{\Omega}_2}(\bar{A},\bar{A})$ to $\mathrm{Hom}_{\Omega_2/N_2}(\bar{A}/\bar{A}N_2,\bar{A}/\bar{A}N_2)$. Since $\bar{A}/\bar{A}N_2 = \sum_{i=1}^{n}\sum_{j=1}^{n(i)} e_{i,j}\Omega_2/N_2$, we can easily see that ϕ is epimorphic. On the other hand, since $\mathrm{Hom}^r_{\Omega_2/N_2}(A/AN_2, A/AN_2)$ is semi-simple, we have $\phi^{-1}(0) \supseteq \bar{N}_1$. Therefore, from the facts that $n = m$ and that the number of simple components of Ω_1/N_1 and Ω_2/N_2 are same by Proposition 4.1, we have $\phi^{-1}(0) = \bar{N}_1$.

Corollary 4.5. *Let Λ_1 and Λ_2 be h-orders such that every simple components of Λ_i/N_i is a matrix ring with same degree n (cf. [9]). Then Λ_1 and Λ_2 belong to the same type if and only if Λ_1 and Λ_2 are isomorphic by an inner-automorphism. For any ideal A in Λ_1, we have that $\tau^l_{\Lambda_1}(A) = \Lambda_1$ if and only if A is principal as a left (right) Λ_1-module. Consequently, every maximal order is isomorphic by an inner-automorphism, and every one-sided ideal is principal (cf. [3, Proposition 3.5]).*

Proof. If $\Lambda_1 = \alpha\Lambda_2\alpha^{-1}$ for some element α in Σ, then Λ_1 and Λ_2 belong to the same type through $\Lambda_1\alpha = \alpha\Lambda_2$. Conversely, we assume that Λ_1 and Λ_2 belong to the same type through A. Then $\Lambda_1/N_1 \approx \mathrm{Hom}^r_{\Lambda_2/N_2}(A/AN_2, A/AN_2)$ by the proposition. Let $A/mA = \sum_{i=1}^{r}\sum_{j=1}^{p(i)} e_{ij}\Lambda_2/\Lambda_2 m$; then $\Lambda_1/N_1 \approx \sum_{i=1}^{r} \otimes (\Delta_i)_{p(i)}$. Hence, by assumption, we have $p(i) = n$ for all i. Therefore, A/mA is isomorphic to $\Lambda_2/m\Lambda_2$ as a right Λ_2-module, which implies that A is a principal ideal, namely $A = \alpha\Lambda_2$. Thus, $\Lambda_1 = \mathrm{Hom}^r{}_{\Lambda_2}(\alpha\Lambda_2, \alpha\Lambda_2) = \alpha\Lambda_2\alpha^{-1}$. Let B be an ideal in Λ_1. If $\tau^l_{\Lambda_1}(B) = \Lambda_1$, then $\mathrm{End}^l_{\Lambda_1}(B)$ and Λ_1 belong to the same type by Lemma 1.1. Therefore, $\mathrm{End}^l_{\Lambda_1}(B) = \Lambda_1$, and B is principal. Let Ω_1 and Ω_2 be maximal orders and $C = C_{\Omega_2}(\Omega_1)$. Since $[\Omega_1/m\Omega_1 : R/m] = [\Omega_2/m\Omega_2 : R/m] = [\Sigma : K]$, and $\Omega_1/N_1 \approx \mathrm{Hom}^r_{\Omega_2/N_2}(C/CN_2, C/CN_2)$, $\Omega_1/N_1 \approx \Delta_n \approx \Omega_2/N_2$.

Theorem 4.6. *For any h-order Λ in Σ, the associated division rings of simple components of $\Lambda/R(\Lambda)$ are isomorphic to a division ring which does not depend on Λ. Let $\Omega \supseteq \Lambda$ be h-orders such that $\Omega/R(\Omega) \approx \sum_{i=1}^{s} \oplus \Delta_{n(i)}$ and $\Lambda/R(\Lambda) \approx \sum_{i=1}^{t} \oplus \Delta_{m(i)}$. Then there is a one-to-one mapping π of $\{1,2,\cdots,s\}$ into $\{1,2,\cdots,t\}$ such that $n(i) \geqq m(\pi(i))$ and this inequality is not equal for some j, where R( ) means the radical of ring.*

Proof. We use the same notations as in the proof of Proposition 4.4. Let $C = C(\Omega)$ and $\Omega/R(\Omega) \approx (\Delta_1(\Omega))_{n(1)} \oplus \cdots + (\Delta_s(\Omega))_{n(s)}$, and $\Lambda/R(\Lambda) \approx (\Delta_1(\Lambda))_{m(1)} + \cdots \oplus (\Delta_s(\Lambda))_{m(s)} \oplus \cdots \oplus (\Delta_t(\Lambda))_{m(t)}$. $\bar{C} = \sum_{i=1}^{s'} \sum_{j=1}^{p(i)} e_{\alpha(i),j}\bar{\Lambda}$ and $\bar{C}/\bar{C}N = \sum_{i=1}^{s'}\sum_{j=1}^{p(i)} e_{\alpha(i),j}\Lambda/N$, where $N = R(\Lambda)$. Then we have a natural epimorphism ϕ of $\bar{\Omega} = \mathrm{Hom}^r_{\bar{\Lambda}}(\bar{C},\bar{C})$ to $\mathrm{Hom}^r_{\bar{\Lambda}/\bar{N}}(\bar{C}/\bar{C}N, \bar{C}/\bar{C}N)$ and $\phi^{-1}(0) \supseteq R(\bar{\Omega})$ (cf. the proof of Proposition 4.5). Since C/CN is Λ/N-module, we have $\sum_{i=1}^{s'}\sum_{j=1}^{p(i)} e_{\alpha(i),j}\Lambda/N = C + N/N \oplus C \cap N/CN = \sum_{i=1}^{s}\sum_{j=1}^{m(i)} e_{i,j}\Lambda/N \oplus C \cap N/CN$ by Theorem 3.3, where we assume that $C + N/N \approx \sum_{i=1}^{s} \oplus (\Delta_i(\Lambda))_{m(i)}$. Hence, $s' \geqq s$. On the other hand, $\mathrm{Hom}^r_{\Lambda/N}(C/CN,\ C/CN)$ has s' simple components, and hence we obtain that $s = s'$ and $\phi^{-1}(0) = R(\bar{\Omega})$. Therefore, $n(i) \geqq m(\pi(i))$ and $\Delta_i(\Omega) \approx \Delta_{\pi(i)}(\Lambda)$.

By Theorem 3.3, each simple component $(\Delta_i(\Lambda))_{m(i)}$ of Λ/N corresponds to a maximal order Ω containing Λ such that $(C(\Omega)+N)/N \approx (\Delta_i(\Lambda))_{m(i)}$. Hence, every associated division ring $\Delta_i(\Lambda)$ is isomorphic to that of $\Omega/R(\Omega)$, which does not depend on Λ by Corollary 4.5. Finally, if we show that $C \cap N/CN \neq (0)$ for $\Omega \neq \Lambda$, then we complete the proof. From an exact sequence: $0 \to C \to \Lambda \to \Lambda/C \to 0$, we have $\mathrm{Tor}_\Lambda^l(\Lambda, \Lambda/N) \to \mathrm{Tor}_\Lambda^l(\Lambda/C, \Lambda/N) \to C \otimes_\Lambda \Lambda/N \to \Lambda/N$. Hence, $C \cap N/CN \approx \mathrm{Tor}_\Lambda^l(\Lambda/C, \Lambda/N)$. If $\mathrm{Tor}_\Lambda^l(\Lambda/C, \Lambda/N) = 0$, then $\mathrm{Tor}_{\hat{\Lambda}}^l(\hat{\Lambda}/\hat{C}, \hat{\Lambda}/\hat{C}) = 0$ by the usual argument, where $\hat{\Lambda}$ means a completion with respect to the maximal ideal of R. Hence, $\hat{\Lambda}/\hat{C}$ is $\hat{\Lambda}$-projective by [7, Theorem 11]. Therefore, we have $\hat{C} = \hat{\Lambda}$ which implies $C = \Lambda$.

5. **Criteria of *h*-orders.** In this section we shall show the converse of Theorem 3.3.

Lemma 5.1. *Let Λ be an order in Σ and Ω a maximal order containing Λ. If Ω is left Λ-projective, then $C(\Omega)$ is a minimal idempotent ideal in Λ.*

Proof. $C = C(\Omega)$ is left Ω-projective, and hence, Λ-projective. By Proposition 1.6, $\tau_\Lambda^l(C) = C$. Hence, C is idempotent by Lemma 1.5. Let C_0 be an idempotent ideal contained in C. Then $\mathrm{End}_\Lambda^r(C_0) = \Omega$. Therefore, C_0 is left Λ-projective. Thus, $C_0 = C(\mathrm{End}^r(C_0)) = C$ by Theorem 1.7.

Lemma 5.2. *Let C be a maximal idempotent ideal in an order Λ such that $\mathrm{End}_\Lambda^r(C) \neq \Lambda$. Then $\mathrm{End}_\Lambda^r(M) \neq \Lambda$ for the maximal ideal M containing C.*

Proof. Let $\Omega_1 = \mathrm{End}_\Lambda^r(M)$ and $\Omega = \mathrm{End}_\Lambda^r(C)$. We have, by Lemma 3.2, $M^n = C$ for some n. We assume $\Omega_1 = \Lambda$. We consider the following two cases: (1) $\tau_\Lambda^r(M) = \Lambda$, (2) $\tau_\Lambda^r(M) = M$.

Case 1. If $\tau_\Lambda^l(M) = \Lambda$, then we have $MM_r^{-1} = M_l^{-1}M = \Lambda$, where $M_r^{-1} = \{x \mid \in \Sigma,\ Mx \subseteq \Lambda\}$, and $M_l^{-1} = \{x \mid \in \Sigma,\ xM \subseteq \Lambda\}$. Hence, M is inversible, which implies that C is also inversible. However, C is an idempotent ideal $\neq \Lambda$. Therefore, $\tau_\Lambda^l(M) \neq \Lambda$, and hence, $\tau_\Lambda^l(M) = M$. By assumptions $\Omega_1 = \Lambda$ and $\tau_\Lambda^r(M) = M$, M is left Λ-projective. Hence, $M = \tau_\Lambda^l(M)M = M^2$ by Lemma 1.1. Thus $M = C$, which is a contradiction to $\Omega \neq \Lambda$.

Case 2. Since $M^n = C$, there exists an integer $i \geqq 2$ such that $M^{i-1} = C$, and $M^i = C$. $\Lambda \supseteq \Omega C = \Omega M^{i-1}M$. Since $\tau_\Lambda^r(M) = M$, we have $\Omega M^{i-1} M \subseteq M$. Hence, $\Omega M^{i-1} \subseteq \mathrm{End}_\Lambda^r(M) = \Lambda$, which implies $M^{i-1} \subseteq C$. Thus, we know $\Omega_1 \neq \Lambda$.

Theorem 5.3. *Let R be a discrete rank one valuation ring with field of quotients K. Let Λ be an order over R in the central simple K-algebra Σ, such that Λ/N has n simple components. We assume that every maximal order $(\supseteq \Lambda)$ is left Λ-projective. If there exists a maximal chain of orders Δ_i containing $\Lambda(\Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_n = \Lambda)$ such that every Δ_i is left Λ-projective and Δ_i has* *precisely i maximal two-sided ideals, then* Λ *is hereditary, and the radical of* Λ *is inversible.*

Proof. We shall prove the theorem by induction on n. If $n=1$, then Λ is maximal and hence, Λ is hereditary. We assume that the theorem is true for order with $n-1$ maximal ideals. Let Λ be an order as in the theorem. Then Δ_{n-1} satisfies the conditions in the theorem by Lemma 1.3. Hence, Δ_{n-1} is hereditary. We denote Δ_{n-1} by Γ_1. Let $\{\Omega_1,\Omega_2,\cdots,\Omega_{n-1}\}$ be the set of maximal orders containing Γ_1 and $D_i = C_\Lambda(\Omega_i)$. Then the D_i's are minimal idempotent and left Λ-projective. Let $C_1 = C(\Gamma_1)$; then $C_1 = \sum_{i=1}^{n-1} D_i$, and $\Gamma_1 = \mathrm{End}^r_\Lambda(C_1)$ by Proposition 1.6. Furthermore, $C_1\Gamma_1$ is idempotent in Γ_1 and $\mathrm{End}^r_{\Gamma_1}(C_1\Gamma_1) \subseteq \bigcap_i \mathrm{End}^r_{\Gamma_1}(D_i\Gamma_1) = \bigcap_i \Omega_i = \Gamma_1$. Since Γ_1 is hereditary, $C_1\Gamma_1 = \Gamma_1$. Therefore, $\tau^l_{\Gamma_1}(C_1) = \Gamma_1$. Let $\Gamma_2 = \mathrm{End}^l_\Lambda(C)$. Then Γ_2 is also hereditary by [3, Theorem A.5], and Γ_2 has $n-1$ maximal ideals. Let $\{\Omega'_1,\Omega'_2,\cdots,\Omega'_{n-1}\}$ be the set of maximal orders containing Γ_2 and $D'_i = C(\Omega'_i)$. Since Λ/N has n simple components and the D_i's and the D'_i's are minimal idempotent in Λ, we may assume that $\Omega_i = \Omega'_i$ for $i \leqq n-2$ and $\Omega_1,\Omega_2,\cdots,\Omega_{n-1},\Omega_n = \Omega'_{n-1}$ are the set of maximal orders containing Λ. Since $C'_2 = C(\Gamma_2) \supseteq D_1 + D_2 + \cdots + D_{n-2} + D_n, C_2 = I(C'_2) + N = D_1 + \cdots + D_{n-2} + D_n$ by Lemma 3.2. Furthermore, $\Gamma' = \mathrm{End}^r_\Lambda(C_2) \supseteq \mathrm{End}^r_\Lambda(C'_2) \supseteq \Gamma_2$ and $\Gamma' \subseteq \bigcap_{i\neq n-1}\Omega_i = \Gamma_2$. Repeating this argument, we have the following set of h-orders $\Gamma_i:(\Gamma_1,C_1,\Gamma_2)$, $(\Gamma_2,C_2,\Gamma_3),\cdots$, $(\Gamma_i,C_i,\Gamma_{i+1})$ and the C_i's are maximal idempotent ideals. Let $D(\Gamma_j)$ be the left conductor of Γ_j; then $D(\Gamma_j) \supset C_{j-1}$. Hence, $I(D(\Gamma_j)) = C_{j-1}$. If $\Gamma_{i+1} = \Gamma_{j+1}$ for $j<i$, then $C_j = I(D(\Gamma_{j+1})) = I(D(\Gamma_i)) = C_i$, and hence, $\Gamma_j = \mathrm{End}^r_\Lambda(C_j) = \mathrm{End}^r_\Lambda(C_i) = \Gamma_i$. Therefore, we assume $\Gamma_{i+1} = \Gamma_1$. By using the same argument as the proof of Theorem 4.3, we shall show that $i=n$. If $i<n$, there exists a maximal order Ω containing all the Γ_i's by the construction of Γ_i. Hence, $C(\Omega)$ is contained in the idempotent ideal $B = I(C_1\cdots C_i)$. Let $\Delta = \mathrm{End}^r_\Lambda(B)$; then $\Delta \supset \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_i$. Since Γ_1 is left Λ-projective and hereditary, there is a one-to-one correspondence between orders Δ' containing Γ_1 and idempotent ideals contained in C. Hence, we have $\mathrm{End}^r_\Lambda(B) = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_i$. It is clear that

$$\mathrm{End}^l_\Lambda(B) \supset \Gamma_2 \cup \Gamma_3 \cdots \cup \Gamma_i \cup \Gamma_1.$$

On the other hand, B is right Λ-projective by Proposition 1.6. Hence, we have a contradiction to Corollary 1.9. Thus, we have proved that for every maximal idempotent ideal C_i, $\mathrm{End}^r_\Lambda(C_i) \neq \Lambda \neq \mathrm{End}^l_\Lambda(C_i)$. Let M_j be a maximal ideal in Λ containing C_j and $C_j = I(M_j) = M_j^t$. Then $\mathrm{End}^r_\Lambda(M_j) = \Gamma_j$ and $\mathrm{End}^l_\Lambda(M_j) = \Gamma_{j+1}$ by Lemma 5.2. Since $C_j\Gamma_j = \Gamma_j$, $M_j\Gamma_j = \Gamma_j$. Let N be the radical of Λ. Then $N = \bigcap_j M_j$ and $\Lambda \supset M_{j-1}\Gamma_j \supseteq N\Gamma_j \supseteq M_{p_1}M_{p_2}\cdots M_{p_{n-2}}M_{j-1}M_j\Gamma_j = M_{p_1}M_{p_2}\cdots M_{p_{n-2}}M_{j-1}$. Therefore, $\tau^l_\Lambda(N) \supseteq \sum_i M_1\cdots M_{i-1}M_{i+1}\cdots M_n + N = \Lambda$. Similarly, we have $\tau^r_\Lambda(N) = \Lambda$. Therefore, N is inversible, and hence, Λ is hereditary by Lemma 3.6.

COROLLARY 5.4. *Let Λ be an order such that Λ contains precisely two maximal ideals. If every maximal order containing Λ is left Λ-projective, then Λ is hereditary.*

Proof. Let Ω be a maximal order containing Λ. Then there are no orders between Ω and Λ, and hence, Λ is hereditary by the theorem.

In the contrast with Lemma 3.6, we have

COROLLARY 5.5. *If every (fractional) idempotent ideal with respect to Λ is left Λ-projective, then Λ is hereditary.*

Proof. We assume that Λ has n maximal ideals. Then first we shall show that there exist precisely n maximal orders Ω_i containing Λ and $\Lambda = \bigcap \Omega_i$. Let $\{\Omega_1, \Omega_2, \cdots, \Omega_r\}$ be the set of maximal orders containing Λ, and $C_i = C(\Omega)$. Since the C_i's are minimal idempotent, $r \leqq n$. If $r < n$, there exists a minimal idempotent ideal $C \neq C_i$ for all $i \leqq r$. Then $\mathrm{End}^l_\Lambda(C)$ is contained in some $\Omega_{\pi(i)} = \mathrm{End}^l_\Lambda(C_i)$. Hence, $C \supseteq D(\mathrm{End}^l_\Lambda(C_i)) = \tau^r_\Lambda(C_i) = C_i$ by Proposition 1.6. It is clear that $C(\bigcap \Omega_i) \supseteq \sum C_i = \Lambda$, and hence, $\Lambda = \bigcap \Omega_i$. Let $D = \sum_{j=1}^{n-1} C_j$; then since D is left Λ-projective, $\Lambda \neq \mathrm{End}^l_\Lambda(D) \subset \bigcap_{j \neq n} \mathrm{End}^1(C_j)$. Therefore, we can prove the corollary by induction on n with Theorem 5.3.

6. **Two-sided ideals with respect to an *h*-order.** In this section, we shall study a group structure of the set of two-sided (fractional) ideals with respect to an h-order Λ.

For this purpose, we quote the following definition (cf. [7, p. 76]):

DEFINITION. For two-sided ideals A, B the product AB is called a characteristic product, if $A' \supseteq A$, $B' \supseteq B$ and $AB = A'B'$; then $A' = A$ and $B' = B$ for any ideals A' and B'.

Let A be an ideal with respect to an h-order Λ. Then $\Omega_1 = \mathrm{End}^r_\Lambda(A)$ and $\Omega_2 = \mathrm{End}^l_\Lambda(A)$ are h-orders containing Λ, and $AA^{-1} = \Omega_1$, $A^{-1}A = \Omega_2$. Let B be another ideal. If $\Omega_2 = \mathrm{End}^l_\Lambda(B)$, then AB is a characteristic product. Because, if $A' \supseteq A$ and $B' \supseteq B$ and $A'B' = AB$, then $AB \subseteq A'B \subseteq A'B'$ and hence, $AB = A'B$. Therefore, $A' = A'BB^{-1} = A\Omega_2 = A$. Conversely, if AB is characteristic, then $A\Omega_2 B = AB$ and hence, $\Omega_2 B = B$, which implies $\Omega_2 \subseteq \mathrm{End}^r_\Lambda(B)$. Similarly, we have $\Omega_2 \supseteq \mathrm{End}^r_\Lambda(B)$ (cf. [2, p. 182, Theorem 4.51]).

Now, let Λ be an h-order over a discrete, rank one valuation ring, which has n-maximal two-sided ideals in Λ. Let Ω_i^{n-j} $(j = 0,1,\cdots,n-1;\ i = 1,2), \cdots, \binom{n}{j})$ be the set of orders containing Λ, and Ω_i^{n-j} has $n-j$ maximal two-sided ideals in Ω_i^{n-j}. By $G_{l,m}^{n-j}$ we denote the set of two-sided fractional ideals A with respect to Λ such that $\mathrm{End}^r_\Lambda(A) = \Omega_l^{n-j}$, and $\mathrm{End}^l_\Lambda(A) = \Omega_m^{n-j}$. Then for $A \in G_{l,m}^{n-j}$, $B \in G_{p,q}^{n-t}$ we have the characteristic product $AB \in G_{l,q}^{n-j}$ if $t = j$ and $m = p$; if not then AB is not characteristic. Let A be an ideal with respect to Λ. Since $\mathrm{End}^r_\Lambda(A)$ and $\mathrm{End}^l_\Lambda(A)$ belong to the same type and hence, A belongs to some $G_{l,m}^{n-j}$. Conversely, since Ω_l^{n-j} and Ω_m^{n-j} belong to the same type, there exists an ideal B that $B \in G_{l,m}^{n-j}$.

THEOREM 6.1. *Let Λ be an h-order over a discrete, rank one valuation ring R in Σ. Let $\Omega_{l,m}^{n-j}$, $G_{l,m}^{n-j}$ be as above. Then the set of two-sided fractional ideals with respect to Λ is a groupoid*[5] *with $G_{l,m}^{n-j}$ and Ω_l^{n-j} as unit element with respect to characteristic product. Furthermore, $G_{l,l}^{n-j}$ is a cyclic group generated by the radical $N_l^{(n-j)}$ of Ω_l^{n-j}.*

Proof. We have observed the first half in the above. We denote $G_{l,l}^{n-j}$, Ω_l^{n-j} by G,Ω. G is, by Lemma 1.1, the set of two-sided ideals with respect to Ω such that $\tau_\Omega^l(A) = \tau_\Omega^r(A) = \Omega$. Hence, it consists of inversible ideals with respect to Ω. Therefore, G is a group. We denote $N_l^{(n-j)}$ by N. Then $N \in G$ by Theorem 5.3. Let $A \in G$ such that $A \subseteq \Lambda$ and $A \nsubseteq N$. We assume that R is complete. Then there exists an idempotent element e in $A + N$, and hence, $e \in \bigcap (A + N)^n \subseteq \bigcap (A + N^n) = A$. Therefore, A^n is idempotent for some n. If R is not complete, then we can use the same argument as in the proof of Lemma 3.2. Since $A^n \in G$, $A = \Omega$. We have proved that N is a maximal two-sided integral ideal in G. For any integral ideal B in G, we can find an integer t such that $N^t \supseteq B$ and $N^{t+1} \nsupseteq B$. Then since $N^{-t}B(\subseteq \Omega)$ is not contained in N, $N^{-t}B = \Omega$. Therefore, $B = N^t$.

7. ***H*-orders over a Dedekind ring.** In the previous sections, we have studied h-orders over a discrete, rank one valuation ring. Now, in this section, we shall deduce properties of h-orders over a Dedekind domain from results in the previous sections.

Let R be a Dedekind domain with field of quotients K and Σ a central simple K-algebra. Let Γ_1 and Γ_2 be orders containing an h-order Λ over R, and Ω_1 a maximal order containing Γ_1. Let $d = C_{\Gamma_2}(\Gamma_1) \cap R = \{x \mid \in R,\ \Gamma_1 x \subseteq \Gamma_2\}$ and $c = C(\Omega_1) \cap R$; then we have $d \supseteq c$. By using prime factors $p_1, \cdots, p_r$ of c, we obtain a multiplicative system $S = R - (p_1 \cup \cdots \cup p_r)$ in R. Then $d_S = C_{\Gamma_{2S}}(\Gamma_{1S}) \cap R_S \neq R_S$ if $d \neq R$. Hence, if $\Gamma_1 \nsubseteq \Gamma_2$, then $\Gamma_{1S} \nsubseteq \Gamma_{2S}$. On the other hand, let Γ' be an order over R_S containing Λ_S. Then $\Gamma' = \mathrm{Hom}_{\Lambda_S}^r(E', E')$ for an idempotent ideal E' in Λ_S. Let $E = E' \cap \Lambda$, then $\Gamma = \mathrm{Hom}_\Lambda^r(E,E)$ is an order containing Λ such that $\Gamma_S = \Gamma'$.

It is clear by Theorem 1.7 that $C(\Omega)$ is a minimal idempotent ideal in Λ and that the set $\{p_1, \cdots, p_r\}$ does depend only on Λ, not on Ω. We say that the p_i's belong to Λ.

Summarizing the above observation, we have

PROPOSITION 7.1. *Let Λ be an h-order over a Dedekind domain R in Σ, and let the set $\{p_i\}$ belong to Λ. Then there is a one-to-one correspondence between orders over R containing Λ and orders over R_S containing Λ_S, which preserves the inclusion, where $S = R - (p_1 \cup \cdots \cup p_r)$.*

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(5) See [7, p. 76, Satz 14].

From this proposition, we may restrict ourselves to the case where R is a semi-local Dedekind domain with maximal ideals $p_1, \cdots, p_r$. Furthermore, we may assume, by the above argument, that Λ_{p_i} is not maximal for each p_i. For a while we assume R is semi-local. Let n and N be the radicals of R and Λ, respectively; then $\Lambda/N = \Lambda/N \otimes_R R/n = \Lambda/N \otimes_R R/p_1 \oplus \cdots \oplus \Lambda/N \otimes_R R/p_r$, and $\Lambda/N \otimes_R R/p_i = \Lambda_{p_i}/N_{p_i}$. On the other hand, we have $\hat{\Lambda} = \Lambda \otimes_R \hat{R} = \Lambda \otimes_R \hat{R}_{p_1} \oplus \cdots \oplus \Lambda \otimes_R \hat{R}_{p_i}$ and $\Lambda/N = \hat{\Lambda}/\hat{N} = \Lambda/N \otimes_R \hat{R}_{p_1} \oplus \cdots \oplus \Lambda/N \otimes_R \hat{R}_{p_r} = \Lambda_{p_i}/N_{p_1} \otimes_R \hat{R}_{p_1} \oplus \cdots \oplus \Lambda_{p_r}/N_{p_r} \otimes_R \hat{R}_{p_r}$, where $\hat{R}$ and the $\hat{R}_p$ are completions of R and R_p with respect to n and pR_p, respectively. Let A be a nonzero idempotent ideal in Λ; then the $((A+N)^{\wedge}/\hat{N}) \otimes \hat{R}_{p_i}$ are nonzero ideals in $\Lambda_{p_i}/N_{p_i} \otimes \hat{R}_{p_i}$. Conversely, if we take nonzero ideals C_i' in $\Lambda/N \otimes \hat{R}_{p_i}$ for each i, we can find idempotent ideals C_i in $\hat{\Lambda}_{p_i}$ such that $C_i \equiv C_i' \bmod \hat{N}_{p_i}$. Hence, $C = \sum_{i=1}^{r} C_i$ is an idempotent ideal in $\hat{\Lambda}$ such that $C + \hat{N} = \Sigma C_i'$ $(= C')$. However, since $\hat{\Lambda}/C = \Sigma \hat{\Lambda}_{p_i}/C_i$, $\hat{\Lambda}/C$ satisfies the minimal condition, C'^n is idempotent and $C'^n + \hat{N} = C + \hat{N}$ for some n. Since $C' \supsetneqq \hat{N}$, there exists an ideal A in Λ such that $\hat{A} = C'$. Hence, A^n is idempotent and $\hat{A}^n_{p_i} \equiv C_i' \bmod \hat{N}_{p_i}$. Therefore, by Lemma 2.4, there is a one-to-one correspondence between idempotent ideals in Λ and ideals A in Λ/N such that $A_{p_i}/N_{p_i} \neq (0)$ for all p_i. Furthermore, from the assumption that the Λ_{p_i}'s are not maximal, every Λ_{p_i}/N_{p_i} is not a simple ring.

We shall come back again to the case where R is a Dedekind domain (not necessarily semi-local).

Let $\{p_1, \cdots, p_r\}$ be the set of prime ideals in R which belong to an h-order Λ. For the set $S_i = R - p_i$, by $\rho(i)$ we denote the number of two-sided ideals $P_{i,j}$ which is a maximal ideal among the set of two-sided ideals A such that $A \cap S_i = \phi$. Then we have a one-to-one correspondence between maximal ideals in Λ_{s_i}, and $P_{i,j}$. Hence, $\rho(i)$ is the number of simple components of Λ_{p_i}/N_{p_i} which is a finite integer $\geqq 2$. Furthermore, it is clear by Theorem 1.7 that for the conductor C of a maximal order containing Λ, we obtain uniquely a simple component $C_p + N_p/N_p$ of Λ_p/N_p for each p and conversely. Similarly, for the conductor C' of a minimal order containing Λ, we can find uniquely a prime ideal p in R and a maximal ideal $\bar{A}_p$ in Λ_p/N_p such that $C_p' + N_p/N_p = \bar{A}_p$ and $C_q' + N_q/N_q = \Lambda_q/N_q$ if $p \neq q$.

Using the above observations and Proposition 3.1 we have a generalization of Theorem 3.3.

THEOREM 7.2. *Let Λ be an h-order over a Dedekind domain R in Σ. Let $\rho(i)$ $(i = 1, \cdots, r)$ be as above. Then there are precisely $\Pi_i \rho(i)$ maximal order Ω_i containing Λ and $\Sigma_i \rho(i)$ minimal orders $\Gamma_{i,j}$ $(i = 1, \cdots, r; j = 1, \cdots, \rho(i))$, containing Λ. For an order Γ $(\supsetneqq \Lambda)$, we have a unique expression of $\Gamma : \Gamma = \bigcup_{l=1}^{\rho \leqq r} \bigcup_{k=1}^{t_l < \rho(l)} \Gamma_{l,jk_l}$* and for order Γ' $(\supseteq \Lambda)$, $\Gamma' = \bigcap_i \Omega_i$ *(not necessarily unique) where Ω_i runs through all maximal orders containing Λ. Consequently, the number of orders containing Λ is equal to $\prod_{i=1}^{r} (2^{\rho(i)} - 1)$.*

Let C_1 and C_2 be idempotent ideals in Λ, then we shall say that C_1 and C_2 belong to the same type if $C_{1p}+N_p/N_p$ and $C_{2p}+N_p/N_p$ have the same number of simple components for each p.

If $\Gamma=\bigcup\bigcup\Gamma_{l,jk_l}$ and $C=C(\Gamma)$, then $C+N/N$ is isomorphic to a unique decomposition: $\sum_{l=1}^{s}\sum_{m=1}^{\rho_l-t_l}\quad \hat{A}_{l,jm}+\sum_{l=s+1}^{r}\Lambda_{pl}/N_{pl}$, where $\hat{A}_{l,j}$ is a simple component of Λ_{pl}/N_{pl}. We call Γ an sth order.

If Γ is a 1st order, then we have the same situation as in the previous section. Therefore, we can prove the following theorem by induction on s as in the proof of Theorem 4.3.

THEOREM 7.3. *Let Λ be as above, and Γ_1, Γ_2 orders containing Λ. Then we have the following equivalent conditions*:

(1) *Γ_1 and Γ_2 belong to the same type.*

(2) *Γ_{1p} and Γ_{2p} belong to the same type for each prime ideal p in R.*

(3) *$C(\Gamma_1)$ and $C(\Gamma_2)$ belong to the same type.*

Finally, we consider a group structure of two-sided ideals with respect to Λ. We use the following well-known lemma:

LEMMA 7.4. *Let E be a finitely generated R-module and A, B submodule in E. Then we have $(A:B)_p=(A_p:B_p)$ for every prime ideal p in R.*

LEMMA 7.5. *For each prime ideal p in R, there exists a two-sided ideal $N(p)$ such that $N(p)_p$ is the radical of Λ_p and $N(p)_q=\Lambda_q$ if $p\neq q$.*

Proof. Let $R(\Lambda_q)$ be the radical of Λ_q and $N(p)=R(\Lambda_p)\cap\Lambda$. Then $N(p)_p=R(\Lambda_p)$. Furthermore, since $N(p)\supseteq R(\Lambda_p)\cap\Lambda\cap R\supseteq p\Lambda\cap R=p$, $N(p)_q\supseteq p_q=R_q$ if $p\neq q$.

Thus, we have the following generalization of [7, p. 74, Satz 9].

THEOREM 7.6. *Let Λ be an h-order over a Dedekind domain R in Σ. Then the set of inversible two-sided ideals A with respect to Λ is an abelian group which is a direct product of cyclic groups generated by $N(p)$.*

Proof. It is clear that A is an inversible ideal if and only if $\tau_\Lambda^l(A)=\tau_\Lambda^r(A)=\Lambda$. By Lemmas 7.4 and 7.5, and Theorem 6.1, we have $\tau_\Lambda^l(N(p))=\tau_\Lambda^r(N(p))=\Lambda$ and hence, $N(p)$ is inversible. Let A be an inversible ideal with respect to Λ. Then $A_p=R(\Lambda_p)^{\rho(p)}$ by Theorem 6.1, and the $\rho(p)$ are equal to zero except a finite number of p. Hence, by Lemmas 7.4 and 7.5, we have $A=\prod N(p)^{\rho(p)}$. Furthermore, $(N(p)N(q))_r=(N(q)N(p))_r$ for every prime ideal r in R. Thus, we prove the theorem.

REMARK 3. We can construct a groupoid structure of two-sided ideals with respect to Λ as in Theorem 6.1.

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