

DIFFERENTIABLE OPEN MAPS ON MANIFOLDS⁽¹⁾

BY

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Introduction. This paper contains a detailed discussion with proofs of results announced in [4].

Let M^n and N^n be n -manifolds without boundary, and let $f: M^n \rightarrow N^n$ be continuous. The map f is *open* if, whenever U is open in M^n , $f(U)$ is open in N^n ; it is *light* if, for every $y \in N^n$, $\dim(f^{-1}(y)) \leq 0$. For $n \geq 2$ there is a canonical light open map $F_{n,d}: E^n \rightarrow E^n$ given by $F_{n,d}(x_1, x_2, \dots, x_n) = (u_1, u_2, x_3, \dots, x_n)$, where

$$u_1 + iu_2 = (x_1 + ix_2)^d \quad (i = \sqrt{-1}; d = 1, 2, \dots).$$

For $n = 2$ it is well known that a nonconstant complex analytic function is open and light. Conversely, Stoilow [12] proved that every light open map is locally topologically equivalent to an analytic map, and thus to some $F_{2,d}$ ($d = 1, 2, \dots$). In fact (1.10), if M^2 is compact and f is C^2 and open, then f has this canonical structure. The main object of this paper is to prove (2.1) that the corresponding conclusion holds for arbitrary n ($n \geq 2$), if we first remove an exceptional set of dimension at most $n - 3$. Examples are given, especially in §3, showing that the exceptional set and some of the hypotheses used are necessary.

DEFINITION. As in [5] the branch set B_f is the set of points in M^n at which f fails to be a local homeomorphism.

NOTATION. If $f: E^n \rightarrow E^p$ is C^r , then f_i will be the i th component real-valued function, and $D_j f_i$ will be the first partial derivative of f_i with respect to its j th coordinate. If y is a point in E^n , then y_i will be its i th coordinate. The symbols M^n and N^p will refer to manifolds of dimensions n and p , respectively. The statement that $f: M^n \rightarrow N^p$ is C^m will imply that the manifolds are also C^m . The set of points in M^n at which the Jacobian matrix of f has rank at most q will be denoted by R_q .

The closure of a set X is denoted by $\text{Cl}[X]$ or \bar{X} , its interior by $\text{int } X$, and the restriction of f to X by $f|X$. A map is a continuous function, the distance between the points x and y is $d(x, y)$, and $S(x, \varepsilon) = \{y: d(x, y) < \varepsilon\}$.

1. General results.

1.1. **LEMMA.** Let $h: E^n \rightarrow E^p$, $h \in C^m$ ($m = 1, 2, \dots$), and let the rank of the Jacobian matrix of h at \bar{x} be at least q ($q = 1, 2, \dots, n - 1$). Then there exist open neighborhoods U of \bar{x} and V of $h(\bar{x})$, and C^m diffeomorphisms (onto)

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$k^1: E \rightarrow U$ and $k^2: V \rightarrow E^p$ such that k^2hk^1 , call it g , has the following properties:

- (1) For each $(p - q)$ -plane α given by g_i constant ($i = 1, 2, \dots, q$), $g^{-1}(\alpha)$ is a (single) $(n - q)$ -plane given by x_j constant ($j = 1, 2, \dots, q$).
- (2) For each x in $g^{-1}(\alpha)$, the rank of the Jacobian matrix of g at x is s if and only if the rank of the Jacobian of $g|g^{-1}(\alpha)$ at x is $s - q$ ($s = q, q + 1, \dots, \min(n, p)$).

Proof. By reordering the variables, both dependent and independent, we may suppose that the determinant $\det[D_jh_i](\bar{x}) \neq 0$ ($i, j = 1, 2, \dots, q$). Let W be a neighborhood of \bar{x} such that $\det[D_jh_i] \neq 0$ on all of W , and let $\bar{h}: W \rightarrow E^q$ be defined by $\bar{h}_i = h_i$ ($i = 1, 2, \dots, q$). Since \bar{h} has maximal rank at every point of W , we may apply the rank theorem [7, pp. 273-274]. Thus, there exists an open n -cell U in W about \bar{x} and C^∞ diffeomorphisms $k^1: E^n \rightarrow U$ and $\bar{k}: h(U) \rightarrow E^q$ such that $(\bar{k}h k^1)_i(x_1, x_2, \dots, x_n) = x_i$ ($i = 1, 2, \dots, q$). Using $V = \bar{h}(U) \times E^{p-q}$, $k_i^2(x) = \bar{k}_i(x_1, x_2, \dots, x_q)$ ($i = 1, 2, \dots, q$), and $k_i^2(x) = x_i$ ($i = q + 1, q + 2, \dots, p$), conclusion (1) follows.

Given $x \in E^n$, let s be the rank of the Jacobian matrix of g at x . If α is the $(p - q)$ -coordinate plane containing $g(x)$, then $J = (D_jg_i(x))$ ($i = q + 1, q + 2, \dots, p$; $j = q + 1, q + 2, \dots, n$) is the Jacobian matrix of $g|g^{-1}(\alpha)$ (as a map into α) at x . Since (by (1)) $D_jg_i(x) = 0$ ($i = 1, 2, \dots, q$; $j = q + 1, q + 2, \dots, n$), and $\det(D_jg_i(x)) \neq 0$ ($i, j = 1, 2, \dots, q$), J has rank $s - q$, yielding (2). (Clearly, the same result holds if we interpret $g|g^{-1}(\alpha)$ as a map into E^p .)

REMARK. If $p = n$ and if $f|U$ has Jacobian determinant non-negative or nonpositive, then, for each $(n - q)$ -cell γ given by conclusion (1), $f|\gamma$ (i.e., $g|\gamma$) has Jacobian determinant non-negative or nonpositive (not "respectively," in general). In particular, if $q = n - 1$, then each map $f|\gamma$ is monotone.

1.2. REMARK. If X is a compact set contained in $E^n = E^{n-r} \times E^r$, and if $\dim(X \cap (E^{n-r} \times \{x\})) \leq q - r$ for each x in E^r , then $\dim X \leq q$.

Let $g: E^{n-r} \times E^r \rightarrow E^r$ be the projection map, and let f be the restriction of g to X . The proof, pointed out to the author by E. Connell, follows from an application of [9, pp. 91-92].

The following result is related to Sard's theorem [11].

1.3. PROPOSITION. If $f: M^n \rightarrow N^p$, f and the manifolds are C^n , then $\dim(f(R_q)) \leq q$ (where R_q is the set of points of M^n at which the Jacobian matrix of f has rank at most q). In particular, $\dim(f(M^n)) \leq n$. If f is also light then $\dim(R_q) \leq q$.

Proof. Clearly, it is sufficient to prove the theorem for $f: E^n \rightarrow E^p$. If X_i is the set of critical points of f_i (the points at which all first partials are zero), then the measure of $f_i(X_i)$ is zero [10, p. 68, (4.3)] ($i = 1, 2, \dots, p$). Thus, $\dim(\bigcap_{i=1}^p f(X_i)) \leq 0$. Since $R_0 = \bigcap_{i=1}^p X_i$, it follows that $\dim(f(R_0)) \leq 0$.

(In fact, for each u in $f(R_0)$, there exists a coordinate p -cube C containing u such that the sides of C are coordinate $(p-1)$ -planes, $\text{diam}(C) < \varepsilon$, and $f(R_0) \cap \text{bdy } C = \emptyset$.)

The proposition follows for $p = 1$ and all n and q ; we proceed by induction on p .

Since each R_q is closed, $f(R_q)$ is the countable union of compact sets; thus it is sufficient to prove that $\dim(f(R_q - R_0)) \leq q$, i.e., to prove the result in the case that the rank is at least one at each point. Furthermore, it suffices to prove the conclusion for $f|U$, where U is the open set given by (1.1) for f , $q = 1$, and an arbitrary point \bar{x} . For each $(n-1)$ -cell γ given in (1.1) and each point x of $R_q \cap \gamma$, $f|_\gamma$ (as a map into the corresponding $(p-1)$ -cell) has rank at most $q-1$ at x . From the inductive hypothesis, $\dim(f(R_q \cap \gamma)) \leq q-1$. Thus $f(R_q \cap U)$ meets each $(p-1)$ -cell of (1.1) in a set of dimension at most $q-1$. Since $R_q \cap U$ is the countable union of compact sets, it follows from (1.2) that $\dim(f(R_q \cap U)) \leq q$; thus $\dim(f(R_q)) \leq q$.

If f is also light, then by [9, pp. 91-92] $\dim(R_q) \leq \dim(f(R_q))$. The condition that $f \in C^n$ is necessary [17] in the above result for $p = 1$.

The following result is, for open maps, an extension of the inverse function theorem.

1.4. THEOREM. *Let $f: E^n \rightarrow E^n$ be open and C' . If the rank of the Jacobian matrix of f at \bar{x} is at least $n-1$, then f is locally a homeomorphism at \bar{x} . In other words, $B_f \subset R_{n-2}$.*

Proof. For $n=1$, the openness alone implies that f is a homeomorphism into. For $n > 1$, let U be the neighborhood of \bar{x} given by (1.1) for $q = n-1$, and let γ be one of the 1-cells. Since $f(\gamma)$ is contained in a 1-cell, and since the restriction $f|_\gamma$ is open [20, p. 147, (7.2)], $f|_\gamma$ is a homeomorphism (into). Thus, $f|U$ is one-to-one and open, so that $\bar{x} \notin B_f$.

1.5. COROLLARY. *If $f: M^n \rightarrow N^n$, f open and C^n , then $\dim(f(B_f)) \leq n-2$. If f is also light, then $\dim(B_f) \leq n-2$.*

The proof follows from (1.2) and (1.4).

1.6. COROLLARY. *Let $f: M^n \rightarrow N^n$, f light and C^n . Then f is open if and only if $B_f \subset R_{n-2}$.*

Proof. If $B_f \subset R_{n-2}$, then $\dim(f(B_f)) \leq n-2$ by (1.2); thus f is open [5, p. 531, (2.4)].

A sufficient condition for openness was given in [14] by Titus and Young. We observe that (for $f \in C^n$) the condition is necessary, and give an independent proof of the sufficiency.

1.7. COROLLARY. *Let $f: E^n \rightarrow E^n$ be C^n and light. Then f is open if and only if the Jacobian determinant J is non-negative or non-positive everywhere.*

Proof. If f is open, then, by (1.3) and (1.6), $\dim(B_f) \leq n - 2$; thus J does not change sign.

Suppose that $J \geq 0$ (or $J \leq 0$). If the rank of the Jacobian matrix of f at \bar{x} is (at least) $n - 1$, then (from the remark after (1)) $f|U$ is one-to-one. For each closed n -cell C in U , $f|C$ is a homeomorphism onto its image; thus $f|\text{int } C$ is a homeomorphism onto its image, which is open by the theorem on invariance of domain. It follows that $x \notin B_f$. Since $B_f \subset R_{n-2}$, the conclusion follows from (1.6).

1.8. THEOREM. *If $f: M^n \rightarrow N^n$, M^n compact, f open and C^n , then f is light.*

In fact, f is a pseudo-covering map [5, pp. 529 and 531, (2.4)].

Proof. By (1.5) $\dim(f(B_f)) \leq n - 2$. By the second paragraph of the proof of [5, p. 531, (2.4)] the restriction of f to $M^n - f^{-1}(f(B_f))$ is a k -to-1 covering map for some k .

Suppose that for some y in $f(B_f)$, $f^{-1}(y)$ contains at least $k + 1$ distinct points y^i ($i = 1, 2, \dots, k + 1$). Then there exist disjoint open neighborhoods U^i of these points and $\bigcap_{i=1}^{k+1} f(U^i)$ is an open set; thus it meets $N^n - f(B_f)$, yielding a contradiction.

REMARK. This result contrasts with the examples by R.D. Anderson [1; 2] of monotone open (not C^n) maps. The compactness of the domain is necessary, as we see in (3.6).

1.9. COROLLARY. *If $f: E^n \rightarrow E^n$ is a C^n light open map, then point inverses are isolated. Moreover, if $f: S^n \rightarrow S^n$ is C^n open with (Brouwer) degree d , then, for each $p \in S^n$, $f^{-1}(p)$ has at most $|d|$ points and $|d|$ is the least such number.*

Proof. The first conclusion follows from (1.3), (1.6), and [5, p. 530, (2.2)]. For the second, since the Jacobian determinant is non-negative or nonpositive [17], $f^{-1}(p)$ has precisely $|d|$ points, for each $p \in S^n - f(B_f)$. In fact, f is a $|d|$ -to-1 pseudo-covering map, and the rest of the conclusion follows from the proof of (1.8).

The corollary is related to [16, p. 329, Theorem A and p. 335, (6a)].

1.10. Stoilow ([12]; cf. [20, p. 198, (5.1)]) proved that a light open map $f: M^2 \rightarrow N^2$ is locally at each point topologically equivalent to the complex analytic map $g(z) = z^d$ ($d = 1, 2, \dots$). (Manifolds are assumed to be without boundary.) For completeness we give now an independent proof in the case that $f \in C^2$. In particular, from (1.7) follows the apparently new result that: If $f: M^2 \rightarrow N^2$, M^2 compact, f open and C^2 , then f has that local structure.

By (1.5) $\dim(f(B_f)) \leq 0$. Given any x in B_f , by restriction [5, p. 529, (1.4) and its proof] there exists a pseudo-covering map g such that its domain V is a compact connected neighborhood of x in E^2 , $g(V)$ is a closed topological disk D , and $(\text{bdy } D) \cap g(B_g) = \emptyset$. We may also suppose [5, p. 530, (2.2), conclusion (1)] that $g^{-1}(g(x)) = x$. Since each component of $f^{-1}(\text{bdy } D)$ ($= \text{bdy } V$) is a simple closed curve, V is a disk-with-holes.

Let U be an open 2-cell about x in $\text{int } V$. Let h be a pseudo-covering map given, as above, for $g|U$ and x ; call its domain E and its range disk D' . If $\text{bdy } E (=f^{-1}(\text{bdy } D'))$ had two or more components (simple closed curves), then U would contain a disk whose image under g contained $D - \text{int}(D')$, contradicting the fact that $U \subset \text{int } V$. Thus E is a topological closed disk itself.

If B_h contains a point $y \neq x$, let γ be an arc in $D' - h(B_h)$, separating $\text{int}(D')$ into two components X and Y such that $h(x) \in X$ and $h(y) \in Y$. Then $h^{-1}(\gamma)$ consists of k mutually disjoint arcs, where k is the degree of h , and thus it separates $\text{int } E$ into $k + 1$ components. Precisely one of these components has image X (since $h^{-1}(h(x)) = x$), so that the other k have image Y , contradicting the fact that Y meets $h(B_h)$.

Thus $\{x\} = B_h$, and the conclusion is evident.

2. The structure theorem. In this section we give a structure theorem for differentiable open maps defined on compact manifolds, or (more generally) differentiable light open maps defined on arbitrary manifolds, comparing them with the maps $F_{n,d}$ defined in the introduction.

2.1. THEOREM. *Let $f: M^n \rightarrow N^n$ be C^n and open ($n \geq 2$); let M^n be compact, or let f be light. Then there exists a closed set E , $\dim E \leq n - 3$, such that for each x in $M^n - E$ there exists a neighborhood U of x on which f is topologically equivalent to one of the canonical maps $F_{n,d}$ ($d = 1, 2, \dots$). Moreover, E is nowhere dense in B_f unless f is a local homeomorphism.*

Trivial examples show that "topologically equivalent" cannot be replaced by "diffeomorphically equivalent." The hypothesis that f is C^n results from the use of (1.3).

Proof. Since f is light (1.8), $\dim(R_{n-3}) \leq n - 3$ (1.3); thus the set E may as well include R_{n-3} . To prove the first part of the theorem we may suppose that $n \geq 3$ and that the rank of the Jacobian matrix is at least $n - 2$ at every point. For each \bar{x} in B_f the restriction $f|U$ of f to some neighborhood U of \bar{x} has the structure of (1.1), where $q = n - 2$ and $p = n$. (We may as well suppose that $f|U$ is the g of (1.1).) Thus the domain and range of $f|U$ are $E^n = E^{n-2} \times C$, where C is the complex plane; for each $v \in E^{n-2}$ the restriction of f to the plane $\{v\} \times C$ is light and open [20, p. 147, (7.2)]. By (1.6) $B_f \subset R_{n-2}$, and by (1.1) $R_{n-2} \cap (\{v\} \times C)$ is the set of points at which $f|(\{v\} \times C)$ has rank 0; thus (1.3)

$$\dim(f(B_{f|(\{v\} \times C)})) \leq 0$$

and $\dim(f(B_f)) \leq n - 2$.

The rest of the proof of the first conclusion uses only the topological properties of $f|U$ found above, and not the differentiability of $f|U$.

Let A be a closed n -cell such that $\bar{x} \in \text{int } A$ and $A \subset U$. Since $\dim(f(B_f)) \leq n - 2$, there exists [5, p. 529, (1.4)] a connected open neighborhood V of \bar{x} such that

the restriction of f to V is a pseudo-covering map g , and $V \subset \text{int } A$. Choose $\bar{w} \in E^{n-2}$ so that $f(\bar{x})$ is in the plane $\{\bar{w}\} \times C$ of (1.1). Since $\text{Cl}[g(B_\theta)] \subset f(B_f \cap A)$, $\text{Cl}[g(B_\theta)]$ meets $\{\bar{w}\} \times C$ in a compact set of dimension 0. Let G be an open disk with center $g(\bar{x})$,

$$G \subset g(V) \cap (\{w\} \times C).$$

Let L be any straight line in $\{\bar{w}\} \times C$ through $g(\bar{x})$, and [9, p. 22, (D)] let a and b be points on opposite sides of $L \cap G$ from $g(\bar{x})$, a and b disjoint from $(\{w\} \times C) \cap \text{Cl}[g(B_\theta)]$. It follows from [9, p. 48, Corollary 1] that there exist arcs Γ_i joining a to b , Γ_i disjoint from $(\{\bar{w}\} \times C) \cap \text{Cl}[g(B_\theta)]$ ($i = 1, 2$), $\Gamma_1 - \{a, b\}$ contained in one component of $G - L$, and $\Gamma_2 - \{a, b\}$ in the other.

Then $\Gamma_1 \cup \Gamma_2$ bounds a topological closed disk $D \subset C$ such that $g(\bar{x}) \in \{\bar{w}\} \times (\text{int } D)$, $\{\bar{w}\} \times D \subset g(V)$, and $\{\bar{w}\} \times (\text{bdy } D)$ is disjoint from the 0-dimensional set $(\{\bar{w}\} \times C) \cap \text{Cl}[g(B_\theta)]$. Thus, for all w sufficiently near \bar{w} , the corresponding disks $\{w\} \times D$ will also be disjoint from $\text{Cl}[g(B_\theta)]$. Let T^{n-2} be such a small closed $(n-2)$ -cell in E^{n-2} for which $\bar{w} \in \text{int}(T^{n-2})$ and $T^{n-2} \times D \subset g(V)$. The restriction of g to the component of $g^{-1}(T^{n-2} \times D)$ containing \bar{x} is also a pseudo-covering map; for convenience we now call this map g , and its domain V .

Each set $g^{-1}(\{w\} \times D)$ is the closure of a region in the plane, each boundary component a simple closed curve. Thus, each $g^{-1}(\{w\} \times D)$ is homeomorphic to the same disk-with-holes H , and we will denote $g^{-1}(\{w\} \times D)$ by H^w .

For each w in T^{n-2} , let $g|H^w$ be denoted by g^w ; and let its branch set be denoted by $B(g^w)$. Clearly, $\bigcup B(g^w) \subset B_g$. Suppose that $x \in \text{int } H^w$ but $x \notin B(g^w)$. Choose an open neighborhood N of x in $\text{int } V$ such that $g^w|(N \cap H^w)$ is a homeomorphism. Let h be a pseudo-covering map whose domain contains x and is contained in N . Then the degree of h is one, and h is a homeomorphism; therefore, $x \notin B_g$. As a result, $\bigcup_w B(g^w) = B_g$.

For each w in T^{n-2} , the light open map g^w is topologically equivalent to a simplicial map [20, p. 198, (5.1)], and it follows from [17] that for some fixed natural number K depending only on H , $B(g^w)$ contains at most K points. Let $\alpha(w)$ be the number of branch points in H^w ($1 \leq \alpha(w) \leq K$). Let Y be any open set in T^{n-2} , and let \bar{y} in Y be a point at which the function α is maximal on Y . Let p^i be the points of $B(g^{\bar{y}})$, and let P^i be mutually disjoint sets open in $H^{\bar{y}}$ such that $P^i \cap g^{-1}(g(p^j)) = \{p^i\}$ ($i, j = 1, 2, \dots, \alpha(\bar{y})$; note that p^i may be in $g^{-1}(g(p^j))$ for $i \neq j$). There exists a disk $\{\bar{y}\} \times D^i$ such that $g(p^i) \in \{\bar{y}\} \times (\text{int}(D^i))$; $\{\bar{y}\} \times D^i \subset \{\bar{y}\} \times D$; and if J^i is the component of $g^{-1}(\{\bar{y}\} \times D^i)$ containing p^i , then $J^i \subset P^i$ [20, p. 131, (4.41)]. Since g is a pseudo-covering map, J^i is a topological 2-disk, and $g|J^i$ is topologically equivalent to the analytic map $\mu(z) = z^d$ ($d = 2, 3, \dots$).

Since $g(B_g)$ is compact and $\bigcup_w B(g^w) = B_g$, there exists a closed $(n-2)$ -cell $W^i \subset Y$, $\bar{y} \in \text{int}(W^i)$, such that $g(B(g^y))$ is disjoint from $\{y\} \times \text{bdy}(D^i)$ for all

$y \in W^i$. If S^i is the component of $g^{-1}(W^i \times D^i)$ containing p^i , we may suppose that W^i is chosen small enough that $S^i \cap H^w$ is connected for all $w \in W^i$ and that the S^i are mutually disjoint ($i = 1, 2, \dots, \alpha(y)$). If $W = \bigcap_i \text{int}(W^i)$, then $B_g \cap H^w \subset \bigcup_i S^i$ ($i = 1, 2, \dots, \alpha(\bar{y}); w \in W$).

Suppose that for some $w \in W^i$, $B(g^w) \cap S^i = \emptyset$; then since $g|S^i$ is a pseudo-covering map and $\{w\} \times D^i$ is simply connected, $g|(H^w \cap S^i)$, and thus $g|S^i$, has degree 1 (i.e., is a homeomorphism). Since $p^i \in B_g \cap S^i$, $B(g^w) \cap S^i \neq \emptyset$, for all $w \in W^i$. Because of the choice of \bar{y} and the fact that the S^i are mutually disjoint, each set $B(g^w) \cap S^i (= B_g \cap S^i \cap H^w)$ ($i = 1, 2, \dots, \alpha(\bar{y})$) is a single point.

Let $\rho^i: W^i \times D^i \rightarrow W^i$ be the projection map, and let $\beta^i = \rho^i|g(B_g \cap S^i)$. Then β^i is continuous, and one-to-one ($(\beta^i)^{-1}(w)$ is the single point of $g(B_g \cap S^i) \cap (\{w\} \times D^i)$). Since $g(B_g \cap S^i)$ is compact, β^i is a homeomorphism onto W^i . Let d^i be the distance from $g(B_g \cap S^i)$ to $W^i \times \text{bdy}(D^i)$, and let Δ^i be the closed disk of radius d^i and center 0 in C . Let $\sigma^i: W^i \times \Delta^i \rightarrow W^i \times D^i$ be the map defined by $\sigma^i(w, x) = (\beta^i)^{-1}(w) + (0, x)$, where $+$ is vector addition, 0 is the origin of E^{n-2} , and $x \in D^i$ (in $C = E^2$). Since σ^i is continuous and one-to-one, $W^i \times \Delta^i$ is compact, σ^i is a homeomorphism (into). Since $\sigma^i(W^i \times \{0\}) = g(B_g \cap S)$, it follows from the theorem on invariance of domain that $g(B_g \cap S^i)$ is a tamely embedded $(n - 2)$ -cell. By [5, p. 533, (4.1)] $g|S^i$ (i.e., $f|S^i$) is topologically equivalent to $F_{n,d}$, for some d ($d = 2, 3, \dots$).

Let $\Omega \cap \text{int}(T^{n-2})$ be the maximal open set (possibly empty) such that $g|g^{-1}(\Omega \times D)$ is locally, at each point, topologically equivalent to one of the maps $F_{n,d}$. To review, we have seen that, for every open set Y in $\text{int}(T^{n-2})$, there exists (of course) a point $\bar{y} \in Y$ such that $\alpha(\bar{y}) \geq \alpha(w)$ for all $w \in Y$; moreover, that there is an open neighborhood W of \bar{y} with $W \subset \Omega$. Thus $\Omega \cap Y \neq \emptyset$. Since Y is an arbitrary open set in $\text{int}(T^{n-2})$, Ω is a dense open set in $\text{int}(T^{n-2})$. Therefore [9, p. 44, Theorem IV 3] its complement F in $\text{int}(T^{n-2})$ has dimension at most $n - 3$.

Let E be the set of points of B_g in $g^{-1}(\{w\} \times \text{int}D)$ for $w \in F$, and let $\pi: \text{int}V \rightarrow \text{int}(T^{n-2})$ ($\text{int}V = g^{-1}(\text{int}(T^{n-2}) \times \text{int}D)$) be the projection map. Then $\pi(E) = F$, and, since $\dim(B(g^w)) = 0$, $\dim(\pi^{-1}(w)) = 0$ for all $w \in F$. Since $g|\text{int}V$ is a pseudo-covering map, π is a closed map and by [9, pp. 91-92] $\dim E \leq n - 3$. This completes the proof of the first conclusion.

For the second conclusion, we will suppose throughout that $B_f \neq \emptyset$. If E is somewhere dense in B_f , then there exists an open set Λ in M^n such that $\Lambda \cap B_f \neq \emptyset$ and $\Lambda \cap B_f \subset E$. By the preceding argument, E is nowhere dense in the set of branch points at which the Jacobian matrix has rank at least $n - 2$. Then $\Lambda \cap B_f \subset R_{n-3}$. Thus, if we still denote $f|\Lambda$ by f , it suffices to prove that $B_f \not\subset R_{n-3}$ (if $B_f \neq \emptyset$).

First suppose that $B_f \subset R_0$. Given $\bar{x} \in B_f$, let g be a pseudo-covering map given by [5, p. 530, (2.2) and p. 529, (1.4)] on a neighborhood V of \bar{x} , $V \subset E^n$, such that $g|g^{-1}(g(B_g))$ is a homeomorphism and $g(V) = E$. Then $g(B_g) \neq g(\bar{x})$ [5, p. 535, (5.6)]. It follows (see the first paragraph of the proof of (1.3)) from A. P. Morse's

theorem [10] that for every point u in $g(B_g) - g(\bar{x})$ (and therefore in $g(R_0)$), there exists a closed n -cube X such that: $q(\bar{x}) \in \text{int } X$, $u \notin X$, its faces are parallel to the coordinate $(n - 1)$ -planes, and those faces are disjoint from $g(R_0)$. The restriction of g to each component of $g^{-1}(\text{bdy } X)$ is a covering map onto $\text{bdy } X$, and therefore that map is a homeomorphism. Since $\bar{x} \in B_g$, the degree of g is at least two; since $V \subset E^n$, $g^{-1}(\text{bdy } X)$ separates V into at least three components, each of which maps onto one of the components of $E^n - \text{bdy } X (= g(V) - \text{bdy } X)$. This contradicts the fact that $g|_{g^{-1}(g(B_g))}$ is one-to-one, so that $B_f \not\subset R_0$.

Thus $B_f \not\subset R_{n-3}$, for $n = 3$. We continue by induction on n . If $n \geq 4$ and $B_f \subset R_{n-3}$, then there exists \bar{x} in B_f at which the Jacobian matrix has rank at least one. We may suppose, by restriction, that the rank is at least one everywhere, and that f is the g of (1.1) for $q = 1$. Let γ be the $(n - 1)$ -plane of (1.1) that contains \bar{x} . Let $h = f|_\gamma$, and let Q_{n-4} be the set of points in γ at which the Jacobian matrix of h has rank at most $n - 4$; then, by the second conclusion of (1.1), $\gamma \cap R_{n-3} = Q_{n-4}$. Since $B_h \subset \gamma \cap B_f$, $B_h \subset Q_{n-4}$, contradicting the inductive hypothesis. Thus $B_f \not\subset R_{n-3}$ for $n \geq 3$ (unless $B_f = \emptyset$), yielding the second conclusion.

The following extension of the inverse function theorem was proved in [4]. For $n = 2$ the analytic function $f(z) = z^2$ is an obvious counterexample.

2.2. COROLLARY. *Suppose that $f: E^n \rightarrow E^n$, $n \geq 3$, $f \in C^n$ and $\dim(R_{n-1}) = 0$ (R_{n-1} is the set of zeros of the Jacobian determinant). Then f is a local homeomorphism.*

2.3. COROLLARY. *If $f: E^n \rightarrow E^n$ is light and C^n , then $B_f = \emptyset$, $\dim(B_f) = n - 2$, or $\dim(B_f) = n - 1$; the last case occurs if and only if f is not open.*

Proof. Since $B_f \subset R_{n-1}$, $\dim(f(B_f)) \leq n - 1$ (by (1.3)); since f is light, $\dim(B_f) \leq n - 1$. If $\dim(f(B_f)) \leq n - 2$, then f is open [5, p. 531, (2.4)], so that either $B_f = \emptyset$ or $\dim(B_f) = n - 2$ (by (2.1)). Thus, $\dim(f(B_f)) = n - 1$ if and only if f is not open [5, p. 531, (2.3)]. If, in this case, $\dim(B_f) < n - 1$, then the Jacobian determinant of f would be either non-negative or nonpositive everywhere; thus (1.7) f would be open. As a result, $\dim(B_f) = n - 1$ if and only if f is not open.

2.4. COROLLARY. *There exists a light open map $f: E^5 \rightarrow E^5$ which is not topologically equivalent to any C^5 map.*

The map is that given by [6, p. 620, (4.3)], so that B_f is not a 3-manifold at any point. If f were equivalent to a C^5 map, then at a dense set of its points B_f would be locally a 3-manifold (2.1).

2.5. REMARKS. Given a C' map $f: E^n \rightarrow E^n$, its directional derivative at x in the direction of the nonzero vector (a_1, a_2, \dots, a_n) is the length of the vector whose j th component is $\sum_{i=1}^n a_i D_{if_j}(x)$. If f is a homeomorphism, it is called quasi-

conformal if (*) there exists $B > 0$ such that, for every point x in E^n and pair of vectors (directions) at x , the ratio of the directional derivatives is less than B . (This definition is equivalent to that given in [8].) A nonconstant complex analytic function f satisfies condition (*) (for $B = 1$) except on B_f , which consists of isolated points. Thus, it would be natural to call quasi-conformal (or quasi-analytic) light maps in E^n ($n > 2$) which satisfy condition (*), except at those points at which all directional derivatives are zero, i.e., R_0 . We now observe that the only such C^n maps are local homeomorphisms (for $n > 2$).

Suppose that f is C^n , light, and not a local homeomorphism. If $R_{n-1} \subset R_0$, then $\dim(R_{n-1}) \leq 0$ (by (1.3)). Thus f is a local homeomorphism (2.2), contradicting the supposition. If $R_{n-1} \not\subset R_0$, then there exists x at which the rank of Jacobian matrix is k , where $0 < k < n$. It follows from the definition of rank that there exist two vectors at x for which one directional derivative is positive, and the other is zero. Thus f does not satisfy condition (*).

We also remark that, except for local homeomorphisms, no C^n light open map is generic in the sense of Thom [14].

3. Some examples. Examples are given now to show that the exceptional set of dimension $n - 3$ in (2.1) and the compactness hypothesis in (1.8) are necessary.

3.1. LEMMA. *Given $\delta_q > 0$ ($q = 1, 2, \dots$), there exists a C^∞ map $\psi: E^1 \rightarrow E^1$ with the following properties:*

- (1) ψ is an even function,
- (2) $\psi(r) = 0$ if and only if $r = 0$,
- (3) $\psi'(r) > 0$ for $r > 0$, and
- (4) the i th derivative $\psi^{(i)}(r) \leq \delta_q$ ($0 \leq r \leq 1/q$; $i = 0, 1, \dots, q$; where $\psi^{(0)} = \psi$).

The proof is omitted.

3.2. LEMMA. *Given $\varepsilon_q > 0$ ($q = 1, 2, \dots$), there exists a C^∞ homeomorphism $h: E^n \rightarrow E^n$ such that on each set $S(0, 1/q)$ (where 0 is the origin) all h_i and all partial derivatives of order at most q are bounded by ε_q .*

Proof. Consider the class \mathfrak{H} of all functions $h: E^n \rightarrow E^n$ such that $h_i(x) = \psi(r) \cdot x_i$, where $r = x_1^2 + x_2^2 + \dots + x_n^2$ and $\psi: E^1 \rightarrow E^1$ is any C^∞ function. On each set $S(0, 1/q)$ there exists constants $\lambda_j > 0$ such that each h_i in \mathfrak{H} and all its partials of order at most q are bounded by $\sum_{j=0}^q \lambda_j |\psi^{(j)}(r(x))|$. Let $\delta_q < \varepsilon_q / \sum_{j=0}^q \lambda_j$ ($q = 1, 2, \dots$), and let ψ_0 be given by (3.1) for $\{\delta_q\}$. Let $h_i(x) = \psi_0(r) \cdot x_i$. That h is a homeomorphism follows from conclusions (2) and (3) of (3.1).

3.3. LEMMA. *Let U and L be, respectively, open and closed subsets of E^n . Let $f: U \rightarrow E^n$ be continuous, C^∞ on $U - L$, and constant on $U \cap L$; let V be a bounded open subset of U such that $\bar{V} \subset U$. Then there exists a homeomorphism $h: E^n \rightarrow E^n$ such that the restriction $hf|V \in C^\infty$.*

Proof. Throughout, symbols such as \bar{V} refer to closure in E^n . Suppose that $f(L) = 0$. Let $X_q = V \cap f^{-1}(S(0, 1/q))$, let $A_q = (X_q \cap V) - X_{q+1}$, and choose α_q ($0 < \alpha_q \leq 1$) less than the distance $d(\bar{A}_q, L)$ ($q = 1, 2, \dots$). We will define h so that the partial derivatives (of all orders) of $hf|V$, call it F , are zero on $L \cap V$; h will be given by (3.2), where we need now specify the ε_q ($q = 1, 2, \dots$).

The component functions (e.g., F_i) will be considered partials of order zero. Suppose h is given by (3.2) for $\varepsilon_q = \varepsilon_{q,0} = 1/q$. For $x \in A_q$, $f(x) \in S(0, 1/q)$, so that $|F_i(x)| < 1/q$ ($i = 1, 2, \dots, n$).

Now suppose that numbers $\varepsilon_{q,m} > 0$ ($q = 1, 2, \dots$; $m = 0, 1, \dots, k$; k fixed) have been defined so that

(1) any homeomorphism given by (3.2) for $\{\varepsilon_{q,k}\}$ will satisfy (a) $|P(x)| < 1/q$, for all $x \in A_q$ and all partials P of F with order m at most the minimum of k and q , and (b) $P(x^0) = 0$, for $x^0 \in L$;

(2) $\varepsilon_{q,m'} < \varepsilon_{q,m}$, whenever $m < m'$; and

(3) $\varepsilon_{q,m} = \varepsilon_{q,q}$, whenever $m > q$.

Let this property of the sequence $\{\varepsilon_{q,m}\}$ ($m = 0, 1, \dots, k$) be called \mathfrak{P}_k ($k = 0, 1, \dots$); we have seen that there exists $\{\varepsilon_{q,0}\}$ satisfying \mathfrak{P}_0 . We proceed by induction. Assuming a sequence $\{\varepsilon_{q,m}\}$ ($m = 0, 1, \dots, k$) satisfying \mathfrak{P}_k , we will find numbers $\varepsilon_{q,k+1}$ ($q = 1, 2, \dots$) such that $\{\varepsilon_{q,m}\}$ ($m = 1, 2, \dots, k + 1$) satisfies \mathfrak{P}_{k+1} .

Given any partial P of F with order k , a natural number j ($j = 1, 2, \dots, n$), and $x^0 \in L$,

$$D_j P(x^0) = \lim_{x_j \rightarrow x_j^0} \frac{P(x) - 0}{x_j - x_j^0}$$

(since $P(x^0) = 0$, by $\mathfrak{P}_k(1)$). Given x in A_q ,

$$\frac{|P(x)|}{|x_j - x_j^0|} \leq \frac{|P(x)|}{\alpha_q}.$$

Now each such P is on $\bar{V} - L$ a sum of products of partials of f and of h , all of orders at most k , each term having at least one partial of h as a factor. Since \bar{A} is compact and $\bar{A}_q \cap L = \emptyset$, there is a uniform bound on the partials of f of order at most k . For $q \leq k$, let $\varepsilon_{q,k+1} = \varepsilon_{q,q}$; for $q > k$, let $\varepsilon_{q,k+1}$ be chosen small enough so that $\varepsilon_{q,k+1} \leq \varepsilon_{q,k}$ and, for any h given by (3.2) for $\varepsilon_{q,k+1}$, $|P(x)|/\alpha_q < 1/q$ (for all $x \in A_q$ and for all partials P of order at most k , a finite number of choices required for each q). Since $L \cup \bigcup_{q=1}^\infty A_q$ is a neighborhood of L , and since $P|L = 0$, all the partials of F of order at most $k + 1$ are 0 on L , for h given by $\{\varepsilon_{q,k+1}\}$. It follows that $\{\varepsilon_{q,m}\}$ ($m = 1, 2, \dots, k + 1$) satisfies \mathfrak{P}_{k+1} .

Positive numbers $\varepsilon_{q,m}$ ($q, m = 1, 2, \dots$) are defined, and the desired h is the one given by (3.2) for $\varepsilon_q = \varepsilon_{q,q}$; all its partials are zero on L .

To prove that $F \in C^\infty$ it is sufficient to prove that each partial P is continuous

on L ; let k be the order of P . By $\mathfrak{F}_k(1)$, $|P(x)| < 1/q$ ($x \in A_q$; $q = k, k + 1, \dots$), so that $P(x) \rightarrow 0$ as $x \rightarrow x^0$, $x^0 \in L$.

3.4. COROLLARY. *There exists $f: E^3 \rightarrow E^3$, $f \in C^\infty$, light and open, such that B_f has a point component.*

The map given in [6, p. 614, (3.3)] is topologically equivalent to a map simplicial except at the origin 0, and thus it is equivalent to a map C^∞ except at 0. From (3.3) we have the desired result.

Although B_f need not be locally connected, it follows from (2.1) that for $f: E^3 \rightarrow E^3$, $f \in C^3$, light and open, each component K of B_f is locally connected. (Suppose that K is not locally connected; then it contains [20, p. 19, (12.3)] a subcontinuum H such that K is not locally connected at any point of H . At each point x of $H - E$, where E is the exceptional set of (2.1), there exists a neighborhood U such that the restriction $f|U$ is a canonical map $F_{n,d}$. Since $H \cap U$ is a tame arc, we have a contradiction.)

The example whose branch set has a Cantor set of point components [6, p. 614] is also equivalent to a C^∞ map. (Appropriate modifications of (3.2) and (3.3) are required.)

For another example of a C^∞ (3-to-1) open map, let z be a complex variable, t real, and let $f: E^3 \rightarrow E^3$ be defined by

$$f(z, t) = (z^3 - 3ze^{-2t} \sin^2 t^{-1}, t).$$

Then (with $z = x + iy$) B_f is the union of the curves $x = \pm e^{-t^{-2}} \sin t^{-1}$ in the $(x - t)$ -plane. Still another example is given in (2.4).

The following remark answers in the negative question II of [13, p. 266].

3.5. REMARK. *There exists a C^∞ 3-to-1 open map $f: E^3 \rightarrow E^3$ which is not topologically equivalent to any real analytic map.*

We use the map above for which B_f has a Cantor set X of point components, or one with a sequence of point components converging to a point. It follows from (1.1) (see the proof of (2.1)) that $X \subset R_0$. Suppose that f is real analytic. Then R_0 is an analytic set (the zeros of $\sum_{i,j} (D_{ij}f)^2$), and thus [3, p. 141] is locally connected. Since $\dim(R_0) = 0$ (1.3), we have a contradiction.

3.6. THEOREM. *There exists a C^∞ open map $f: E^2 \rightarrow E^2$ which is not light.*

Proof. The domain of f will actually be the square S given by $|x| < 1$ and $|y| < 1$; let L be the intersection of the y -axis with S . Let r^j ($j = 1, 2, \dots$) be any countable dense subset of $L - \{0\}$, and let $h: S \rightarrow S$ be given by $h(x, y) = (x, xy)$.

If $X_{j,k}$ ($j, k = 1, 2, \dots$) are the subsets of $h(S)$ defined by

$$2^{-2^{j-1}(2k-1)} \leq x \leq \left(\frac{3}{2}\right) \cdot 2^{-2^{j-1}(2k-1)},$$

then their closures are mutually disjoint and each

$$X_{j,k} \subset S(0, 3 \cdot 2^{-2^{j-1}(2k-1)}).$$

Let $g: h(S) \rightarrow E^2$ be a map such that

- (1) $g|(h(S) - \{0\})$ is a C^∞ local homeomorphism,
- (2) $g|(h(S) - \bigcup_{j,k} X_{j,k})$ is the identity map,
- (3) $g(X_{j,k}) \subset S(0, 3 \cdot 2^{-2^{j-1}(2k-1)})$, and
- (4) there exists a point $p^{j,k}$ common to $h^{-1}(X_{j,k})$ and the line $y = r^j$ such that $g(h(p^{j,k})) = 0$ ($j, k = 1, 2, \dots$).

(By (1) and (3) g is continuous at 0.)

Given any neighborhood U of r^j , there exists $p^{j,k} \in U$; since $g(h(r^j)) = 0$, it follows from conditions (1) and (4) that $gh(r^j) \in \text{int}(gh(U))$. Since the r^j are dense in L , and since $gh|(S - L)$ is a (C^∞) local homeomorphism, gh is open.

The result follows from (3.3).

REMARK. Given any three natural numbers j, k , and n such that (1) $0 \leq j \leq \min(k - 1, n - 2)$, (2) $1 \leq k \leq n - 1$, and (3) $n \geq 2$, modifications of the above argument yield a nonlight C^∞ open map $f: E^n \rightarrow E^n$ for which $\dim(f(B_r)) = j$ and $\dim(B_r) = k$.

3.7. REMARK. If $f: E^n \rightarrow E^n$ is C^n open, but not light, then for every k ($k = 1, 2, \dots$) there exists x such that $f^{-1}(x)$ consists of isolated points, at least k in number. The proof is similar to that of (1.8).

4. **A counterexample to a statement of Stoilow.** In [13] S. Stoilow states that, if $f: E^3 \rightarrow E^3$ is light open, then $\dim(B_r) \leq 1$. His proof employs the following lemma [13, pp. 263-264]: Let $x \in E^3$, and let B_ρ be the geometric ball of radius ρ and center $f(x)$. Then there exists $r > 0$ and a compact neighborhood D of x such that $f|D$ is open and $f(D) = B_r$. There exist positive numbers ε_1 and ε_2 such that the number of components of $f^{-1}(B_\rho)$ is the same for all ρ with $0 < \varepsilon_1 < \rho < \varepsilon_2$. Moreover, for any such set of numbers $\varepsilon_1, \varepsilon_2$, and ρ , each component of $f^{-1}(\text{bdy}(B_\rho))$ is a 2-manifold. The last statement is false in general.

It appears that a modification of the proof of (2.1) using this statement would yield (2.1) for $n = 3$ and f light open but not necessarily differentiable. For this reason it seems worthwhile to give a counterexample here.

We write E^3 as $E^1 \times C$, where C is the complex plane, and let X_m be the set of (t, z) such that either $|t| \leq 2^{-m}$ and $|z| \leq 2^{-m}$, or $2^{-m} \leq t \leq 3 \cdot 2^{-m-1}$ and $|z - 2^{-m-1}| \leq 2^{-m-1}$ ($m = 1, 2, \dots$). Then $X_{m+1} \subset \text{int}(X_m)$, and there exists a homeomorphism $h: E^3 \rightarrow E^3$ such that $K_m = h(\text{bdy}(X_m))$ is a geometric 2-sphere about the origin 0. The map $hF_{3,2}$ is the desired counterexample f , since $f^{-1}(K_m) (= F_{3,2}^{-1}(\text{bdy}(X_m)))$ is not a 2-manifold while $f^{-1}(h(X_m))$ is connected ($m = 1, 2, \dots$).

Stoilow uses a characterization of compact 2-manifolds in E^3 due to Wilder [18, Theorem 21], and the sets $f^{-1}(K_m)$ fail to satisfy the first conclusion of that theorem. With a suitable modification of the sets X_m , the sets $f^{-1}(K_m)$ also fail to satisfy the second conclusion.

Added in Proof. J. Väisälä has kindly pointed out to the author the following simple example of a C^∞ map $f: E^2 \rightarrow E^2$ which is open but not light (cf. 3.6). For $z = x + iy$ and $x \neq 0$, $f(z) = \exp(-z/x^3)$; $f(iy) = 0$. Except on the imaginary axis f is a local homeomorphism.

In *Images of critical sets*, Ann. of Math. (2) **68** (1958), 247–259, Arthur Sard considers maps f of $U \cdots E^n$ into E^p . He proves under very general differentiability hypotheses that if R_k is the countable union of sets of finite Hausdorff $(k+1)$ -measure, then the $(k+1)$ -measure of $f(R_k)$ is 0. It follows [9, p. 104] that $\dim(f(R_k)) \leq k$. Thus, in this case (1.3) is the consequence of a more general result. In general, however, R_k need not be the countable union of sets of finite $(k+1)$ -measure.

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