ASYMPTOTIC BEHAVIOR OF THE EIGENVALUES OF CERTAIN INTEGRAL EQUATIONS(1)

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1. We shall be concerned with the behavior of the eigenvalues of the integral equation

$$\int V(\mathbf{x})^{1/2}k(\mathbf{x}-\mathbf{y})\,V(\mathbf{y})^{1/2}f(\mathbf{y})d\mathbf{y} = \lambda f(\mathbf{x}), \tag{1}$$

where k is a function integrable over E_d (Euclidean space of dimension d) having ultimately positive Fourier transform, and where V is a bounded non-negative function with bounded support. Roughly, the main result is as follows. Let

$$K(\xi) = \int k(\mathbf{x})e^{-i\xi\cdot\mathbf{x}}\,d\mathbf{x}.$$

Then the number of eigenvalues of (1) which exceed ε is asymptotic, as $\varepsilon \to 0$, to $(2\pi)^{-d}$ times the measure in $E_d \times E_d$ of

$$\{(\mathbf{x},\xi) : V(\mathbf{x}) \cdot K(\xi) > \varepsilon\}.$$

Actually we consider an equation more general than (1) which we shall now describe. Let $K(\xi)$ be a bounded non-negative function tending to zero as $|\xi| \to \infty$. The operator T_0 on $L_2(E_d)$ we define by

$$(T_0 f)\hat{}\,(\xi) = K(\xi)\hat{f}(\xi),$$

where the circumflex denotes Fourier transformation. If

$$K(\xi) = \int k(\mathbf{x})e^{-i\xi\cdot\mathbf{x}}\,d\mathbf{x} \qquad k \in L_1(E_d)$$

then T_0 is just convolution by k. However we shall not insist that this be the case.

Let $V(\mathbf{x})$ be a bounded non-negative function with bounded support and denote by $M_V^{1/2}$ the operator on $L_2(E_d)$ which is multiplication by $V(\mathbf{x})^{1/2}$. We shall denote by $\lambda_1 \geqq \lambda_2 \geqq \cdots$ the positive eigenvalues of the positive semi-definite operator $M_V^{1/2}T_0M_V^{1/2}$. In case K is the Fourier transform of an L_1 function k then this is just the integral operator in equation (1). The result (Theorem II) is as

Received by the editors November 19, 1962.
(1) Supported in part by NSF grant 18837.
(2) Sloan Foundation fellow.

follows. Let $\phi_0(\alpha)$ $(0 < \alpha < \infty)$ be a nonincreasing function equimeasurable with the function $V(\mathbf{x}) \cdot K(\xi)$ on $E_d \times E_d$. Then with further conditions imposed (growth and regularity conditions imposed on K and ϕ_0, V assumed properly Riemann integrable) we have

$$\lambda_n \sim \phi_0((2\pi)^d n) \qquad n \to \infty.$$

The theorem as stated is not too useful. In the case of equation (1) it requires that k be a positive definite function (or at least, if Ω is the support of V, that k be the restriction to $\Omega - \Omega$ of a positive definite function on E_d). However there is a somewhat stronger result (Theorem II′) for which this is not required. With K, V, ϕ_0 as before, let K' be any bounded function asymptotic to K as $|\xi| \to \infty$ and let $\lambda_1' \geqq \lambda_2' \geqq \cdots$ be the positive eigenvalues of $M_V^{1/2} T_0' M_V^{1/2}$, where T_0' is the operator arising from K' just as T_0 arose from K. Then we still have

$$\lambda_n' \sim \phi_0((2\pi)^d n) \qquad n \to \infty.$$

The special case of greatest interest (and, as far as we know, essentially the only case which has been considered before) is equation (1) with $k(\mathbf{x}) = |\mathbf{x}|^\alpha$ $(\alpha > -d, \alpha \neq 0, 2, 4, \cdots)$. If Ω is the support of V then k is equal on $\Omega - \Omega$ to a function whose Fourier transform is asymptotically

$$2^{d+\alpha}\pi^{d/2} \frac{\Gamma\left(\frac{d+\alpha}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)} |\xi|^{-d-\alpha}$$

It follows that the eigenvalues of (1) in this case satisfy

$$\text{(2)} \qquad \lambda_n \sim \pi^{-\alpha/2} \left(\frac{2}{d}\right)^{(d+\alpha)/d} \frac{\Gamma\left(\frac{d+\alpha}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right) \Gamma\left(\frac{d}{2}\right)^{(d+\alpha)/d}} \left\{ \int V(x)^{d/(d+\alpha)} dx \right\}^{(d+\alpha)/d} n^{-(d+\alpha)/d}.$$

If $\alpha = 2 - d$ the kernel k is that which is associated with Laplace's equation. Thus (2) in this case, with V a characteristic function, is essentially due to Weyl [4]. In case V is the characteristic function of an interval in one dimension and $\alpha > -\frac{1}{2}$, (2) was obtained by Rosenblatt [3]. In case $-1 < \alpha < 0$ and $d = 1$, (2) was obtained by Kac [1] using probabilistic methods which extend to higher dimensions.

2. We shall concern ourselves first with the periodic analogue of the situation described in §1. Let $c_{\mathbf{n}}$ $(\mathbf{n} = (n_1, \cdots, n_d)$ with n_i integral) be real and tend to zero as $|\mathbf{n}| \to \infty$. If I_d denotes the cube

$$\{\mathbf{x}: |x_i| \leqq \pi, \quad i = 1, \cdots, d\}$$

we associate with $\{c_{\mathbf{n}}\}$ the operator T on $L_2(I_d)$ given by

$$Tf(\mathbf{x}) = \sum c_{\mathbf{n}} f_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{x}},$$

where

$$f(\mathbf{x}) = \sum f_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{x}}.$$

We shall be interested in the eigenvalues of the operator $M_{V^{1/2}} T M_{V^{1/2}}$ on $L_2(I_d)$, where V is a Riemann integrable function defined on I_d.

There is one trivial case. If $V(\mathbf{x}) \equiv 1$ then $M_{V^{1/2}} T M_{V^{1/2}} = T$ has eigenvalues $c_{\mathbf{n}}$. What follows is the derivation of the behavior of the eigenvalues for general V from this special case.

We make the following assumptions concerning $\{c_{\mathbf{n}}\}$:

(i) $c_{\mathbf{n}} \geqq 0$;

(ii) *with all n_i fixed but n_{i_0}, $c_{\mathbf{n}}$, as a function of n_{i_0}, is nondecreasing between $-\infty$ and some $\bar{n} = \bar{n}(i_0)$ and nonincreasing between $\bar{n}$ and ∞*;

(iii) *if $|\mathbf{n}|, |\mathbf{m}| \to \infty$ and $|\mathbf{n}| = O(|\mathbf{m}|)$ then $c_{\mathbf{m}} = O(c_{\mathbf{n}})$*;

(iv) *if $|\mathbf{n}|, |\mathbf{m}| \to \infty$ and $|\mathbf{n}| = o(|\mathbf{m}|)$ then $c_{\mathbf{m}} = o(c_{\mathbf{n}})$.*

The crucial lemma, which will allow us to pass from the case $V(x) \equiv 1$ to the general case, is the following.

MAIN LEMMA. *Let Ω_1 and Ω_2 be nonoverlapping intervals (rectangular parallelepipeds with edges parallel to the coordinate axes) contained in I_d. Denote by P_i $(i = 1,2)$ multiplication by the characteristic function of Ω_i. Let $N^+(\varepsilon)$ and $N^-(\varepsilon)$ be the number of eigenvalues of $P_1TP_2 + P_2TP_1$ which are respectively $> \varepsilon$ and $< -\varepsilon$. Denote by $\Psi(\varepsilon)$ the number of lattice points $\mathbf{n}$ for which $c_{\mathbf{n}} > \varepsilon$. Then if $\{c_n\}$ satisfies* (i)–(iv) *we have*

$$N^{\pm}(\varepsilon) = o(\Psi(\varepsilon)) \qquad \varepsilon \to 0.$$

The proof of this will be preceded by five subsidiary lemmas. The first is an easy estimate of an integral involving an exponential sum.

LEMMA 1. *Assume $\{c_{\mathbf{n}}\}$ satisfies* (i) *and* (ii) *and that the lattice points $\mathbf{n}$ for which $c_{\mathbf{n}} \neq 0$ are contained in a sphere of radius r $(r > 2)$. Then*

$$\int_{I_d} |\mathbf{x}| \quad \left| \sum c_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{x}} \right|^2 d\mathbf{x} \leqq A_1 r^{d-1} \log r \max c_{\mathbf{n}}^2,$$

where A_1 is a constant depending only on d.

Proof. Consider first the case $d = 1$. It is no loss of generality to assume that c_n *is* nondecreasing from $-\infty$ to 0 and nonincreasing from 0 to ∞. Thus $\max c_n = c_0$. Let

$$S_n(x) = \sum_0^n e^{ikx} = e^{inx/2}\frac{\sin\frac{1}{2}(n+1)x}{\sin\frac{1}{2}x}.$$

We have on the one hand (since there are at most $2r+1$ nonvanishing c_n)

$$\left|\sum_0^\infty c_n e^{inx}\right| \leqq (2r+1)c_0$$

and on the other

$$\left|\sum_0^\infty c_n e^{inx}\right| = \left|\sum_0^\infty \Delta c_n S_{n+1}(x) - c_0\right|$$

$$\leqq \csc\frac{1}{2}x \sum_0^\infty \Delta c_n + c_0 = \left(\csc\frac{1}{2}x + 1\right)c_0.$$

Therefore with appropriate constants A, A' (recall $r > 2$)

$$\int_{|x|\leqq r^{-1}} |x| \left|\sum_0^\infty c_n e^{inx}\right|^2 dx \leqq Ac_0^2$$

and

$$\int_{r^{-1}\leqq |x| \leqq \pi} |x| \left|\sum_0^\infty c_n e^{inx}\right|^2 dx \leqq A' \log r c_0^2.$$

It follows that

$$\int_{-\pi}^{\pi} |x| \left|\sum_0^\infty c_n e^{inx}\right|^2 dx \leqq A'' \log r\, c_0^2,$$

and a similar inequality holds for the analogous integral involving $\sum_{-\infty}^{-1}$.

Next consider an arbitrary d. Since

$$|\mathbf{x}| \leqq d^{1/2}(|x_1| + \cdots + |x_d|)$$

it suffices to prove the inequality with $\mathbf{x}$ replaced by x_1. Now

$$\int_{I_d} |x_1| \left|\sum c_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{x}}\right|^2 d\mathbf{x}$$

$$= (2\pi)^{d-1} \sum_{n_2,\dots,n_d} \int_{-\pi}^{\pi} |x_1| \left|\sum_{n_1} c_{n_1,\dots,n_d} e^{in_1x_1}\right|^2 dx_1.$$

The integral on the right vanishes except for at most $A''' r^{d-1}$ choices of $n_2,\cdots,n_d$ and (by the inequality for $d=1$) the integral is in any case at most $A^{iv}\log r \max c_{\mathbf{n}}^2$. The desired inequality follows.

In the next three lemmas we deduce from (i)–(iv) certain properties of the function Ψ. We shall assume in these lemmas that (i)–(iv) hold.

LEMMA 2. *There is a constant A_2 such that the lattice points* $\mathbf{n}$ *for which* $c_{\mathbf{n}} > \varepsilon$ *are contained in a sphere of volume* $A_2\Psi(\varepsilon/2)$.

Proof. It follows from (iii) that there is a constant $a>0$ such that $|c_{\mathbf{m}}-c_{\mathbf{n}}|\leqq c_{\mathbf{m}}$ whenever $|\mathbf{m}-\mathbf{n}|\leqq a|\mathbf{n}|$ and $|\mathbf{n}|$ is sufficiently large. Thus if $c_{\mathbf{n}}>\varepsilon$ for a certain $\mathbf{n}$, we have $c_{\mathbf{m}}>\varepsilon/2$ for all $\mathbf{m}$ in a sphere of radius $a|\mathbf{n}|$, i.e., for at least $a'|\mathbf{n}|^d$ lattice points $\mathbf{m}$. Thus $\Psi(\varepsilon/2)\geqq a'|\mathbf{n}|^d$, which implies that $\mathbf{n}$ lies in a sphere of volume $A_2\Psi(\varepsilon/2)$ about the origin. Although this is proved for ε sufficiently small, the constant A_2 can be adjusted so that the statement holds for all ε.

LEMMA 3. *If $\delta\to 0$ and $\varepsilon=o(\delta)$ then $\Psi(\delta)=o(\Psi(\varepsilon))$.*

Proof. We first note the following elementary fact. Given a set S of s distinct lattice points, there are at least $\frac{3}{2}s$ lattice points $\mathbf{m}$ whose distance from S is at most $\frac{1}{2}(|\mathbf{m}|+1)$. This is easy if $d=1$ and may be proved by induction for $d>1$ by taking sections of S. We omit the details. Now let

$$S=\{\mathbf{n}:c_{\mathbf{n}}>\delta\}.$$

There are at least $\frac{3}{2}\Psi(\delta)$ lattice points $\mathbf{m}$, each within $\frac{1}{2}(|\mathbf{m}|+1)$ of a point $\mathbf{n}_{\mathbf{m}}$ of S. If $|\mathbf{m}|\geqq 2$ this implies

$$\frac{1}{4}\leqq\frac{|\mathbf{n}_{\mathbf{m}}|}{|\mathbf{m}|}\leqq\frac{7}{4}$$

from which we deduce, by (iii), that $c_{\mathbf{m}}\geqq c_{\mathbf{n}_{\mathbf{m}}}/A>\delta/A$ for some constant A. Thus if M is the number of lattice points satisfying $|\mathbf{m}|<2$ we have

$$\Psi(\delta/A)\geqq\frac{3}{2}\Psi(\delta)-M\geqq\frac{4}{3}\Psi(\delta)$$

for sufficiently small δ. For any positive power p therefore,

$$\Psi(\delta)\leqq\left(\frac{3}{4}\right)^p\Psi(\delta/A^p).$$

If $\varepsilon=o(\delta)$ we have $\Psi(\delta/A^p)\leqq\Psi(\varepsilon)$ for sufficiently small δ and so

$$\Psi(\delta)\leqq\left(\frac{3}{4}\right)^p\Psi(\varepsilon).$$

Since p can be made arbitrarily large we have the result.

LEMMA 4. *For any constant Δ we have $\Psi(\varepsilon)=O(\Psi(\Delta\varepsilon))$ as $\varepsilon\to 0$.*

Proof. Suppose the contrary. Then for some sequence of ε's approaching 0 we have $\Psi(\Delta\varepsilon)=o(\Psi(\varepsilon))$. Since there are $\Psi(\varepsilon)$ lattice points $\mathbf{m}$ satisfying $c_{\mathbf{m}}>\varepsilon$, one of these, call it $\mathbf{m}_\varepsilon$, must satisfy $|\mathbf{m}_\varepsilon|\geqq a\Psi(\varepsilon)^{1/d}$. (Here a is a positive constant.) Since the lattice points $\mathbf{n}'$ satisfying $c_{\mathbf{n}'}>2\Delta\varepsilon$ lie inside a sphere of volume

$A_2\Psi(\Delta\varepsilon)$ there is some point $\mathbf{n}_\varepsilon$ such that $c_{\mathbf{n}_\varepsilon} \leqq 2\Delta\varepsilon$ and $|\mathbf{n}_\varepsilon| \leqq A\Psi(\Delta\varepsilon)^{1/d}$. Our assumption $\Psi(\Delta\varepsilon) = o(\Psi(\varepsilon))$ gives $|\mathbf{n}_\varepsilon| = o(|\mathbf{m}_\varepsilon|)$. But $c_{\mathbf{n}_\varepsilon} < 2\Delta c_{\mathbf{m}_\varepsilon}$ so $c_{\mathbf{m}_\varepsilon} \neq o(c_{\mathbf{n}_\varepsilon})$ and this contradicts (iv).

The final subsidiary lemma is well known. Given a self-adjoint completely continuous operator A we denote by $N^{\pm}(\varepsilon, A)$ the number of eigenvalues of A which are respectively $> \varepsilon$ and $< -\varepsilon$.

LEMMA 5. *Let A_i $(i = 1, \cdots, i_0)$ be self-adjoint and completely continuous. Then if $\varepsilon = \sum \varepsilon_i$ we have*

$$N^{\pm}(\varepsilon, \textstyle\sum A_i) \leqq \sum N^{\pm}(\varepsilon_i, A_i).$$

Proof. If $\lambda^+_{n,i}$ denotes the nth largest positive eigenvalue of A_i and λ^+_n that of A, we have for any $n_1, \cdots, n_{i_0}$

$$\lambda^+_{\sum n_i - i_0 + 1} \leqq \sum \lambda^+_{n_i, i}.$$

This follows from the minimax characterization of the eigenvalues. See for example [2, §95]. If we set $n_i = N^+(\varepsilon_i, A_i)$ we obtain

$$N^+(\varepsilon, \textstyle\sum A_i) \leqq \sum N^+(\varepsilon_i, A_i)$$

and the other inequality is obtained similarly.

3. We can now prove the main lemma. Let $0 < \varepsilon < \delta$. (Later δ will be chosen as a specific function of ε.) We write $c_{\mathbf{n}} = c_{\mathbf{n},1} + c_{\mathbf{n},2} + c_{\mathbf{n},3}$ as follows:

$$c_{\mathbf{n},1} = \begin{cases} c_{\mathbf{n}} & \text{if } c_{\mathbf{n}} \leqq \varepsilon, \\ \varepsilon & \text{otherwise,} \end{cases}$$

$$c_{\mathbf{n},2} = \begin{cases} c_{\mathbf{n}} - \delta & \text{if } c_{\mathbf{n}} > \delta, \\ 0 & \text{otherwise,} \end{cases}$$

$$c_{\mathbf{n},3} = \begin{cases} c_{\mathbf{n}} - \varepsilon & \text{if } \varepsilon < c_{\mathbf{n}} \leqq \delta, \\ \delta - \varepsilon & \text{if } c_{\mathbf{n}} > \delta, \\ 0 & \text{if } c_{\mathbf{n}} \leqq \varepsilon. \end{cases}$$

Each of the sequences $\{c_{\mathbf{n},i}\}$ $(i = 1, 2, 3)$ gives rise to an operator T_i just as $\{c_{\mathbf{n}}\}$ gave rise to T. Denote by $N_i^+(\varepsilon)$ the number of eigenvalues of $P_1T_iP_2 + P_2T_iP_1$ greater than ε. It follows from Lemma 5 that

$$N^+(3\varepsilon) \leqq N_1^+(2\varepsilon) + N_2^+(0) + N_3^+(\varepsilon).$$

Since T_1 is an operator of norm at most ε, $P_1T_1P_2 + P_2T_1P_1$ has norm at most 2ε. Thus $N_1^+(2\varepsilon) = 0$.

Since T_2 is an operator of rank $\Psi(\delta)$, $P_1T_2P_2 + P_2T_2P_1$ has rank at most $2\Psi(\delta)$. Thus $N_2^+(0) \leqq 2\Psi(\delta)$.

It remains to estimate $N_3^+(\varepsilon)$. The square of the Hilbert-Schmidt norm of

$P_1T_3P_2 + P_2T_3P_1$ is at least $\varepsilon^2 N_3^+(\varepsilon)$. Thus, since T_3 is convolution by $(2\pi)^{-d}\sum c_{\mathbf{n},3}e^{i\mathbf{n}\cdot\mathbf{x}}$,

$$\begin{aligned}\varepsilon^2 N_3^+(\varepsilon) &\leqq 2(2\pi)^{-2d}\int_{\Omega_1}\int_{\Omega_2}\left|\sum c_{\mathbf{n},3}e^{i\mathbf{n}\cdot(\mathbf{x}-\mathbf{y})}\right|^2 d\mathbf{x}d\mathbf{y}\\ &= 2(2\pi)^{-2d}\int_{\Omega_1}d\mathbf{y}\int_{\Omega_2-\mathbf{y}}\left|\sum c_{\mathbf{n},3}e^{i\mathbf{n}\cdot\mathbf{x}}\right|^2 d\mathbf{x}\\ &= 2(2\pi)^{-2d}\int_{\Omega_2-\Omega_1}\left|\Omega_1\cap(\Omega_2-\mathbf{x})\right|\,\left|\sum c_{\mathbf{m},3}e^{i\mathbf{n}\cdot\mathbf{x}}\right|^2 d\mathbf{x},\end{aligned}$$

where $|\cdots|$ denotes measure. If we recall that Ω_1 and Ω_2 are nonoverlapping intervals contained in I_d we see that

$$\left|\Omega_1\cap(\Omega_2-\mathbf{x})\right| \leqq (2\pi)^{d-1}\begin{cases}\min_i\,(2\pi - x_i)\\ \min_i\,(2\pi + x_i)\\ \sum |x_i|.\end{cases}$$

Therefore for some constant A

$$\left|\Omega_1\cap(\Omega_2-\mathbf{x})\right| \leqq A\sum\left|\sin\frac{1}{2}x_i\right| \qquad x\in I_d - I_d$$

and so

$$\begin{aligned}\varepsilon^2 N_3^+(\varepsilon) &\leqq A'\int_{I_d-I_d}\sum\left|\sin\frac{1}{2}x_i\right|\,\left|\sum c_{\mathbf{n},3}e^{i\mathbf{n}\cdot\mathbf{x}}\right|^2 d\mathbf{x}\\ &= 2^d A'\int_{I_d}\sum\left|\sin\frac{1}{2}x_i\right|\,\left|\sum c_{\mathbf{n},3}e^{i\mathbf{n}\cdot\mathbf{x}}\right|^2 d\mathbf{x}.\end{aligned}$$

since the integrand is of period 2π in each x_i. Consequently

$$\varepsilon^2 N_3^+(\varepsilon) \leqq A''\int_{I_d}|\mathbf{x}|\,\left|\sum c_{\mathbf{n},3}e^{i\mathbf{n}\cdot\mathbf{x}}\right|^2 d\mathbf{x}.$$

It follows from Lemmas 1 and 2 that we have, for sufficiently small ε,

$$\varepsilon^2 N_3^+(\varepsilon) \leqq A'''\Psi(\varepsilon/2)^{(d-1)/d}\log\Psi(\varepsilon/2)\delta^2.$$

Combining the three estimates obtained and replacing ε by $\varepsilon/3$,

$$\begin{aligned}N^+(\varepsilon) &\leqq 2\Psi(\delta) + 9A'''\Psi(\varepsilon/6)^{(d-1)/d}\log\Psi(\varepsilon/6)\,\delta^2/\varepsilon^2\\ &\leqq 2\Psi(\delta) + \Psi(\varepsilon/6)^{1-1/2d}\,\delta^2/\varepsilon^2\end{aligned}$$

if ε is sufficiently small and $\delta > \varepsilon/3$. From Lemma 4 it follows that

$$\Psi(\varepsilon/6)^{1-1/2d} = o(\Psi(\varepsilon)).$$

Now we set

$$\delta = \varepsilon\left(\frac{\Psi(\varepsilon)}{\Psi(\varepsilon/6)^{1-1/2d}}\right)^{1/4}.$$

Then

$$\Psi(\varepsilon/6)^{1-1/2d}\delta^2/\varepsilon^2 = (\Psi(\varepsilon)\Psi(\varepsilon/6)^{1-1/2d})^{1/2} = o(\Psi(\varepsilon)).$$

Also $\lim \delta/\varepsilon = \infty$ so by Lemma 3 we have $\Psi(\delta) = o(\Psi(\varepsilon))$. Thus $N^+(\varepsilon) = o(\Psi(\varepsilon))$, and a similar argument gives $N^-(\varepsilon) = o(\Psi(\varepsilon))$.

4. Now that the main lemma is proved we can proceed in a straightforward way to the asymptotic distribution of the eigenvalues. For a subset Ω of I_d, P_Ω will denote the projection operator on $L_2(I_d)$ which is multiplication by the characteristic function of Ω. $\{c_{\mathbf{n}}\}$ and T are as above and we assume (i)–(iv) hold.

LEMMA 6. *Let Ω be an interval. Then for each δ in* $0 < \delta < 1$,

$$\limsup_{\varepsilon\to 0} \frac{N^+(\varepsilon, P_\Omega T P_\Omega)}{\Psi((1-\delta)\varepsilon)} \leqq (2\pi)^{-d}\,|\Omega|,$$

$$\liminf_{\varepsilon\to 0} \frac{N^+(\varepsilon, P_\Omega T P_\Omega)}{\Psi((1+\delta)\varepsilon)} \geqq (2\pi)^{-d}\,|\Omega|.$$

Proof. Let $2\pi r_i$ $(i = 1, \cdots, d)$ denote the lengths of the edges of Ω. We assume first that $1/r_i = q_i$ are integers. Then we can find $\Omega_1, \cdots, \Omega_{q_1\ldots q_d}$, nonoverlapping translates of Ω whose union is I_d. We have

$$T = \sum_{j,k} P_{\Omega_j} T P_{\Omega_k}$$

and it follows from Lemma 5 that

$$\begin{aligned} N^+((1-\delta)\varepsilon, T) &\geqq N^+\left(\varepsilon, \sum_j P_{\Omega_j} T P_{\Omega_j}\right) - N^+\left(\delta\varepsilon, -\sum_{j\neq k} P_{\Omega_j} T P_{\Omega_k}\right) \\ &= N^+\left(\varepsilon, \sum_j P_{\Omega_j} T P_{\Omega_j}\right) - N^-\left(\delta\varepsilon, \sum_{j\neq k} P_{\Omega_j} T P_{\Omega_k}\right). \end{aligned} \tag{3}$$

Each of the operators $P_{\Omega_j} T P_{\Omega_j}$ has the same spectrum as $P_\Omega T P_\Omega$ and the sum $\sum P_{\Omega_j} T P_{\Omega_j}$ is direct. Therefore

$$N^+\left(\varepsilon, \sum_j P_{\Omega_j} T P_{\Omega_j}\right) = q_1 \cdots q_d N^+(\varepsilon, P_\Omega T P_\Omega).$$

Since the eigenvalues of T are the quantities $\{c_{\mathbf{n}}\}$,

$$N^+((1-\delta)\varepsilon, T) = \Psi((1-\delta)\varepsilon).$$

Thus (3) gives

$$q_1 \cdots q_d N^+(\varepsilon, P_\Omega T P_\Omega) \leqq \Psi((1-\delta)\varepsilon) + N^-\left(\delta\varepsilon, \sum_{j\neq k} P_{\Omega_j} T P_{\Omega_k}\right).$$

By Lemma 5 we have

$$N^-\left(\delta\varepsilon, \sum_{j\neq k} P_{\Omega_j}TP_{\Omega_k}\right) \leqq \sum_{j<k} N^-\left(\delta\varepsilon/\binom{q_1\cdots q_d}{2}, P_{\Omega_j}TP_{\Omega_k}+P_{\Omega_k}TP_{\Omega_j}\right)$$

and this is $o(\Psi((1-\delta)\varepsilon))$ by the Main Lemma and Lemma 4. Hence

$$\limsup \frac{N^+(\varepsilon, P_\Omega TP_\Omega)}{\Psi((1-\delta)\varepsilon)} \leqq (q_1\cdots q_d)^{-1} = (2\pi)^{-d}|\Omega|,$$

which is the first inequality in our special case. The second is proved similarly.

Next we assume $r_i = p_i/q_i$ with p_i, q_i integers. Let Ω_0 be an interval with edges $2\pi/q_i$ (and for which, therefore, the lemma has been proved), and let $\Omega_1, \cdots, \Omega_{p_1\ldots p_d}$ be nonoverlapping translates of Ω_0 whose union is Ω. Then

$$\begin{aligned} N^+(\varepsilon, P_\Omega TP_\Omega) &\leqq N^+\left((1-\delta)\varepsilon, \sum_j P_{\Omega_j}TP_{\Omega_j}\right) + N^+\left(\delta\varepsilon, \sum_{j\neq k} P_{\Omega_j}TP_{\Omega_k}\right) \\ &= p_1\cdots p_d N^+((1-\delta)\varepsilon, P_{\Omega_0}TP_{\Omega_0}) + o(\Psi((1-\delta)^2\varepsilon)) \\ &\leqq \frac{p_1\cdots p_d}{q_1\cdots q_d}\Psi((1-\delta)^2\varepsilon) + o(\Psi((1-\delta)^2\varepsilon)), \end{aligned}$$

and so

$$\limsup \frac{N^+(\varepsilon, P_\Omega TP_\Omega)}{\Psi((1-\delta)^2\varepsilon)} \leqq \frac{p_1\cdots p_d}{q_1\cdots q_d} = (2\pi)^{-d}|\Omega|,$$

which is the first of the desired inequalities except that $1-\delta$ is replaced by $(1-\delta)^2$, a matter of no importance. The second inequality is proved similarly.

To remove the restriction that each r_i be rational, observe that $\Omega_1 \subset \Omega_2$ implies

$$N^+(\varepsilon, P_{\Omega_1}TP_{\Omega_1}) \leqq N^+(\varepsilon, P_{\Omega_2}TP_{\Omega_2}).$$

This follows from the minimax characterization of the eigenvalues. We can find Ω_1 and Ω_2 with edges which are rational multiples of π in such a way that $\Omega_1 \subset \Omega \subset \Omega_2$ and that the ratio $|\Omega_2|/|\Omega_1|$ is as close to 1 as desired. The inequalities for Ω now clearly follow.

The above lemma essentially determines the behavior of the eigenvalues when V is the characteristic function of an interval. We now pass to the more general situation. The characteristic function of Ω will be denoted by χ_Ω.

LEMMA 7. *Let Ω_j be finitely many nonoverlapping intervals, $V_j \geqq 0$ and $V(\mathbf{x}) = \sum V_j\chi_{\Omega_j}(\mathbf{x})$. Then for each δ in $0<\delta<1$,*

$$\limsup_{\varepsilon\to 0} \frac{N^+(\varepsilon, M_V{}^{1/2}TM_V{}^{1/2})}{\sum|\Omega_j|\Psi((1-\delta)\varepsilon/V_j)} \leqq (2\pi)^{-d},$$

$$\liminf_{\varepsilon\to 0} \frac{N^+(\varepsilon, M_V{}^{1/2}TM_V{}^{1/2})}{\sum|\Omega_j|\Psi((1+\delta)\varepsilon/V_j)} \geqq (2\pi)^{-d}.$$

Proof. We have

$$N^+(\varepsilon, M_V^{1/2} T M_V^{1/2} \leqq N^+\Big((1-\delta)\varepsilon, \sum_j V_j P_{\Omega_j} T P_{\Omega_j}\Big) + N^+\Big(\delta\varepsilon, \sum_{j \neq k} V_j^{1/2} V_k^{1/2} P_{\Omega_j} T P_{\Omega_k}\Big).$$

It follows from the main lemma and Lemma 4 that the last term is $o(\Psi(\varepsilon))$ and so also

$$o\big(\sum |\Omega_j| \Psi((1-\delta)^2 \varepsilon / V_j)\big).$$

Since the sum $\sum_j V_j P_{\Omega_j} T P_{\Omega_j}$ is direct

$$N^+\Big((1-\delta)\varepsilon, \sum_j V_j P_{\Omega_j} T P_{\Omega_j}\Big) = \sum N^+((1-\delta)\varepsilon / V_j, P_{\Omega_j} T P_{\Omega_j})$$

and by Lemma 6 this is at most

$$(2\pi)^{-d} \sum |\Omega_j| \Psi((1-\delta)^2 \varepsilon / V_j)(1 + o(1)).$$

This proves the first inequality, with $1-\delta$ replaced by $(1-\delta)^2$, and the second follows similarly.

Denote by Λ_d the set of lattice points in E_d. The cartesian product $I_d \times \Lambda_d$ has a natural measure: the product measure obtained from Lebesgue measure on I_d and from the measure which assigns to each point of Λ_d the measure 1. Given our sequence $\{c_{\mathbf{n}}\}$ and a function $V(\mathbf{x})$ on I_d we shall write

$$\Phi(\varepsilon) = |\{(\mathbf{x}, \mathbf{n}) : V(\mathbf{x}) \cdot c_{\mathbf{n}} > \varepsilon\}|.$$

Note that in case $V = \sum V_j \chi_{\Omega_j}$ as in Lemma 7 we have $\Phi(\varepsilon) = \sum |\Omega_j| \Psi(\varepsilon / V_j)$.

Lemma 8. *Assume $V(\mathbf{x})$ is Riemann integrable on I_d. Then for any δ in* $0 < \delta < 1$,

$$\limsup_{\varepsilon \to 0} \frac{N^+(\varepsilon, M_V^{1/2} T M_V^{1/2})}{\Phi((1-\delta)\varepsilon)} \leqq (2\pi)^{-d},$$

$$\liminf_{\varepsilon \to 0} \frac{N^+(\varepsilon, M_V^{1/2} T M_V^{1/2})}{\Phi((1+\delta)\varepsilon)} \geqq (2\pi)^{-d}.$$

Proof. Let $\eta > 0$ and find a partition of I_d into nonoverlaping intervals Ω_j such that, with

$$m_j = \inf_{\Omega_j} V(\mathbf{x}) \qquad M_j = \sup_{\Omega_j} V(\mathbf{x})$$

we have

$$\sum (M_j - m_j) |\Omega_j| < \eta^2 \delta. \tag{4}$$

Let

$$V_1 = \sum m_j \chi_{\Omega_j} \qquad V_2 = \sum M_j \chi_{\Omega_j}.$$

It follows from the minimax characterization of the eigenvalues that

$$N^+(\varepsilon, M_{V_1^{1/2}} T M_{V_1^{1/2}}) \leqq N^+(\varepsilon, M_{V^{1/2}} T M_{V^{1/2}}) \leqq N^+(\varepsilon, M_{V_2^{1/2}} T M_{V_2^{1/2}}).$$

By the first inequality of Lemma 7 we have for sufficiently small ε

$$N^+(\varepsilon, M_{V_2^{1/2}} \; T M_{V_2^{1/2}}) \leqq (1+\eta)(2\pi)^{-d} \sum |\Omega_j| \Psi((1-\delta)\varepsilon/M_j).$$

We now write $\sum = \sum^{(1)} + \sum^{(2)} + \sum^{(3)}$; to which sum a given index j belongs is determined as follows:

$$\sum{}^{(1)} : \; M_j < \eta,$$

$$\sum{}^{(2)} : \; M_j \geqq \eta \text{ and } m_j < (1-\delta) M_j,$$

$$\sum{}^{(3)} : \; M_j \geqq \eta \text{ and } m_j \geqq (1-\delta) M_j.$$

We have

$$(1+\eta)(2\pi)^{-d} \sum{}^{(1)} |\Omega_j| \Psi((1-\delta)\varepsilon/M_j) \leqq (1+\eta)\Psi((1-\delta)\varepsilon/\eta).$$

From (4) we deduce $\sum^{(2)}|\Omega_j| \leqq \eta$. Therefore

$$(1+\eta)(2\pi)^{-d} \sum{}^{(2)} |\Omega_j| \Psi((1-\delta)\varepsilon/M_j) \leqq (1+\eta)\eta(2\pi)^{-d} \Psi((1-\delta)\varepsilon/\sup V).$$

Now it follows from Lemma 4 that (except in the trivial case $V \equiv 0$) the ratio $\Psi(\varepsilon)/\Phi(\varepsilon)$ is bounded above and below. Hence for an appropriate constant A we have

$$(1+\eta)(2\pi)^{-d} \sum{}^{(2)} |\Omega_j| \Psi((1-\delta)\varepsilon/M_j) \leqq A(1+\eta)\eta\Phi((1-\delta)^2\varepsilon).$$

Finally

$$(1+\eta)(2\pi)^{-d} \sum{}^{(3)} |\Omega_j| \Psi((1-\delta)\varepsilon/M_j) \leqq (1+\eta)(2\pi)^{-d} \Phi((1-\delta)^2\varepsilon).$$

Therefore, putting the three inequalities together,

$$\begin{aligned} N^+(\varepsilon, M_{V^{1/2}} T M_{V^{1/2}}) \leqq \; & (1+\eta)\Psi((1-\delta)\varepsilon/\eta) \\ & + ((2\pi)^{-d} + A\eta)(1+\eta)\Phi((1-\delta)^2\varepsilon), \end{aligned}$$

and so

$$\begin{aligned} \limsup_{\varepsilon \to 0} \frac{N^+(\varepsilon, M_{V^{1/2}} T M_{V^{1/2}})}{\Phi((1-\delta)^2\varepsilon)} \leqq \; & (1+\eta) \limsup_{\varepsilon \to 0} \frac{\Psi((1-\delta)\varepsilon/\eta)}{\Phi((1-\delta)^2\varepsilon)} \\ & + ((2\pi)^{-d} + A\eta)(1+\eta). \end{aligned}$$

Now $\eta > 0$ is arbitrarily small. The first inequality of the lemma will therefore be proved if we can show

$$\lim_{\eta\to 0} \limsup_{\varepsilon\to 0} \frac{\Psi(\varepsilon/\eta)}{\Psi(\varepsilon)} = 0$$

(recall the boundedness of $\Psi(\varepsilon)/\Phi(\varepsilon)$). This, however, is equivalent to the statement of Lemma 3.

The second inequality is proved similarly.

5. We now state and prove the theorems which give the asymptotic behavior of the eigenvalues in the periodic case.

THEOREM I. *Assume* $\{c_{\mathbf{n}}\}$ *satisfies* (i)–(iv) *and* V *is Riemann integrable. Let* $\phi(\alpha)$ $(0 < \alpha < \infty)$ *be a nonincreasing function equimeasurable with the function* $V(\mathbf{x}) \cdot c_{\mathbf{n}}$ *on* $I_d \times \Lambda_d$. *Assume further that*

$$\alpha \sim \beta \to \infty \text{ implies } \phi(\alpha) \sim \phi(\beta). \tag{5}$$

Then if $\lambda_1 \geqq \lambda_2 \geqq \cdots$ *are the positive eigenvalues of* $M_V{}^{1/2}\, T M_V{}^{1/2}$ *we have*

$$\lambda_n \sim \phi((2\pi)^d n) \qquad n \to \infty.$$

Proof. We show first that with Φ as above we have $\phi(\Phi(\varepsilon)) \sim \varepsilon$ as $\varepsilon \to 0$. Of course were Φ to be continuous and strictly decreasing we would have $\phi(\Phi(\varepsilon)) = \varepsilon$ but this is generally not the case. It is true though that $\alpha < \Phi(\varepsilon)$ implies $\phi(\alpha) \geqq \varepsilon$, so in particular $\phi((1-\varepsilon)\Phi(\varepsilon)) \geqq \varepsilon$. Since $(1-\varepsilon)\Phi(\varepsilon) \sim \Phi(\varepsilon)$ the assumption (5) gives

$$\liminf \frac{\phi(\Phi(\varepsilon))}{\varepsilon} \geqq 1$$

and a similar argument gives

$$\limsup \frac{\phi(\Phi(\varepsilon))}{\varepsilon} \leqq 1.$$

Next, by the first inequality of Lemma 8 we have $(0 < \delta < 1)$

$$(\varepsilon)(2\pi)^{-d} N^{+}(\varepsilon, M_V{}^{1/2} T M_V{}^{1/2}) \leqq \Phi((1-\delta)\varepsilon),$$

where $\rho(\varepsilon) \to 1$ as $\varepsilon \to 0$. Set $\varepsilon = (1 - n^{-1})\lambda_n$. Then since $\varepsilon < \lambda_n$,

$$\rho((1-n^{-1})\lambda_n)(2\pi)^{-d} n \leqq \Phi((1-\delta)(1-n^{-1})\lambda_n)$$

and so

$$\phi(\rho((1-n^{-1})\lambda_n)(2\pi)^{-d} n) \geqq \phi(\Phi((1-\delta)(1-n^{-1})\lambda_n)).$$

Using (5) and the facts that $\rho(\varepsilon) \to 1$ and $\phi(\Phi(\varepsilon)) \sim \varepsilon$, we conclude

$$\liminf \frac{\phi((2\pi)^{-d} n)}{\lambda_n} \geqq 1 - \delta.$$

Since $\delta > 0$ was arbitrarily small the lim inf is at least 1. Similarly the lim sup is at most 1 and the theorem is proved.

The next theorem shows that the conclusion of Theorem I holds if $\{c_{\mathbf{n}}\}$ is replaced by any asymptotic sequence $\{c'_{\mathbf{n}}\}$. The sequence $\{c_{\mathbf{n}}\}$, the corresponding operator T, and the function ϕ are as before.

THEOREM I′. *Assume the hypotheses of Theorem* I *hold. Let* $\{c'_{\mathbf{n}}\}$ *satisfy* $c'_{\mathbf{n}} \sim c_{\mathbf{n}}$ $(|n| \to \infty)$ and *let* T' *be the operator on* $L_2(I_d)$ *corresponding to* $\{c'_{\mathbf{n}}\}$. *Then if* $\lambda'_1 \geqq \lambda'_2 \geqq \cdots$ *are the positive eigenvalues of* $M_V{}^{1/2}T'M_V{}^{1/2}$ *we have*

$$\lambda'_n \sim \phi((2\pi)^d n) \qquad n \to \infty.$$

Proof. Let $\delta > 0$. There is a sequence $\{c''_{\mathbf{n}}\}$ vanishing on all but finitely many (say N) lattice points so that

$$c'_{\mathbf{n}} \leqq (1+\delta)c_{\mathbf{n}} + c''_{\mathbf{n}}$$

for all $\mathbf{n}$. Then if T'' is the operator corresponding to $\{c''_{\mathbf{n}}\}$,

$$T' \leqq (1+\delta)T + T'',$$

and so, if $\lambda''_1 \geqq \lambda''_2 \geqq \cdots$ are the positive eigenvalues of $M_V{}^{1/2}T''M_V{}^{1/2}$,

$$\lambda'_n \leqq (1+\delta)\lambda_{n-N} + \lambda''_{N+1} = (1+\delta)\lambda_{n-N}$$

since $M_V^{1/2}T''M_V^{1/2}$ is of rank N. By Theorem I,

$$\lambda_{n-N} \sim \phi((2\pi)^d(n-N)) \sim \phi((2\pi)^d n)$$

by (5). Therefore

$$\limsup \frac{\lambda'_n}{\phi((2\pi)_d n)} \leqq 1+\delta.$$

Since δ was arbitrarily small the lim sup is at most 1; and similarly the lim inf is at least 1.

6. We now consider the situation as described in §1. $K(\xi) = K(\xi_1,\cdots,\xi_d)$ is a bounded function on E_d which tends to zero as $|\xi| \to \infty$. Conditions replacing (i)–(iv) above are:

(i′) $K(\xi) \geqq 0$;

(ii′) with all ξ_i fixed but ξ_{i_0}, $K(\xi)$, as a function of ξ_{i_0}, is nondecreasing between $-\infty$ and some $\bar{\xi} = \bar{\xi}(i_0)$ and nonincreasing between $\bar{\xi}$ and ∞;

(iii′) if $|\xi|, |\eta| \to \infty$ and $\xi \sim \eta$ (i.e. $|\xi - \eta| = o(|\eta|)$) then $K(\xi) \sim K(\eta)$;

(iv′) if $|\xi|, |\eta| \to \infty$ and $|\xi| = o(|\eta|)$ then $K(\eta) = o(K(\xi))$.

T_0 is the operator on $L_2(E_d)$ associated with K as described in §1.

THEOREM II. *Assume K satisfies* (i′)–(iv′) *and V is properly Riemann integrable. Let $\phi_0(\alpha)$ $(0 < \alpha < \infty)$ be a nonincreasing function equimeasurable with the function $V(\mathbf{x}) \cdot K(\xi)$ on $E_d \times E_d$. Assume further that*

$$\alpha \sim \beta \to \infty \text{ implies } \phi_0(\alpha) \sim \phi_0(\beta). \tag{6}$$

Then if $\lambda_1 \geqq \lambda_2 \geqq \cdots$ are the positive eigenvalues of $M_V{}^{1/2}T_0M_V{}^{1/2}$ we have

$$\lambda_n \sim \phi_0((2\pi)^d n) \qquad n \to \infty.$$

Proof. For any $a > 0$ the eigenvalues of $M_V{}^{1/2}T_0M_V{}^{1/2}$ are unchanged if $V(\mathbf{x})$ is replaced by $V(a\mathbf{x})$ and $K(\xi)$ by $K(\xi/a)$. The function ϕ_0 is also unchanged. Because of this, and because V has bounded support, we may assume that V vanishes outside the cube $\{\mathbf{x}: |x_i| \leqq \pi/4\}$. Then $M_V{}^{1/2}T_0M_V{}^{1/2}$ may be identified in a natural way with an operator on L_2 of the cube $I_d: \{\mathbf{x}: |x_i| \leqq \pi\}$. (Of course it may also be identified with an operator on L_2 of the smaller cube which supports V, but we prefer to ignore this.) We shall see that it is in fact an operator of the type considered in the previous section and to which Theorem I′ can be applied.

Let $u(\mathbf{x})$ be an infinitely differentiable function on E_d which satisfies

$$u(\mathbf{x}) = 1 \qquad \text{if all } |x_i| \leqq \pi/2,$$
$$u(\mathbf{x}) = 0 \qquad \text{if any } |x_i| > \pi,$$

and set

$$U(\xi) = \int u(\mathbf{x}) e^{-i\xi\cdot\mathbf{x}} d\mathbf{x}.$$

Let us assume first that

$$K(\xi) = \int k(\mathbf{x}) e^{-i\xi\cdot\mathbf{x}} d\mathbf{x}, \qquad k \in L_1.$$

Then $M_V{}^{1/2}T_0M_V{}^{1/2}$ is the integral operator on $L_2(I_d)$ with kernel

$$V(\mathbf{x})^{1/2}k(\mathbf{x}-\mathbf{y})V(\mathbf{y})^{1/2} = V(\mathbf{x})^{1/2}k(\mathbf{x}-\mathbf{y})u(\mathbf{x}-\mathbf{y})V(\mathbf{y})^{1/2}$$

since $u(\mathbf{x}-\mathbf{y}) = 1$ whenever $V(\mathbf{x}) \cdot V(\mathbf{y}) \neq 0$. This is also equal to

$$V(\mathbf{x})^{1/2}k_1(\mathbf{x}-\mathbf{y})V(\mathbf{y})^{1/2},$$

where $k_1(\mathbf{x})$ is the function of period 2π in each x_i which is equal to $k(\mathbf{x})\,u(\mathbf{x})$ in $|x_i| \leqq \pi$. Now $k_1(\mathbf{x}-\mathbf{y})$ is the kernel of the operator T on $L_2(I_d)$ associated with the sequence $\{c_\mathbf{n}\}$, where

$$c_\mathbf{n} = \int_{I_d} k_1(\mathbf{x}) e^{-i\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} = \int_{E_d} u(\mathbf{x})k(\mathbf{x}) e^{-i\mathbf{n}\cdot\mathbf{x}} d\mathbf{x}$$

since $u(\mathbf{x}) = 0$ outside I_d. Thus

$$M_V{}^{1/2}T_0M_V{}^{1/2} = M_V{}^{1/2}TM_V{}^{1/2}, \tag{7}$$

where T is the operator on $L_2(I_d)$ associated with the sequence

$$c_{\mathbf{n}} = (2\pi)^{-d} \int U(\xi)K(\mathbf{n} - \xi)\,d\xi.$$

Now this last statement holds even if K is not the Fourier transform of an L_1 function. For we can find a sequence $K^{(j)}(j \to \infty)$ satisfying

(a) each $K^{(j)}$ is the Fourier transform of an L_1 function,

(b) $K^{(j)} \to K$ boundedly and pointwise.

Then we have

$$M_V{}^{1/2}T_0^{(j)}M_V{}^{1/2} = M_V{}^{1/2}T^{(j)}M_V{}^{1/2}, \tag{8}$$

where $T_0^{(j)}$ is the operator on $L_2(E_d)$ corresponding to $K^{(j)}$ and $T^{(j)}$ is the operator on $L_2(I_d)$ corresponding to

$$c_{\mathbf{n}}^{(j)} = (2\pi)^{-d} \int U(\xi)K^{(j)}(\mathbf{n} - \xi)\,d\xi.$$

It follows from (b) that $T_0^{(j)} \to T_0$ and $T^{(j)} \to T$ strongly as $j \to \infty$. Thus (8) gives (7) in the general case.

Let $\phi(\alpha)$ $(0 < \alpha < \infty)$ be nonincreasing and equimeasurable with the function $V(\mathbf{x}) \cdot K(\mathbf{n})$ on $I_d \times \Lambda_d$. We shall show that $c_{\mathbf{n}} \sim K(\mathbf{n})$; since $K(\mathbf{n})$ satisfies (i)–(iv), Theorem I′ will give $\lambda_n \sim \phi((2\pi)^d n)$. Now

$$\begin{aligned} c_{\mathbf{n}} &= (2\pi)^{-d} \int U(\xi)K(\mathbf{n} - \xi)\,d\xi \\ &= K(\mathbf{n}) + (2\pi)^{-d} \int U(\xi)[K(\mathbf{n} - \xi) - K(\mathbf{n})]\,d\xi \end{aligned}$$

since $(2\pi)^{-d} \int U(\xi)\,d\xi = u(\mathbf{0}) = 1$. For any $\delta > 0$

$$\begin{aligned} |c_{\mathbf{n}} - K(\mathbf{n})| &\leq (2\pi)^{-d} \int_{|\xi| \leq \delta|\mathbf{n}|} |U(\xi)| \; |K(\mathbf{n} - \xi) - K(\mathbf{n})|\,d\xi \\ &\quad + (2\pi)^{-d} \int_{|\xi| \geq \delta|\mathbf{n}|} |U(\xi)| \; |K(\mathbf{n} - \xi) - K(\mathbf{n})|\,d\xi \\ &\leq \sup_{|\xi| \leq \delta|\mathbf{n}|} |K(\mathbf{n} - \xi) - K(\mathbf{n})| + O(|\mathbf{n}|^{-p}), \end{aligned}$$

$p > 0$ being arbitrary. The bound $O(|\mathbf{n}|^{-p})$ for the last integral arises from the fact that u is infinitely differentiable and has bounded support. Since, as follows from (iii′), $|\mathbf{n}|^{-p} = o(K(\mathbf{n}))$ for some sufficiently large p, we obtain

$$|c_{\mathbf{n}} - K(\mathbf{n})| \leqq \sup_{|\xi| \leqq \delta|\mathbf{n}|} |K(\mathbf{n} - \xi) - K(\mathbf{n})| + o(K(\mathbf{n})).$$

It also follows from (iii′) that, $\varepsilon > 0$ being given, δ can be chosen so small that

$$\sup_{|\xi| + \delta|\mathbf{n}|} |K(\mathbf{n} - \xi) - K(\mathbf{n})| \leqq \varepsilon K(\mathbf{n}).$$

Thus

$$\limsup \left| \frac{c_{\mathbf{n}}}{K(\mathbf{n})} - 1 \right| \leqq \varepsilon$$

and since $\varepsilon > 0$ was arbitrary $c_{\mathbf{n}} \sim K(\mathbf{n})$.

We have established so far that $\lambda_n \sim \phi((2\pi)^d n)$; we shall be through if we can prove $\phi(\alpha) \sim \phi_0(\alpha)$ as $\alpha \to \infty$. Given $\delta > 0$ the following is true for $|\mathbf{n}|$ sufficiently large: If $V(\mathbf{x}) \cdot K(\mathbf{n}) > \varepsilon$ then $V(\mathbf{x}) \cdot K(\xi) > (1 - \delta)\varepsilon$ for all ξ in the cube $|\xi_i - n_i| \leqq \frac{1}{2}$. This follows from (iii′). Thus there is an N such that

$$|\{(\mathbf{x},\xi) : V(\mathbf{x}) \cdot K(\xi) > (1 - \delta)\varepsilon\}| > |\{(\mathbf{x},\mathbf{n}) : V(\mathbf{x}) \cdot K(\mathbf{n}) > \varepsilon\}| - N$$

for all ε. Given α, let $(1 - \delta)\varepsilon > \phi_0(\alpha)$. Then

$$|\{(\mathbf{x},\xi) : V(\mathbf{x}) \cdot K(\xi) > (1 - \delta)\varepsilon\}| \leqq \alpha$$

and so

$$\alpha + N > |\{(\mathbf{x},\mathbf{n}) : V(\mathbf{x}) \cdot K(\mathbf{n}) > \varepsilon\}|,$$

which implies $\phi(\alpha + N) \leqq \varepsilon$. This holds for all $\varepsilon > (1 - \delta)^{-1}\phi_0(\alpha)$ and so

$$\frac{\phi_0(\alpha)}{\phi(\alpha + N)} \geqq 1 - \delta$$

or alternatively

$$\frac{\phi_0(\alpha - N)}{\phi(\alpha)} \geqq 1 - \delta.$$

Since $\alpha - N \sim \alpha$ as $\alpha \to \infty$ (6) gives

$$\liminf \frac{\phi_0(\alpha)}{\phi(\alpha)} \geqq 1 - \delta.$$

Since $\delta > 0$ was arbitrarily small the lim inf is at least 1; similarly the lim sup is at most 1. This completes the proof of Theorem II.

Finally we state and prove the analogue of Theorem I′. The functions K and ϕ_0 and the operator T_0 are as before.

THEOREM II$'$. *Assume the hypotheses of Theorem* II *hold. Let $K'(\xi)$ satisfy $K'(\xi) \sim K(\xi)(|\xi| \to \infty)$ and let T_0' be the operator on $L_2(E_d)$ corresponding to K'. Then if $\lambda_1' \geqq \lambda_2' \geqq \cdots$ are the positive eigenvalues of $M_V{}^{1/2} T_0' M_V{}^{1/2}$ we have*

$$\lambda_n' \sim \phi_0((2\pi)^d n) \qquad n \to \infty.$$

Proof. This is only slightly less simple than the proof of Theorem I$'$. Given $\delta > 0$ and $p > 0$ we can find a function K'' of the form $A/(1 + |\xi|^p)$ (with A depending on δ and p) so that

$$K'(\xi) \leqq (1 + \delta) K(\xi) + K''(\xi)$$

for all ξ. Therefore with the obvious notation

$$\lambda_n' \leqq (1 + \delta)\lambda_{n-N} + \lambda_{N+1}''.$$

Now λ_{N+1}'' is at most A sup V times the $(N + 1)$ st eigenvalue of $PT_0'''P$, where P is multiplication by the characteristic function of the support of V and T_0''' is the operator on $L_2(E_d)$ corresponding to the function $1/(1 + |\xi|^p)$. Applying Theorem I to this case we obtain $\lambda_{N+1}'' \sim A' N^{-p/d}$ as $N \to \infty$. Now (6) implies $\phi_0(\alpha) \geqq \alpha^{-\Delta}$ for some $\Delta > 0$ and α sufficiently large. Thus if we set $p = 3\Delta d$ and $N = [n^{1/2}]$ we shall have $\lambda_{N+1}'' = o(\phi_0((2\pi)^d n))$. Since

$$\lambda_{n-N} = \lambda_{n-[n^{1/2}]} \sim \phi_0((2\pi)^d(n - [n^{1/2}])) \sim \phi_0((2\pi)^d n)$$

by Theorem I and (6), we have

$$\lambda_n' \leqq (1 + 2\delta)\phi_0((2\pi)^d n)$$

for n sufficiently large. Since $\delta > 0$ was arbitrary this gives $\limsup \lambda_n'/\phi_0((2\pi)^d n) \leqq 1$. Similarly $\liminf \geqq 1$.

7. Now that all the unpleasant details are behind us, we state what we think is the correct theorem in the subject.

Let G be a locally compact abelian group, $\hat{G}$ its character group. Adjust the Haar measures $d\mathbf{x}$ and $d\xi$ in G and $\hat{G}$, respectively, so that Fourier transformation F,

$$Ff(\xi) = \int \xi(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x},$$

is a unitary operator from $L_2(G)$ to $L_2(\hat{G})$. Let V, K be bounded non-negative functions on G and $\hat{G}$, respectively, both tending to zero at infinity; let $\lambda_1 \geqq \lambda_2 \geqq \cdots$ be the positive eigenvalues of the operator

$$M_V{}^{1/2} F^* M_K F M_V{}^{1/2}$$

on $L_2(G)$. Then if $\psi(\alpha)$ $(0 < \alpha < \infty)$, a nonincreasing function equimeasurable with the function $V(\mathbf{x}) \cdot K(\xi)$ on $G \times \hat{G}$, is sufficiently regular and does not approach zero too rapidly as $\alpha \to \infty$, we ought to have $\lambda_n \sim \psi(n)$.

What we have considered were the two cases in which G was I_d (i.e., the d-fold product of the circle group with itself) and E_d. The assumptions (5) and (6) are of the type we have in mind for ψ. In addition to saying something about regularity they imply $\psi(\alpha) \geqq \alpha^{-\Delta}$ for some positive Δ. That it is necessary to have some restriction on the rapidity with which ψ may approach zero (although $\psi(\alpha) \geqq \alpha^{-\Delta}$ for some $\Delta > 0$ is probably much too restrictive) can be seen by taking (in the case $G = \hat{G} = E_1$) both K and V to be characteristic functions of intervals. This corresponds to an integral equation like

$$\int_{-1}^{1} \frac{\sin(x-y)}{x-y} f(y)\, dy = \lambda f(x) \qquad |x| \leqq 1.$$

Then $\psi(\alpha)$ is ultimately zero but the operator is certainly not of finite rank so $\lambda_n \sim \psi(n)$ is false.

It is probably not necessary to have a restriction on the slowness with which ψ may approach zero. The condition (iv) implies that $\psi(\alpha) = O(\alpha^{-\delta})$ for some $\delta > 0$ but if the c's have somewhat more regularity than we have assumed (in dimension one, if $\{c_n\}$ is even and convex for $n \geqq 0$) one can make a better estimate than that given by Lemma 1 and then dispense with (iv) altogether.

The following is the reason our assumptions on K and V seem unnecessary. The two operators

$$M_V{}^{1/2} F^* M_K F M_V{}^{1/2}, \qquad M_K{}^{1/2} F M_V F^* M_K{}^{1/2}$$

have exactly the same spectrum since one is of the form A^*A and the other AA^* with an appropriate operator A. Thus (in case $G = \hat{G} = E_d$) the roles of K and V are interchangeable. Since in our proofs there was no assumption made on both K and V (the assumptions on K and V were of entirely different sorts) no assumptions of any kind ought to be necessary.

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