INTEGRAL REPRESENTATIONS OF DIHEDRAL GROUPS OF ORDER $2p$

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Introduction. Information about the integral representations of finite groups has been obtained to varying extents. For Z the ring of rational integers and G the cyclic group of prime order, the ZG-modules were studied by Diederichsen [3] and Reiner [11], who showed that there were finitely many indecomposable ZG-modules and determined them completely. The finiteness of the number of indecomposables in the case where G is cyclic of order p^2 was shown, for $p = 2$, by Troy [16] and for any p by Heller and Reiner [5] and by Knee [8], while Oppenheim [10] and Knee [8] established the finiteness of the number of indecomposables for G cyclic, of square free order. Heller and Reiner [5; 6] and Jones [7] established that the number of indecomposable ZG-modules is finite if and only if all p-Sylow subgroups of G are cyclic of order at most p^2. Here, as well as throughout this paper, we shall mean by a ZG-module one which is finitely generated and Z-free.

In this paper we shall classify all finitely-generated S-free SG-modules where G is the dihedral group of order $2p$, p an odd prime, and S is Z or Z_{2p} the semi-local ring formed by the intersection of Z_p and Z_2, respectively the rings of p-integral and 2-integral elements in Q the rational field. $Z_{2p} = \{r/s \in Q\colon (s,2p) = 1\}$. Taking θ to be a primitive pth root of unity, we shall denote by $K = Q(\theta)$ the cyclotomic field of degree $p-1$ over Q and by $K_0 = Q(\theta + \theta^{-1})$ the real subfield of K. R_0 and R shall be the integral closures of S in K_0 and K, respectively. Letting $\mathfrak{h}$ denote the group of automorphisms of K with fixed field K_0, we may form Λ the twisted group ring of $\mathfrak{h}$ with coefficients in R.

§1 of this paper is devoted to a characterization of R-projective Λ-modules of finite R-rank. The results of this section are then applied in the second section to show that there are precisely $7h + 3$ nonisomorphic, indecomposable SG-modules where h is the ideal class number of R_0. In §3 it is shown that although a Krull-Schmidt theorem is not obtainable for SG-modules, invariants may be obtained which determine an SG-module up to $Z_{2p}G$-isomorphism. The final section deals with projective SG-modules. Here an isomorphism is established between the projective class group of SG and the ideal class group of R_0.

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1. **Modules of the twisted group ring.** The group $\mathfrak{h}$ of automorphisms of K having fixed field K_0 is of order 2 with generator a where $a\theta = \theta^{-1}$ the complex conjugate of θ. We shall henceforth denote θ^{-1} by $\bar{\theta}$. It follows that $ax = \bar{x}$ for every $x \in K$. The twisted group ring of $\mathfrak{h}$ with coefficients in R is given by $\Lambda = R + Ra$ where $a(r_1 + r_2 a) = \bar{r}_1 a + \bar{r}_2$ for $r_1 + r_2 a \in \Lambda$.

Every Λ-module can be regarded as an R-module. We shall call a Λ-module M R-projective if M is a projective R-module.

PROPOSITION 1.1. *Every R-projective Λ-module is Λ-projective.*

Proof. Let M be any Λ-module which is R-projective. Let $\phi: R \to \Lambda$ be the natural map. Define $M_\phi = \Lambda \otimes_R M$ where $a(\lambda \otimes m) = a\lambda \otimes m$ for $\lambda \in \Lambda$, $m \in M$. Since M is R-projective, M_ϕ is Λ-projective [1, p. 30]. Consider the exact sequence of Λ-modules

$$0 \to \ker g \to M_\phi \xrightarrow{g} M \to 0 \tag{1.1}$$

where $g(\lambda \otimes m) = \lambda m$. Take $\rho = \theta/(\theta + \bar{\theta})$, a unit in R such that $\rho + \bar{\rho} = 1$ and define $f: M \to M_\phi$ by $f(m) = (1 \otimes \rho m) + (a \otimes \rho a m)$. f is a Λ-homomorphism. For any $m \in M$, $gf(m) = g(1 \otimes \rho m) + g(a \otimes \rho(am)) = \rho m + a\rho a m = \rho m + \bar{\rho} m = m$, whence the sequence (1.1) splits. It follows that M is isomorphic as a Λ-module to a direct summand of the projective Λ-module M_ϕ and is thus Λ-projective.

Any ideal I in Λ, considered as an R-module, is a submodule of the free R-module Λ. Since R is dedekind, R is an hereditary ring and I is thus R-projective. By Proposition 1.1 I is Λ-projective. This establishes

PROPOSITION 1.2. *Λ is an hereditary ring.*

It follows that every submodule of a free Λ-module is a direct sum of modules, each isomorphic to a left ideal in Λ [1, p. 13]. It remains for us to characterize the ideals in Λ.

DEFINITION. An R-ideal A in K is said to be ambiguous if and only if $A = \bar{A}$, that is, if and only if whenever $x \in A$, $\bar{x} \in A$.

Since a is the automorphism of K given by $ax = \bar{x}$ for each $x \in K$, an ambiguous ideal in K can be considered as an ideal of Λ of R-rank one under the action of a given by $ar = \bar{r}$ for $r \in A$. Conversely, any Λ-module I of R-rank one is isomorphic as an R-module to an R-ideal A in K. Since $ax \in I$ for each $x \in I$, the isomorphism is an isomorphism of Λ-modules if and only if $ar = \bar{r} \in A$ for each $r \in A$. We have thus shown

PROPOSITION 1.3. *An ideal I in Λ has R-rank one if and only if I is Λ-isomorphic to an ambiguous R-ideal in K.*

Now assume I is any ideal in Λ having R-rank two. Consider $I^* = K_0 \otimes_{R_0} I$. I^* is a module over $\Lambda^* = K_0 \otimes_{R_0} \Lambda \cong K + Ka$, where $ax = \bar{x}a$ for $x \in K$. Since $K_0 \subset K$ is the fixed field of a, Λ^* is the crossed product algebra of K over K_0 with respect to $\mathfrak{h}$. It follows that Λ^* is a simple algebra over K_0 and, in fact, a simple ring with minimum condition. Thus any Λ^*-module is isomorphic to a direct sum of minimal left ideals of Λ^* and all minimal left ideals of Λ^* are isomorphic. In particular, if K is made a Λ^*-module by defining $(x_1 + x_2 a)x = x_1 x + x_2 \bar{x}$ where $x \in K$ and $x_1 + x_2 a \in \Lambda^*$, we see that K, being a field, is an irreducible Λ^*-module. It follows that any Λ^*-module is isomorphic to a direct sum of copies of K, that is, there exists a K-basis for I^*, (e_1, e_2), such that $I^* \cong Ke_1 \oplus Ke_2$. Let $I_2 = I \cap Ke_2$. I_2 is invariant under the action of a and is thus a Λ-submodule of I having R-rank one. By Proposition 1.3 I_2 is isomorphic to an ambiguous R-ideal in K. I/I_2, considered as the quotient of two Λ-modules, is a Λ-module of R-rank one and hence is isomorphic to an ambiguous R-ideal in K. As such it is Λ-projective. It follows that the exact sequence of Λ-modules,

$$0 \to I_2 \to I \to I/I_2 \to 0$$

splits and I/I_2 is isomorphic to a direct summand of I. Hence I is isomorphic to a direct sum of two ambiguous R-ideals in K. We have shown

THEOREM 1.1. *Every ideal I in Λ is Λ-isomorphic to either an ambiguous R-ideal in K or a direct sum of two ambiguous R-ideals in K, depending on whether I has R-rank one or two.*

Let us now characterize ambiguous R-ideals in K.

DEFINITION. Two ideals A and B in K will be called real-equivalent if and only if there exists an $\alpha \in K_0$ such that $A = B\alpha$.

Real-equivalence is an equivalence relation on the set of ambiguous R-ideals in K. We have immediately

LEMMA 1.1. *Two ambiguous ideals in K yield isomorphic ideals in Λ if and only if they are real-equivalent.*

Proof. Let A and B be ambiguous R-ideals in K which are Λ-modules under the action $ax = \bar{x}$ for $x \in A$, $ay = \bar{y}$ for $y \in B$. Let ϕ be a Λ-isomorphism of A and B. Since ϕ is an R-isomorphism, it must be given by multiplication by an element $\alpha \in K$, that is, $B = A\alpha$ and $\phi(x) = x\alpha \in B$ for $x \in A$. Isomorphism as Λ-modules implies α is real since $a\phi(x) = \phi(ax)$ if and only if $\bar{x}\alpha = \overline{x\alpha}$, that is, if and only if $\alpha = \bar{\alpha}$ which implies $\alpha \in K_0$. The converse is trivial.

Since for any ideal $A \subset K$ we may find an element $z \in S \subset K_0$ such that $Az \subset R$, we may now restrict our attention to ambiguous ideals in R.

LEMMA 1.2. *An ideal in R is ambiguous if and only if it can be written in the form* $(1-\theta)^\varepsilon WR$ *where W is an ideal in* R_0 *and* $\varepsilon = 0$ *or* 1.

Proof. Let $A \subset R$ be an ambiguous ideal and consider its factorization into prime ideals in R. If P is a prime ideal and $P \mid A$, then $\bar{P} \mid A$, and we have the following two possibilities:

(i) $P \neq \bar{P}$. In this case P and $\bar{P}$ occur to the same exponent in the factorization of A, so that A has a factor $(\bar{P}P)^e$ for some integer $e > 0$. We can write $\bar{P}P = VR$ for some ideal $V \subset R_0$.

(ii) $P = \bar{P}$. Then since $\bar{P}P = VR$ for some ideal $V \subset R_0$, $P^2 = VR$ and V cannot have more than one type of prime ideal divisor in R_0. If V is not prime in R_0, then $V = W^2$ where $W \subset R_0$, W is a prime ideal and $P = WR$. If, on the other hand, V is prime in R_0, $VR = P^2$ implies that V ramifies from K_0 to K. The only prime which so ramifies is p, whence $P = (1-\theta)R$ and $P^2 = VR$.

Combining (i) and (ii) establishes the lemma in one direction.

Conversely, for any $Y \subset R_0$, $Y = \bar{Y}$. Then $YR = \overline{YR}$ and since $(1-\theta)/(1-\bar{\theta})$ is a unit in R, $\overline{(1-\theta)YR} = (1-\bar{\theta})YR = (1-\bar{\theta})\cdot[(1-\theta)/(1-\bar{\theta})]YR$ implies that $\overline{(1-\theta)YR} = (1-\theta)YR$.

We note that $(1-\theta)^\varepsilon YR$ and $(1-\theta)^\varepsilon XR$ are real-equivalent for $\varepsilon = 0$ or $\varepsilon = 1$ if and only if X and Y are in the same ideal class of R_0, and further that XR and $(1-\theta)YR$ are never real-equivalent for any ideals X and $Y \subset R_0$. We thus have

THEOREM 1.2. *There are precisely* $2h$ *nonisomorphic, indecomposable,* Λ*-modules of R-rank* 1. *These arise from the ambiguous ideals of R where h is the ideal class number of* R_0.

If $\{U_i : 1 \leqq i \leqq h\}$ is a complete set of representatives of the h distinct ideal classes of R_0, then $\{U_iR,\ (1-\theta)U_iR : 1 \leqq i \leqq h\}$ is a complete set of representatives of the classes of real-equivalent ambiguous R-ideals in K. We note we may choose the set of U_i for $i = 1, \cdots, h$ such that $U_i \dotplus U_j = R_0$ for $i \neq j$. Further, since $(1-\theta)U_iR = (\bar{\theta}-\theta)U_iR$, we may choose our $2h$ nonisomorphic, indecomposable, Λ-modules to be given by U_iR and $(\bar{\theta}-\theta)U_iR$ for $1 \leqq i \leqq h$ where $a \cdot u = \bar{u}$ for $u \in U_iR$ and $a(\bar{\theta}-\theta)u = -(\bar{\theta}-\theta)\bar{u}$ for $(\bar{\theta}-\theta)u \in (\bar{\theta}-\theta)U_iR$.

Our above remarks have already established

PROPOSITION 1.4. *If I and J are ideals in* Λ *of R-rank one, then* $I \cong (\bar{\theta}-\theta)^{\varepsilon_i}U_iR$ *and* $J \cong (\bar{\theta}-\theta)^{\varepsilon_j}U_jR$, $1 \leqq i, j \leqq h$, *where* ε_i *and* ε_j *are each either* 0 *or* 1. *I and J are* Λ*-isomorphic if and only if* $i = j$.

LEMMA 1.3. *If* U_i *and* U_j *are representatives of distinct ideal classes of* R_0, $(\bar{\theta}-\theta)^{\varepsilon_i}U_iR \dotplus (\bar{\theta}-\theta)^{\varepsilon_j}U_jR \cong (\bar{\theta}-\theta)^{\varepsilon_i}R \dotplus (\bar{\theta}-\theta)^{\varepsilon_j}U_iU_jR$ *where* ε_i *and* ε_j *may each be taken to be either* 0 *or* 1.

Proof. U_i and U_j may be chosen such that $U_i \dotplus U_j = R_0$. Then there exist

$\alpha \in U_i$ and $\beta \in U_j$ such that $\alpha + \beta = 1$. The map ϕ defined by $\phi(x,y) = (x + y, \beta x - \alpha y)$ for $(x,y) \in (\bar\theta - \theta)^\varepsilon U_i R \dotplus (\bar\theta - \theta)^\varepsilon U_j R$ where ε is fixed as 0 or 1 is a Λ-isomorphism of $(\bar\theta - \theta)^\varepsilon U_i R \dotplus (\bar\theta - \theta)^\varepsilon U_j R$ and $(\bar\theta - \theta)^\varepsilon R \dotplus (\bar\theta - \theta)^\varepsilon U_i U_j R$. If $\varepsilon_i \neq \varepsilon_j$, since $(\bar\theta - \theta)^2 U_j$ is a member of the same ideal class of R_0 as U_j, we can choose U_i such that $U_i \dotplus (\bar\theta - \theta)^2 U_j = R_0$. Then there are $\alpha \in U_i$ and $\beta \in (\bar\theta - \theta)^2 U_j$ such that $\alpha + \beta = 1$. The map ϕ of $U_i R \dotplus (\bar\theta - \theta) U_j R$ onto $R \dotplus (\bar\theta - \theta) U_i U_j R$ given by $\phi(x,y) = (x + y, \beta x - \alpha y)$ for $x \in U_i R$ and $y \in (\bar\theta - \theta) U_j R$ is a Λ-isomorphism of the two direct sums.

We remark at this point that if S is the semilocal ring Z_{2p}, then R_0 and R, being dedekind domains with only finitely many prime ideals, are principal ideal domains and $h = 1$. In light of this remark Lemma 1.3 is trivially true for the case where $S = Z_{2p}$.

LEMMA 1.4. *If $M \cong \sum_{i=1}^n (\bar\theta - \theta)^{\varepsilon_i} U_i R$ where each $\varepsilon_i = 0$ or 1 and U_i is an ideal in R_0, then the class of $\prod_{i=1}^n U_i$ in R_0 is an invariant of M.*

Proof. Considering $\mathrm{Hom}_\Lambda(R,M)$ as an R_0-module, we see that $\mathrm{Hom}_\Lambda(R,M) \cong \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_n$ where $\mathfrak{A}_i$, $1 \leq i \leq n$, are ideals in R_0 and the class of $\prod_{i=1}^n \mathfrak{A}_i$ in R_0 is an invariant of $\mathrm{Hom}_\Lambda(R,M)$. On the other hand, Hom is an additive functor so that

$$\mathrm{Hom}_\Lambda(R,M) \cong \sum_{i=1}^n \mathrm{Hom}_\Lambda(R,(\bar\theta - \theta)^{\varepsilon_i} U_i R).$$

If $f \in \mathrm{Hom}_\Lambda(R,(\bar\theta - \theta)^{\varepsilon_i} U_i R)$, f is determined by $f(1) \in (\bar\theta - \theta)^{\varepsilon_i} U_i R$. Since $af(1) = f(a \cdot 1) = f(1)$; $f(1) \in K_0$ hence $\mathrm{Hom}_\Lambda(R,(\bar\theta - \theta)^\varepsilon U_i R) \cong (\bar\theta - \theta)^\varepsilon U_i R \cap K_0$ under the mapping $f \to f(1)$. But $(\bar\theta - \theta)^{\varepsilon_i} U_i R \cap K_0 = \rho_i R_0 U_i$ where $\rho_i R_0$ is a principal ideal in R_0. Thus $\mathrm{Hom}_\Lambda(R,M) \cong \sum_{i=1}^n \rho_i R_0 U_i$. It follows that the class of $\prod \mathfrak{A}_i$ in R_0 is the same as the class of $\prod U_i$ in R_0 and the class of $\prod U$ in R_0 is an invariant of M.

We note that if $M = \Lambda$, $f \in \mathrm{Hom}_\Lambda(R,\Lambda)$ is determined by $f(1) = r + \bar r a$ for $r \in R$. The mapping $r \to r + \bar r a = f(1)$ is an isomorphism of $\mathrm{Hom}_\Lambda(R,\Lambda)$ and R as R_0-modules. Since $af(1) = f(a \cdot 1) = f(1)$, $\mathrm{Hom}_\Lambda(R,\Lambda) \cong R \cap K_0 = R_0$. It follows that the class of principal ideals in R_0 is an invariant of Λ, that is,

$$\Lambda \cong (\bar\theta - \theta)^{\varepsilon_1} R \dotplus (\bar\theta - \theta)^{\varepsilon_2} R \tag{1.2}$$

where each of $\varepsilon_1, \varepsilon_2$ are 0 or 1.

It is now clear that any R-projective Λ-module of R-rank n is isomorphic as a Λ-module to a direct sum of ambiguous ideals in R of the two types $U_i R$ and $(\bar\theta - \theta) U_i R$. Note that if we choose basis elements e_1 and e_2 such that $ae_1 = e_1$ and $ae_2 = -e_2$, we may replace $U_i R$ and $(\bar\theta - \theta) U_i R$ by the modules $U_i R e_1$ and $U_i R e_2$ where a acts semi-linearly on an element of $U_i R$. The action of a considered as a semi-linear transformation on a Λ-module M having v factors of the type U_iRe_1 and $n-v$ of the type U_iRe_2 is given by the diagonal matrix $\mathbf{M} = \mathrm{diag}[I_v, -I_{n-v}]$ where I_v is the $v \times v$ identity matrix and I_{n-v} is the $n-v \times n-v$ identity matrix. We must determine when the two R-projective Λ-modules M and N are isomorphic. Clearly, since isomorphism as Λ-modules implies isomorphism as R-modules, M and N must have the same R-rank n. Lemma 1.4 tells us the class of $\prod U_{i_v}$ in R_0 is the same for M and N. Now let v and u be the numbers of summands of type U_iRe_1 in M and N, respectively. Let $\mathbf{M} = \mathrm{diag}[I_v, -I_{n-v}]$ and $\mathbf{N} = \mathrm{diag}[I_u, -I_{n-u}]$ and suppose $u \neq v$. M is Λ-isomorphic to N if and only if there exists a unimodular matrix $\mathbf{C}$ over R such that $\bar{\mathbf{C}}\mathbf{M}\mathbf{C}^{-1} = \mathbf{N}$ where $\bar{\mathbf{C}} = [\bar{\gamma}_{ij}]$ if $\mathbf{C} = [\gamma_{ij}]$. Let P be the maximal prime ideal in the local ring R_P, the integral closure of Z_p in K. $(\bar{\theta} - \theta)$ is not a unit in R_P, whence $\bar{\theta} \equiv \theta \bmod P$, and $\bar{\mathbf{C}} \equiv \mathbf{C} \bmod P$. If $\mathbf{C}$ is unimodular over R, $\mathbf{C}$ is unimodular over R_P and $\mathbf{C}\mathbf{M}\mathbf{C}^{-1} \equiv \mathbf{N} \bmod P$ where $\mathbf{C}$ is unimodular over R_P/P. But R_P/P is a field of characteristic $p \neq 2$ and such a $\mathbf{C}$ cannot exist. Hence, M is not Λ-isomorphic to N. It follows that the number of summands of M of type U_iR is an invariant.

Consolidating the results of this section we see that we have established

THEOREM 1.3. *If M is any R-projective Λ-module of R-rank n,*

$$M \cong \sum_{v=1}^{v} U_{i_v}R \dotplus \sum_{\mu=1}^{n-v} (\bar{\theta} - \theta) U_{i_\mu}R,$$

where U_{i_v}, U_{i_μ} are ideals in R_0 and the action of a is given by conjugation. M is determined up to Λ-isomorphism by n, v, and the ideal class of $(\prod_v U_{i_v})(\prod_\mu U_{i_\mu})$ in R_0.

2. **Indecomposable SG-modules.** Let G be the dihedral group generated by a and b under the defining relations $a^2 = b^p = 1$ and $ab = b^{p-1}a$. We note that SG is the twisted group ring $S[b] + S[b]a$. Taking $\Phi_p(X)$ to be the cyclotomic polynomial of degree $p-1$ and $R = S[\theta]$, we see that the correspondence $b \to \theta$ induces an SG-isomorphism between $SG/\Phi_p(b)SG$ and $R + Ra = \Lambda$, where b acts on Λ as multiplication by θ and $a\lambda = \bar{\lambda}a$ for $\lambda \in \Lambda$.

Let M be any finitely generated, S-torsion free, SG-module. Define $M_0 = \{m \in M : \Phi_p(b)m = 0\}$. M_0 is a pure SG-submodule of M annihilated by $\Phi_p(b)$ and we can therefore consider M_0 as a Λ-module. Being a finitely generated, R-torsion free Λ-module, M_0 is Λ-projective. It follows from §1 that M_0 is Λ-isomorphic, and hence SG-isomorphic, to a direct sum of ambiguous ideals in R, $M_0 \cong A_1 \dotplus \cdots \dotplus A_n$ where $A_i = (\bar{\theta} - \theta)^{\varepsilon} U_iR$ for $\varepsilon = 0$ or 1 and a and b act on A_i by conjugation and multiplication by θ, respectively. M_0 is determined up to SG-isomorphism by the number of ideals of each of the two types U_iR and $(\bar{\theta} - \theta)U_jR$, and the ideal class of $\prod_{i=1}^{n} Zl_i$ in R_0.

On the other hand, since $(b-1)$ annihilates M/M_0, M/M_0 is an $S[a]$-module.

It follows from [11] that $M/M_0 \cong S^{(r)} \dotplus S'^{(s)} \dotplus L^{(t)}$, where S, S' and L are defined as SG-modules by

$$S: \quad ax = x, \ x \in S,$$
$$S': \quad \{x \in S\} \quad \text{with } ax = -x \text{ for } x \in S',$$
$$L: \quad \{(x_1e_1 + x_2e_2): \ x_i \in S\} \quad \text{with } ax_1e_1 = x_1e_2, \ ax_2e_2 = x_2e_1,$$

the action of b being trivial. M/M_0 is determined up to SG-isomorphism by the numbers (r), (s) and (t) of each type of summand.

It is readily seen that the problem of classifying SG-modules reduces to one of determining the extensions of $S^{(r)} \dotplus S'^{(s)} \dotplus L^{(t)}$ by $A_1 \dotplus \cdots \dotplus A_n$.

For any pair of SG-modules X and Y, we can obtain from the S-module $X \dotplus Y$, an SG-module denoted by $(X, Y; F)$ by choosing a pair of homomorphisms $F_g \in \mathrm{Hom}_S(Y, X)$ such that $g(x, y) = (gx + F_g(y), gy)$ where $g = a, b$. The pair (F_a, F_b) determine a map $F \in \mathrm{Hom}_S(SG, \mathrm{Hom}_S(Y, X))$ which will be called a binding homomorphism of X and Y. Clearly, due to the defining relations of G, an $F \in \mathrm{Hom}_S(SG, \mathrm{Hom}_S(Y, X))$ is a binding homomorphism if and only if

$$\text{(i)} \qquad aF_a(y) + F_a(ay) = 0,$$

$$\text{(2.1) (ii)} \qquad \sum_{i=0}^{p-1} b^{p-1-i}F_b(b^i y) = 0,$$

$$\text{(iii)} \qquad aF_b(y) + F_a(by) = b^{p-1}F_a(y) - b^{p-1}F_b(b^{p-1}ay),$$

for $y \in Y$. The totality of all binding homomorphisms of X and Y $B(Y, X)$ is an additive subgroup of $\mathrm{Hom}_S(SG, \mathrm{Hom}_S(Y, X))$.

Definition. If X and Y are SG-modules and F, $F' \in B(Y, X)$, we shall say F and F' are strongly equivalent, denoted by $F \approx F'$, if there exists an $E \in \mathrm{Hom}_S(Y, X)$ such that $F_g(y) - F'_g(y) = gE(y) - Eg(y)$ for all $y \in Y$, and $g \in G$. We will say F and F' are equivalent, denoted by $F \sim F'$ if $(X, Y; F) \cong {}_{SG}(X, Y; F')$.

Clearly, $F \approx F'$ implies $F \sim F'$. We remark further that if $(X, Y; F)$ is an SG-module with $F \approx 0$, then $(X, Y; F) \cong X \dotplus Y$ (SG-direct sum).

We refer the reader to [13] for the proof of the following

Proposition 2.1. *Let X and Y be arbitrary SG-modules and $F, F' \in B(Y, X)$. If there exist SG-isomorphisms α of X onto X and β of Y onto Y such that $\alpha F \approx F'\beta$, then $F \sim F'$. Further, if* $\mathrm{Hom}_{SG}(X, Y) = 0$, *the converse is also true.*

Strong equivalence is an equivalence relation under which $B(Y, X)$ may be partitioned into classes of strongly equivalent binding homomorphisms. These classes form an S-module customarily denoted by $\mathrm{Ext}^1_{SG}(Y, X)$. In order to determine the extensions of M/M_0 by M_0, we shall first consider separately the extensions of S, S', and L by A_i. We shall adopt the notation Hom and Ext for Hom_{SG} and Ext^1_{SG}. Further, since in considering $A_i = (\bar{\theta} - \theta)^e U_i R$, the class of U_i in R_0 is of no consequence, we shall merely write A_i or A_i', depending on whether $\varepsilon = 0$ or $\varepsilon = 1$. Note that for $x \in A_i$, $ax = \bar{x}$, while for $x \in A_i'$, $ax = -\bar{x}$.

Treating SG as a left SG-module we obtain the exact sequences

$$\text{(i)} \qquad 0 \to I \xrightarrow{\psi_1} SG \xrightarrow{\phi_1} S \to 0,$$

$$\text{(2.2) (ii)} \qquad 0 \to I' \xrightarrow{\psi_2} SG \xrightarrow{\phi_2} S' \to 0,$$

$$\text{(iii)} \qquad 0 \to SG(b-1) \xrightarrow{\psi_3} SG \xrightarrow{\phi_3} L \to 0.$$

(i) is obtained by taking $\phi_1 : SG \to S$ to be defined by $\phi_1(a) = \phi_1(b) = 1$. It is easily verified that $I = SG(b-1) + S(a-1)$. Taking $\phi_2(a) = -1$, $\phi_2(b) = 1$, we see that $I' = SG(b-1) + S(a+1)$. To obtain (iii) we observe that if $Y = S\Phi_p(b) + Sa\Phi_p(b)$, then $\tau(\Phi_p(b)) = e_1, \tau(a\Phi_p(b)) = e_2$ is an SG-isomorphism of Y and L. Taking $\eta : SG \to Y$ given by $\eta(z) = z\Phi_p(b)$ for $z \in SG$, we see that $SG(b-1)$ is the kernel of η and

$$0 \to SG(b-1) \xrightarrow{\psi_3} SG \xrightarrow{\eta} Y \to 0$$

is exact. We need only take $\phi_3 = \tau\eta$ to obtain (iii).

LEMMA 2.1. *There exist S-isomorphisms*
(i) $\mathrm{Ext}(S, A_i) \cong (0) \cong \mathrm{Ext}(S', A_i')$,
(ii) $\mathrm{Ext}(L, A_i) \cong A_i/(\theta - 1)A_i \cong \mathrm{Ext}(S', A_i)$,
(iii) $\mathrm{Ext}(L, A_i') \cong A_i'/(\theta - 1)A_i' = \mathrm{Ext}(S, A_i')$.

Proof. (We shall prove the lemma only for $\mathrm{Ext}(S, A_i)$ and $\mathrm{Ext}(S, A_i')$. The proofs of the other results are similar.) Since SG is a free SG-module, we obtain from (2.2) the exact sequences

$$\text{(i)} \quad \cdots \to \mathrm{Hom}(SG, A_i) \xrightarrow{\psi_1^*} \mathrm{Hom}(I, A_i) \to \mathrm{Ext}(S, A_i) \to 0,$$

$$\text{(i')} \quad \cdots \to \mathrm{Hom}(SG, A_i') \xrightarrow{\psi_1^*} \mathrm{Hom}(I, A_i') \to \mathrm{Ext}(S, A_i') \to 0,$$

where ψ_1^* arises from the inclusion map ψ_1 in (2.2) by $(\psi_1^* f)x = f(\psi_1(x))$ for $x \in I$ and $f \in \mathrm{Hom}(SG, A_i)$ or $\mathrm{Hom}(SG, A_i')$, as the case may be. It follows that $\mathrm{Ext}(S, A_i) \cong \mathrm{Hom}(I, A_i)/\text{image of } \psi_1^*$ in (i) and $\mathrm{Ext}(S, A_i') \cong \mathrm{Hom}(I, A_i)/\text{image of}$ ψ_1^* in (i'). An $f \in \mathrm{Hom}(I, A_i')$ is determined by its action on $(a-1)$ and $(b-1)$. If $f(a-1) = x$ and $f(b-1) = y$ where $x, y \in A_i'$, then $(b-1)(f(a-1)) = f((b-1)(a-1))$ and $(b-1)(a-1) = [a(b^{p-2} + b^{p-3} + \ldots + 1) - 1](b-1)$ imply that $(\theta - 1)x = \theta\bar{y} - y$. It follows that x depends wholly on the choice of y, enabling us to specify f by specifying $f(b-1) = y$. Further, since $y \equiv \bar{y} \bmod (1-\theta)A_i'$, we have $y(\theta - 1) \equiv 0 \bmod (1-\theta)A_i'$ and we see that our choice of y may be arbitrary in A_i'. By associating f with $f(b-1) = y$ we obtain $\mathrm{Hom}(I, A_i') \cong A_i'$. Now consider $\psi_1^*(\mathrm{Hom}(SG, A_i'))$. Certainly $\mathrm{Hom}(SG, A_i') \cong A_i'$. If $h \in \mathrm{Hom}(SG, A_i')$,

$(\psi_1^* h)(b-1) = h(b-1) = (b-1)h(1)$ and the image of ψ_1^* in (i′) is isomorphic to $(b-1)A_i'$. Since b acts as multiplication by θ, $\mathrm{Ext}(S,A_i') \cong A_i'/(\theta-1)A_i'$.

Replacing A_i' by A_i and again using the relationship $(b-1)(f(a-1)) = f((b-1)(a-1))$ for $f \in \mathrm{Hom}(I,A_i)$, we obtain $(\theta-1)x = -\theta\bar{y} - y$. Then $y(1+\theta) \equiv 0 \bmod (\theta-1)A_i$ from which it follows that $y \in (\theta-1)A_i$ and $\mathrm{Hom}(I,A_i) \cong (\theta-1)A_i$. Thus $\mathrm{Ext}(S,A_i) \cong (\theta-1)A_i/(\theta-1)A_i \cong (0)$.

Since $A_i'/(\theta-1)A_i' \cong A_i/(\theta-1)A_i \cong S/pS$, we see that the number of extensions of S, S' or L by A_i or A_i' is, in all cases, either 1 or p. If there is only one extension we have a decomposable SG-module. In the case where the number of extensions is p, taking representatives of $A_i'/(\theta-1)A_i'$ to be given by $\{jn_0 : 0 \leqq j \leqq p-1,\ n_0 \in A_i', n_0 \notin (\theta-1)A_i'\}$, we shall denote the representatives of the p inequivalent classes of binding homomorphisms so obtained by $F^{(j)}$, $j = 0,\cdots,p-1$. We now consider $(A_i',S;F^{(j)})$ for $j = 0,\cdots,p-1$. An $F \in B(S,A_i')$ is determined by the action of the pair (F_a,F_b) on $1 \in S$. From (2.1(iii)) and the defined actions of a and b on S and A_i', we have $\overline{-F_b(1)} + \bar{\theta}F_b(1) = (\bar{\theta}-1)F_a(1)$. It follows that $F_a(1)$ and hence F_1 is determined by $F_b(1)$. We shall choose $F_b^{(j)}(1) = jn_0$ for $j = 0,\cdots,p-1$ and show that $F^{(j)} \sim F^{(1)}$ for $0 \leqq j \leqq p-1$. Both S and A_i' are irreducible SG-modules, hence $\mathrm{Hom}(S,A_i') = 0$ and, by Proposition 2.1, we need only find SG-automorphisms α and β of A_i' and S such that $\alpha F^{(1)} \approx F^{(j)}\beta$. This will be the case if and only if $(\alpha F^{(1)} - F^{(j)}\beta)(1) \in (\theta-1)A_i'$. Take β to be the identity automorphism of S and α to be left multiplication by the $u = (x\bar{x})^{1/2}$ with $x = (\theta^j - 1)/(\theta-1)$ so that u is a unit of R_0 and hence of R. Then $\alpha F_b^{(1)}(1) - F_b^{(j)}\beta(1) = (u-j)n_0$ which, since $u \equiv \bmod(\theta-1)$ implies that $(u-j)n_0 \in (\theta-1)A_i'$ for $0 < j \leqq p-1$. It follows that there exists up to isomorphism at most one indecomposable module arising from an extension of S by A_i'.

To establish that $(A_i',S;\ F^{(1)})$ is indeed indecomposable we note that by Proposition 2.1 if $F^{(1)} \sim F^{(0)}$ there would exist SG-automorphisms α and β of A_i' and S such that $(\alpha F_b^{(1)} - F_b^{(0)}\beta)(1) \in (\theta-1)A_i'$. But $F^{(0)} \approx (0)$ implies that $\alpha(n_0) \in (\theta-1)A_i'$. Since α must be multiplication by a unit of R, $n_0 \in (\theta-1)A_i'$ a contradiction of our original choice of $F^{(1)}$. We have shown

Proposition 2.2. *There exists one indecomposable SG-module arising from an extension of S by A_i'. This module, denoted by $(A_i',S;F)$, is defined by $\{(x,y): x \in A_i', y \in S\}$ where the action of G is given by $a(x,y) = (\bar{x} + F_a(y),y)$, $b(x,y) = (\theta x + F_b(y),y)$.*

$$F_b(y) = yn_0,\ F_a(y) = y(-\bar{n}_0 + \bar{\theta}n_0)/(\bar{\theta}-1) \textit{ for } n_0 \in A_i',\ n_0 \notin (\theta-1)A_i'.$$

In a similar manner, employing the same automorphisms α and β, we can show the existence of precisely one indecomposable SG-module of each of the types $(A_i,S';F)$, $(A_i,L;F)$ and $(A_i',L;F)$. In the case of the last two types we need only note the class of $F^{(j)}$, $0 \leqq j \leqq p-1$, in $B(L,A_i)$ or $B(L,A_i')$ under strong equivalence is uniquely determined by $(F_a^{(j)}(e_1), F_b^{(j)}(e_2)) = (jn_0, jn_0)$ where $n_0 \in A_i$, $n_0 \notin (\theta - 1)A_i$ and $n_0 \notin P_i$ for any prime ideal factor P_i of $2R$. In view of the existence of only one indecomposable module for each extension of S, S' or L by A_i or A_i', we shall hereafter drop the F and refer to the nontrivial extension only by the pair of modules involved.

We shall now determine the extensions of M/M_0 by M_0 which yield indecomposable SG-modules M. Note that if M is any finitely generated Z-free ZG-module, we can form the associated $Z_{2p}G$-module $M_{2p} = Z_{2p} \otimes_Z M$. When $S = Z_{2p}$, the class number h of R_0 is one and $A_i = R$, $A_i' = (\bar{\theta} - \theta)R = R'$ for $1 \leqq i \leqq h$. In this case M_0 simplifies to the form $R^{(u)} \dot{+} R'^{(v)}$. Since a theorem due to Reiner [14] tells us that M is a decomposable ZG-module if and only if M_{2p} is decomposable as a $Z_{2p}G$-module, we shall for the remainder of this section, except where it is expressly stated to the contrary, assume $S = Z_{2p}$.

Let M be an indecomposable SG-module. M is the extension of $S^{(s)} \dot{+} S'^{(t)} \dot{+} L^{(w)}$ by $R^{(u)} \dot{+} R'^{(v)}$. Since M is indecomposable, we cannot have all of (s), (t), and (w) equal to 0 unless one of (u) and (v) is 0 and the other is 1. Similarly if $(v) = (u) = 0$ one and only one of (s), (t) and (w) is equal to 1 and the other two are 0. Assuming now that neither all of (s), (t) and (w) nor all of (u) and (v) are 0, we have

Case 1. $w \neq 0$. There are two subcases: (i) $s = t = 0$ and (ii) either one or both of s and t are nonzero.

(i) If $s = t = 0$, M arises from the extension of $L^{(w)}$ by $R^{(u)} \dot{+} R'^{(v)}$. Letting $\sum_{i=1}^{w} SGx_i$ be a free SG-module with basis $\{x_1, \cdots, x_w\}$, and adding w copies of the exact sequence (2.2.(iii)), we obtain the exact sequence

$$0 \to \sum_{i=1}^{w} SG(b-1)x_i \xrightarrow{\tau} \sum_{i=1}^{w} SGx_i \to \sum_{i=1}^{w} Lx_i \to 0.$$

Then $\operatorname{Ext}(L^{(w)}, R^{(u)} \dot{+} R'^{(v)}) \cong \operatorname{Hom}(\sum SG(b-1)x_i, R^{(u)} \dot{+} R'^{(v)})/\text{image of } \tau^*$. Let $\{a_1, \cdots, a_u\}$ and $\{b_1, \cdots, b_v\}$ be bases for $R^{(u)}$ and $R'^{(v)}$ respectively such that $R^{(u)} = Ra_1 \oplus \cdots \oplus Ra_u$ and $R'^{(v)} = R'b_1 \oplus \cdots \oplus R'b_v$. An

$$F \in \operatorname{Hom}(\sum SG(b-1)x_i, R^{(u)} \dot{+} R'^{(v)})$$

is given by

$$F((b-1)x_i) = \sum_{j=1}^{u} r_{ji}a_j + \sum_{k=1}^{v} r'_{ki}b_k, \qquad 1 \leqq i \leqq w,\ r_{ji} \in R,\ r'_{ki} \in R'.$$

The class of F in $\operatorname{Ext}(L^{(w)}, R^{(u)} \dot{+} R'^{(v)})$ corresponds to a pair of matrices $F_\rho = (\rho_{ji})$ and $F'_\rho = (\rho'_{ki})$ where the entries ρ_{ji} and ρ'_{ki} are in $\operatorname{Ext}(L, R)$ and $\operatorname{Ext}(L, R')$, respectively. In particular, since there is, up to isomorphism, only one indecomposable module arising from each extension, ρ_{ji} and ρ'_{ki} can be taken to be either 0 or 1.

A change of basis of $R^{(u)}$, leaving a_j fixed for some $j \neq 1$ and replacing a_1

by $a_1 - \lambda a_j$, will replace ρ_{ji} by $(\rho_{ji} - \lambda\rho_{1i})$, $1 \leqq i \leqq w$. On the other hand, since, a change of basis of $L^{(w)}$ leaving $x_1, x_3, \cdots, x_w$ unchanged, but replacing x_2 by $x_2 - \lambda x_1$, replaces $(b-1)x_2$ by $(b-1)x_2 - \lambda(b-1)x_1$ and hence ρ_{j2} and ρ'_{k2} by $\rho_{j2} - \lambda\rho_{j1}$ and $\rho'_{k2} - \lambda\rho'_{k1}$, respectively. We will identify F with its class in $\mathrm{Ext}(L^{(w)}, R^{(u)} \dotplus R'^{(v)})$ and speak of $F(x_i)$ rather than $F((b-1)x_i)$.

Consider first the $(u \times w)$ matrix $F_\rho = (\rho_{ji})$. There must be a nonzero element $\rho_{1i} = 1$ in the first row of F_ρ, since otherwise a factor of Ra_1 would split off and M would be decomposable. Renumber the basis elements of $L^{(w)}$ if necessary, to place this element in the $(1,1)$ position. We may assume hereafter $\rho_{11} = 1$. A change of basis of $R^{(u)}$ which results in replacing ρ_{j1} by $\rho_{j1} - \lambda\rho_{11}$ where $\lambda = \rho_{j1}$ is either 1 or 0 for $2 \leqq j \leqq u$ can be performed. F_ρ now will have all entries in its first column, with the exception of ρ_{11}, equal to 0. By changing the basis of $L^{(w)}$ we may now make the $(1,2), \cdots, (1,w)$ entries 0. Repeating this process we may diagonalize F_ρ to obtain $F_\rho = \mathrm{diag}[I_m, 0]$. If $m < u$, a factor $Ra_{m+1} \oplus \cdots \oplus Ra_u$ would be a direct summand of M, contradicting the indecomposability of M. Therefore, we may assume $m = u$, and

$$\text{(2.3)}\qquad \begin{aligned} F(x_i) &= a_i + \sum_{k=1}^{v} \rho'_{ki} b_k, \qquad 1 \leqq i \leqq u, \\ F(x_i) &= \sum_{k=1}^{v} \rho'_{ki} b_k, \qquad u+1 \leqq i \leqq w. \end{aligned}$$

We note that although the diagonalization process will change the values o coefficients of the b_k's, these coefficients are elements of $\mathrm{Ext}(L, R')$ and, as such, may be taken to be 0 and 1; thus we retain the notation ρ'_{ki} for these coefficients

Now consider $F_{\rho'}$. If $v = 0$ (2.3) tells us that $M = (R, L)^{(u)} \oplus L^{(w-u)}$ contradicting the indecomposability of M. Thus assume $v \neq 0$. There exists a nonzero entry in the last column of $F_{\rho'}$ since otherwise L or (R, L) would be a direct summand of M, depending on whether $u < w$ or $u = w$. We may renumber the basis elements $b_1, \cdots, b_v$ such that $\rho'_{1w} = 1$. A change of basis of $R'^{(v)}$, replacing b_1 by $b_1 - \lambda b_k$ where $\lambda = \rho'_{kw}$, $2 \leqq k \leqq v$ will reduce the entries of the last column of $F_{\rho'}$ to 0 for $2 \leqq k \leqq v$, that is, $F(x_w) = \delta_{uw} a_u + b_1$ where $\delta_{uw} = 0$ if $u < w$, $\delta_{uw} = 1$ if $u = w$. A change of basis of $L^{(w)}$, replacing x_i by $(x_i - \rho'_{1w} x_w)$ for $1 \leqq i \leqq w-1$ will give us

$$\text{(2.4)}\qquad \begin{aligned} F(x_i) &= a_i - \delta_{uw}\rho'_{1i} a_u + \sum_{k=2}^{v} \rho'_{ki} b_k, \qquad 1 \leqq i \leqq u, \\ F(x_i) &= \delta_{uw}\rho'_{1i} a_u + \sum_{k=2}^{v} \rho'_{ki} b_k, \qquad u+1 \leqq i \leqq w-1. \\ F(x_w) &= \delta_{uw} a_u + b_1. \end{aligned}$$

If $u \neq w$, $\delta_{uw} = 0$ and (2.4) becomes

$$F(x_i) = a_i + \sum_{k=2}^{v} \rho'_{ki}b_k, \qquad 1 \leqq i \leqq u,$$

$$F(x_i) = \sum_{k=2}^{v} \rho'_{ki}b_k, \qquad u+1 \leqq i \leqq w-1,$$

$$F(x_w) = b_1,$$

whence a factor (R',L) is a direct summand of M. If $u = w$, $\delta_{uw} = 1$ and (2.4) is given by

$$F(x_i) = a_i - \rho'_{1i}a_n + \sum_{k=2}^{v} \rho'_{ki}b_k, \qquad 1 \leqq i \leqq w-1,$$

$$F(x_w) = aw + b_1.$$

Replacing a_i by $a'_i = a_i - \rho'_{1i}a_n$ for $1 \leqq i \leqq n-1$ and taking $a'_n = a_n$ we finally obtain

$$F(x_i) = a'_i + \sum_{k=2}^{v} \rho'_{ki}b_k, \qquad 1 \leqq i \leqq w-1,$$

$$F(x_w) = a'_n + b_1$$

whence $(L, R \dotplus R')$ is a direct factor of M. In either case M is now decomposable. Thus if M is an indecomposable module obtained by an extension of $L^{(w)}$ by $R^{(u)} \dotplus R'^{(v)}$, we have $\max(u,v,w)=1$. We have already seen M is indecomposable if $w = 1$ and one or both of u, v are equal to 0; or if $w = 0$ and one of u and v is 0. That M is indecomposable when $u = v = w = 1$ follows from the indecomposability of the group ring (cf. [15]) and the fact that SG has S-rank $2p$.

(ii) If $w \neq 0$ and one or both of s and t is nonzero, then M is an extension of $S^{(s)} \dotplus S'^{(t)} \dotplus L^{(w)}$ by $R^{(u)} \dotplus R'^{(v)}$. Noting that by Lemma 2.1 $\mathrm{Ext}(S, R) = \mathrm{Ext}(S', R') = 0$, we see that the class of an $F \in \mathrm{Ext}(S^{(s)} \dotplus S'^{(t)} \dotplus L^{(w)}, R^{(u)} \dotplus R'^{(v)})$ is determined by the four matrices

$$\begin{aligned} F_\tau &= (\tau_{ij})_{(u \times t)}, \\ F_{\tau'} &= (\tau'_{ij})_{(v \times s)}, \\ F_\rho &= (\rho_{ij})_{(u \times w)}, \\ F_{\rho'} &= (\rho'_{ij})_{(v \times w)} \end{aligned} \tag{2.5}$$

with entries in $\mathrm{Ext}(S',R)$, $\mathrm{Ext}(S,R')$, $\mathrm{Ext}(L,R)$ and $\mathrm{Ext}(L,R')$, respectively. In particular, these entries may be taken to be 0 or 1.

We suppose first that M is the indecomposable module arising from an extension of $S \oplus L$ by R'. The matrix representation of M has the form

$$\begin{bmatrix} R' & 1 & 1 \\ & S & 0 \\ & & L \end{bmatrix}.$$

But L is the extension of S' by S [11] whence, noting that Ext $(S', R') = 0$, it follows after suitable manipulation of bases that M is determined by the two matrices $F = (\tau'_{ij}), i = 1, j = 1,2$ and $E = (\rho'_{11})$ with nonzero entries in $\mathrm{Ext}_{SG}(S, R')$ and $\mathrm{Ext}_{S[a]}(S', S)$, respectively; that is, M now has a representation of the form

$$\begin{bmatrix} R' & 1 & 1 & 0 \\ & S & 0 & 0 \\ & & S & 1 \\ & & & S' \end{bmatrix}$$

and F is the matrix corresponding to an extension of $S^{(2)}$ by R'. If $S^{(2)} = Sz_1 \oplus Sz_2$ and $R' = R'b_1$, $F(z_1) = b_1$ and $F(z_2) = b_1$. A change of bases to z'_i where $z'_1 = z_1$ and $z'_2 = z_2 - z_1$ makes $F = (1\ 0)$ whence M becomes $(R', S) \oplus L$. A similar argument will show that the extension of $S' \oplus L$ by R cannot be indecomposable. To return to the more general situation we suppose we have the four matrices indicated in (2.5). Let the bases for $R^{(u)}$, $R'^{(v)}$ and $L^{(w)}$ be as in (i) and take $S^{(s)} = Sy_1 \oplus \cdots \oplus Sy_s$. Then identifying maps with matrices

$$\text{(2.6)} \qquad \begin{aligned} F(y_l) &= \sum_{k=1}^{v} \tau'_{kl} b_k, && 1 \leqq l \leqq s, \\ F(x_j) &= \sum_{i=1}^{u} \rho_{ij} a_i + \sum_{k=1}^{v} \rho'_{kj} b_k, && 1 \leqq j \leqq w. \end{aligned}$$

There exists a nonzero element $\tau'_{k1} = 1$ in the first column of $F_{\tau'}$ since otherwise Sy_1 would be a direct summand of M. Renumbering the b_k such that $\tau'_{11} = 1$ and using the same process as in (i), we may diagonalize $F_{\tau'}$ to obtain, $F(y_l) = b_l$ for $1 \leqq l \leqq m$. If $m < s$, $Sy_{m+1} \oplus \cdots \oplus Sy_s$ would be a direct summand of M. We may therefore assume $m = s$. (2.6) becomes

$$F(y_l) = b_l, \qquad 1 \leqq l \leqq s, \qquad s \leqq v,$$

$$F(x_j) = \sum_{k=1}^{v} \rho'_{kj} b_k + \sum_{i=1}^{u} \rho_{ij} a_i, \qquad 1 \leqq j \leqq w.$$

We note as in (i) that although ρ'_{kj} may change under the diagonalization the values remain 0 or 1. Now consider $F_{\rho'}$. Again the absence of a nonzero element $\rho'_{1k} = 1$ in the first row would cause $(R'b_1, Sy_1)$ to be a direct summand of M.

Renumbering the x_j, if necessary, we place ρ'_{1k} in the $(1,w)$ position. Fixing x_w and replacing x_j by $x_j - \rho_{1j}x_w$ for $1 \leqq j \leqq w-1$, we obtain

$$F(y_l) = b_l,$$

$$F(x_j) = \sum_{k=2}^{v} \rho'_{kj} b_k + \sum_{i=1}^{u} \rho_{ij} a_i, \qquad 1 \leqq j \leqq w-1,$$

$$F(x_w) = b_1 + \sum_{k=2}^{v} \rho_{kw} b_k + \sum_{i=1}^{u} \rho_{iw} a_i.$$

Thus an extension of $S \oplus L$ by R' appears in M. As we have already seen, we may then split off (R', S), making M decomposable.

Case 2, $w = 0$. Then M is an extension of $S^{(s)} \dot{+} S'^{(t)}$ by $R^{(u)} \dot{+} R'^{(v)}$.

The class of an $F \in \mathrm{Ext}(S^{(s)} \dot{+} S'^{(t)}, R^{(u)} \dot{+} R'^{(v)})$ is seen to be given by a pair of matrices $F_\tau = (\tau_{ij})$ and $F_{\tau'} = (\tau'_{ij})$ where $\tau_{ij} \in \mathrm{Ext}(S, R')$ and $\tau'_{ij} \in \mathrm{Ext}(S', R)$. Use of the methods in Case 1 quickly results in the diagonalization of both of these matrices to obtain the result that M is decomposable.

Allowing S to be Z or Z_{2p} and returning to the notation $A'_i = (\bar{\theta} - \theta)U_iR$ $A_i = U_iR$ where U_i is a representative of an ideal class of R_0, we see we have established the existence of five types of indecomposable SG-modules arising from nontrivial extensions of M/M_0 by M_0:

(2.7) (U_iR, S'), (U_iR, L), $((\bar{\theta} - \theta)U_iR, S)$, $((\bar{\theta} - \theta)U_iR, L)$, $(R \dot{+} (\bar{\theta} - \theta)U_jR, L)$.

We can now state the following

PROPOSITION 2.3. *There exist h nonisomorphic, indecomposable, SG-modules of each of the five types listed in* (2.7). *These are obtained by allowing U_i to range through the complete set of representatives of the h ideal classes of R_0.*

Proof. The existence of isomorphisms $(U_iR, S') \cong (U_jR, S')$, $((\bar{\theta} - \theta)U_iR, S) \cong ((\bar{\theta} - \theta)U_jR, S)$ or $((\bar{\theta} - \theta)^\varepsilon U_iR, L) \cong ((\bar{\theta} - \theta)^\varepsilon U_jR, L)$ for $\varepsilon = 0$ or 1 would imply by Proposition 2.1 the existence of SG-isomorphisms and hence of Λ-isomorphisms of $(\bar{\theta} - \theta)^\varepsilon U_iR$ and $(\bar{\theta} - \theta)^\varepsilon U_jR$ for $\varepsilon = 0$ or 1. We have noted in §1 that such isomorphisms will exist if and only if U_i and U_j are in the same ideal class of R_0. Similarly, if $(R \dot{+} (\bar{\theta} - \theta)U_iR, L) \cong (R \dot{+} (\bar{\theta} - \theta)U_jR, L)$ where U_i and U_j are in distinct ideal classes of R_0, we have a Λ-isomorphism of $R \dot{+} (\bar{\theta} - \theta)U_iR$ and $R \dot{+} (\bar{\theta} - \theta)U_jR$. Lemma 1.4 tells us that this is impossible. We now have

THEOREM 2.1. *There exist precisely $7h + 3$ nonisomorphic, indecomposable, SG-modules where h is the ideal class number of R_0. If $\{U_i : i = 1, \cdots, h\}$ is a full set of representatives of ideal classes of R_0, h of these indecomposables come from each of the following types of modules*: U_iR, $(\bar{\theta} - \theta)U_iR$, (U_iR, S'),

$((\bar{\theta}-\theta)U_iR,S)$, (U_iR,L), $((\bar{\theta}-\theta)U_iR,L)$ and $(R \dotplus (\bar{\theta}-\theta)U_iR,L)$, by taking $i=1,\cdots,h$. The additional three modules are S, S' and L.

3. **Nonuniqueness of decomposition.** The decomposition of an SG-module, into indecomposables is certainly nonunique in the case where $S=Z$ and $h \neq 1$ since here we already have by Lemma 1.3

$$U_iR \dotplus U_jR \cong R \dotplus U_iU_jR.$$

Let us therefore consider the situation which occurs when $S=Z_{2p}$. Since $h=1$, there exists only one ideal class of R_0 so that nonuniqueness is not immediate.

If M is any SG-module, we may form the associated Z_pG and Z_2G-modules $M_p = Z_p \otimes_S M$ and $M_2 = Z_2 \otimes_S M$, respectively. For any two SG-modules M and M', $M \cong M'$ if and only if $M_p \cong M'_p$ and $M_2 \cong M'_2$.

Under extension of the ground ring from S to Z_p, L decomposes into the direct sum $Z_p \oplus Z'_p$ and it follows that $(R,L)_p \cong (R_p,Z_p) \oplus Z'_p$, $(R',L)_p \cong (R'_p,Z_p) \oplus Z'_p$ and $(R \dotplus R',L)_p \cong (R_p,Z'_p) \oplus (R_p,Z'_p)$. In extending S to Z_2, we find that although Z_2, Z'_2, L_2, R_2 and R'_2 remain indecomposable, in each case our extensions of Z_2, Z'_2 and L_2 by R_2 and R'_2 split into direct sums of the modules involved. Thus if M is an SG-module which has the decomposition $M \cong (R,L) \oplus (R',L)$ and M' is an SG-module having the decomposition $M' \cong L \oplus (R \dotplus R',L)$,

$$M_p \cong (R_p,Z'_p) \oplus Z_p \oplus (R'_p,Z_p) \oplus Z'_p \cong M'_p$$

and

$$M_2 \cong R_2 \oplus L_2 \oplus R'_2 \oplus L_2 \cong M'_2,$$

whence $M \cong M'$ as SG-modules.

Although the decomposition of SG-modules into sums of indecomposables does not even preserve the S-rank of the summands, we may still obtain certain invariants for a direct sum decomposition. We shall make use of

THEOREM 3.1 (KRULL-SCHMIDT). *In any decomposition of a Z_pG-module M_p into a direct sum of indecomposables, the indecomposable summands are uniquely determined by M_p up to Z_pG-isomorphism and order of occurrence.*

Proof. Let Q^* denote the p-adic completion of Q and Z^* the ring of integral elements in Q^*. For any Z_pG-module M_p, we may form the associated Z^*G-module $M_p^* = Z^* \otimes_{Z_p} M_p$. We have (Maranda [9]; see also [2]).

(3.1) $M_p^* \cong M_p'^*$ as Z^*G-modules if and only if $M_p \cong M'_p$ as Z_pG-modules.

Further, since QR_p, QZ_p, QR'_p and QZ'_p are irreducible QG-modules which remain irreducible under extension to Q^*G-modules, a theorem due to Heller [4] tells us that M_p is decomposable if and only if M_p^* is decomposable as a Z^*G-module, The Krull-Schmidt theorem holds for Z^*G-modules (see [12]). The result now follows from (3.1).

Now consider the following chart, which shall represent a direct sum decomposition of M. The left-hand column gives the number of summands of each type of indecomposable SG-module appearing in the decomposition. The two columns on the right are corresponding decompositions for M_p and M_2.

Number of Summands	M	M_p	M_2
s_1	S	Z_p	Z_2
s_2	S'	Z'_p	Z'_2
l	L	$Z_p \oplus Z'_p$	L_2
r_1	R	R_p	R_2
r_2	R'	R'_p	R'_2
u_1	(R, S')	(R_p, Z'_p)	$R_2 \oplus Z'_2$
u_2	(R', S)	(R'_p, Z_p)	$R'_2 \oplus Z_2$
v_1	(R, L)	$(R_p, Z'_p) \oplus Z_p$	$R_2 \oplus L_2$
v_2	(R', L)	$(R'_p, Z_p) \oplus Z'_p$	$R'_2 \oplus L_2$
t	$(R \dot{+} R', L)$	$(R_p, Z'_p) \oplus (R'_p, Z_p)$	$R_2 \oplus R'_2 \oplus L_2$

Theorem 3.1 tells us that the number of various types of indecomposable summands in a decomposition of M_p is invariant. We thus have as invariants for M_p and hence for M

$$s_1 + l + v_1,\ s_2 + l + v_2,\ r_1,\ r_2,\ u_1 + v_1 + t,\ u_2 + v_2 + t.$$

From the structure of M/M_0 as an $S[a]$-module, we see that the total number each of S, S' and L appearing in summands of M is also an invariant of M, whence we have the additional invariants $s_1 + u_2$ and $s_2 + u_1$. It is a simple exercise to verify that these eight invariants determine M up to SG-isomorphism if $S = Z_{2p}$. We can now easily show, taking S to be either Z or Z_{2p},

THEOREM 3.2. *Any finitely generated, S-free, SG-module M can be written*

$$\begin{aligned} M \cong S^{(s_1)} \dot{+} S'^{(s_2)} \dot{+} L^{(l)} \dot{+} (U_{i_\delta} R)^{(r_1)} \dot{+} ((\bar{\theta} - \theta) U_{i_\varepsilon} R^{(r_2)} \dot{+} (U_{i_\zeta} R, S')^{(u_1)} \\ \dot{+}\ ((\bar{\theta} - \theta) U_{i_\eta} R, S)^{(u_2)} \dot{+} (U_{i_\lambda} R, L)^{(v_1)} \dot{+} ((\bar{\theta} - \theta) U_{i_\mu} R, L)^{(v_2)} \\ \dot{+}\ (R \dot{+} (\bar{\theta} - \theta) U_{i_\nu} R, L)^{(t)}, \end{aligned}$$

where $1 \leqq i \leqq h$, *and the invariants*: $s_1 + l + v_1$, $s_2 + l + v_2$, $u_1 + v_1 + t$, $u_2 + v_2 + t$, $s_2 + u_1$, $s_1 + u_2$, r_1, r_2, *and the ideal class of*

$$(\prod_{\delta} U_{i_\delta})(\prod_{\varepsilon} U_{i_\varepsilon})(\prod_{\eta} U_{i_\eta})(\prod_{\zeta} U_{i_\zeta})(\prod_{\lambda} U_{i_\lambda})(\prod_{\mu} U_{i_\mu})(\prod_{\nu} U_{i_\nu})$$

in R_0 determine M up to $Z_{2p}G$-isomorphism.

4. **The group ring and projective modules.** SG considered as a left SG-module is indecomposable [15] of S rank $2p$ and hence must be a module of the form $(R \dot{+} (\bar{\theta} - \theta)U_i R, L)$. It is necessary only to determine the class of U_i in R_0. Since $SG/\Phi_p(b)SG \cong \Lambda$, we see that $\Lambda \cong R \dot{+} (\bar{\theta} - \theta)U_i R$.

In §1 (1.2) we remarked that the class of ideals in R_0 which is an invariant of Λ is the class of principal ideals in R_0. It follows immediately that $SG \cong (R \dot{+} (\bar{\theta} - \theta)R, L)$.

To simplify the notation throughout the rest of this section, we shall denote $R \dot{+} (\bar{\theta} - \theta)U_i R$ by M_i for $1 \leqq i \leqq h$, having renumbered the U_i, if necessary, such that the ideal class of U_1, $[U_1] = [R_0]$ the class of principal ideals in R_0. Thus $R \dot{+} (\bar{\theta} - \theta)R = M_i$. We shall denote $(R \dot{+} (\bar{\theta} - \theta)U_i, L)$, that is, (M_i, L), by X_i for $1 \leqq i \leqq h$. M_{ij} will be used to indicate $R \dot{+} (\bar{\theta} - \theta)U_i U_j R$ and $X_{ij} = (M_{ij}, L)$.

Let $\mathscr{F}$ denote the class of all finitely generated, free SG-modules and let $\mathscr{P}$ be the class of all finitely generated, projective SG-modules. Then $\mathscr{F} \subset \mathscr{P}$, and we may define an equivalence relation on $\mathscr{P}$ as follows:

Definition. P_1 and P_2 in $\mathscr{P}$ are equivalent if and only if there exist F_1 and F_2 in $\mathscr{F}$ such that $P_1 \dot{+} F_1 \cong P_2 \dot{+} F_2$, as SG-modules.

We shall denote the equivalence class of P in $\mathscr{P}$ by $\{P\}$. By $\{0\}$ we shall mean the set of all $P \in \mathscr{P}$ such that $P \dot{+} F \in \mathscr{F}$ for some $F \in \mathscr{F}$; and by $-\{P\}$, the class of all $P' \in \mathscr{P}_\rho$ such that $P \dot{+} P' \in \mathscr{F}$. The set of classes of $\mathscr{P}$ under this relation form a group called the projective class group. We have

Theorem 4.1. (*Swan* [15]; *see also* [2].) *If P is a projective SG-module, P can be written $P = P_0 \dot{+} F$ where F is a free SG-module and P_0 is a projective ideal of SG.*

If P_0 is a projective ideal of SG, $QP_0 \cong QG$. Then P_0 must have S-rank $2p$. The h nonisomorphic left ideals of SG, X_i, $1 \leqq i \leqq h$, constitute a complete set of indecomposable SG-modules having S-rank $2p$. We shall show each X_i to be a projective ideal of SG.

X_i is projective if and only if $\mathrm{Ext}(X_i, A) = 0$ for each SG-module A. $\mathrm{Ext}(X_i, A) = 0$ if and only if $Z_q \otimes_S \mathrm{Ext}(X_i, A) = 0$ for each prime $q \mid [G:1]$.

But $Z_q \otimes_S X_i \cong Z_q G$ and $\mathrm{Ext}_{Z_qG}(Z_q G, Z_q A) = 0$ whence, since $Z_q \otimes_S \mathrm{Ext}(X_i, A) \cong \mathrm{Ext}_{Z_qG}(Z_q \otimes_S X_i, Z_q \otimes_S A)$, it follows that X_i is projective.

Lemma 4.1. $X_i \dot{+} X_j \cong X_1 \dot{+} X_{ij}$ *for* $1 \leqq i, j \leqq h$.

Proof. If $X_i = (M_i, L;\ F^{(i)})$ and $X_j = (M_j, L;\ F^{(j)})$ where $F^{(i)} \in B(L, M_i)$ and $F^{(j)} \in B(L, M_j)$, then

$$X_i \dot{+} X_j \cong (M_i \dot{+} M_j,\ L \dot{+} L; F)$$

where $F_g(l_1,l_2) = (F_g^{(i)}(l_1), F_g^{(j)}(l_2))$ defines an $F \in B(L \dot{+} L, M_i \dot{+} M_j)$. The map ϕ of $M_i \dot{+} M_j$ onto $M_1 \dot{+} M_{ij}$ given by

$$\phi((r_1,m_i),(r_2,m_j)) = ((r_1, m_i + m_j),\ (r_2, \alpha m_i - \beta m_j)),$$

where $(r_1,m_i) \in M_i$ and $(r_2,m_j) \in M_j$ and α and β are elements of U_i and U_j, respectively, chosen so that $\alpha + \beta = 1$, is an SG-isomorphism. It follows that the map ψ of $(M_i \dot{+} M\ ,L \dot{+} L; F)$ onto $(M_1 \dot{+} M_{ij}, L \dot{+} L;\ F')$ given by $\psi(M_i \dot{+} M_j, L \dot{+} L; F) = (\phi(M_i \dot{+} M_j), L \dot{+} L;\ \phi F)$ is also an SG-isomorphism, F', of course, being $\phi F \in B(L \dot{+} L, M_1 \dot{+} M_{ij})$. Since $(M_1 \dot{+} M_{ij}, L \dot{+} L; F')$ is decomposable, it may be written as a direct sum of indecomposables involving only R, $(\bar{\theta} - \theta)R$, $(\bar{\theta} - \theta)U_iU_jR$ and L, or S and S'. Since it is isomorphic to a direct sum of projective modules, we must have $(M_1 \dot{+} M_{ij},\ L \dot{+} L) \cong (M_1, L) \dot{+} (M_{ij}, L)$ implying that $X_i \dot{+} X_j \cong X_1 \dot{+} X_{ij}$.

LEMMA 4.2. *$X_i \dot{+} X_j \cong X_k \dot{+} X_l$ if and only if $[U_i][U_j] = [U_k][U_l]$.*

Proof. By Lemma 4.1, $X_i \dot{+} X_j \cong X_1 \dot{+} X_{ij} \cong (M_1 \dot{+} M_{ij}, L \dot{+} L)$. Thus if $X_i \dot{+} X_j \cong X_k \dot{+} X_l$, there exists an SG-isomorphism of $M_1 \dot{+} M_{ij}$ onto $M_1 \dot{+} M_{kl}$. By Lemma 1.4, $[U_i][U_j] = [U_k][U_l]$. The converse is an immediate consequence of Lemma 4.1.

If P is any projective SG-module, by Theorem 4.1 $P = X_i \dot{+} F$ for some $1 \leqq i \leqq h$ and $\{X_i\}$, the projective class of X_i is the same as $\{P\}$. If $X_1 \dot{+} X_i \cong X_1 \dot{+} X_j$, by Lemma 4.2 $[U_i] = [U_j]$. Then by Proposition 2.3 $X_i \cong X_j$. Since $X_1 \cong SG$ and $\{0\} = \{X_1\}$, it follows that

PROPOSITION 4.1. *$X_i \dot{+} F_i \cong X_j \dot{+} F_j$, where F_i, F_j are free SG-modules of equal SG-rank, if and only if $X_i \cong X_j$.*

Further, $X_i \in \{0\}$ if and only if $X_i \dot{+} F = F'$. But this is the case if and only if $X_i \dot{+} F \cong X_1 \dot{+} F''$ and by Proposition 4.1 $X_i \cong X_1$ and conversely, that is,

PROPOSITION 4.2. *$X_i \in \{0\}$ if and only if X_i is a free SG-module.*

We are now able to establish the main result of this section.

THEOREM 4.2. *There are h projective classes of SG-modules given by $\{X_i\}$ for $i = 1, \cdots, h$. In particular, the projective class group of SG is isomorphic to the ideal class group of R_0.*

Proof. Let ρ be a mapping of the projective class group of SG into the ideal class group of R_0 given by $\rho: \{X_i\} \to [U_i]$. By Proposition 4.1 $X_i \dot{+} F_i \cong X_j \dot{+} F_j$ implies $[U_i] = [U_j]$, so ρ is well defined. Since $X_i \dot{+} X_j = X_1 \dot{+} X_{ij}$ and $\{X_1\} = \{0\}$, we have $\rho: \{X_i \dot{+} X_j\} \to [U_i][U_j]$. But $\rho(\{X_i\}) \cdot \rho(\{X_j\}) = [U_i][U_j]$; hence ρ is a homomorphism. ρ is obviously onto. That it is 1–1 follows from Proposition 4.1; thus ρ is the desired isomorphism.

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