HOMOTOPY CLASSIFICATION OF MAPS BY COHOMOLOGY HOMOMORPHISMS [1]

BY
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1. **Introduction.** Let f be a map from a space X to a space Y. To what extent is the homotopy class of f determined by the cohomology homomorphisms induced by it? If X is a complex of dimension n and Y an n-sphere, then by the Hopf Theorem f is determined up to homotopy by its induced cohomology homomorphism with integer coefficients. Another such example is given by F. Peterson [14]. Let X be a complex of dimension $\leqq 2n$ and let ω denote a complex n-plane bundle over X. Suppose that X has no torsion in its even dimensional cohomology groups. Peterson shows that the bundle ω is then determined by its Chern classes. Here Y is the classifying space $B_{U(n)}$, and f is the characteristic map for ω.

We consider arcwise connected spaces with basepoint, and denote by $[X, Y]$ the set of homotopy classes of (basepoint preserving) maps from X to Y. Let G be an abelian group and let $\bar{H}^*(X;G)$ denote the reduced singular cohomology groups of X with coefficients in G. Define a function (see [20, p. 14])

$$[X, Y] \xrightarrow{\lambda_G} \operatorname{Hom}(\bar{H}^*(Y;G),\ \bar{H}^*(X;G))$$

by setting $\lambda_G[f] = f^*$, where $[f]$ denotes the homotopy class of f and f^* its induced cohomology homomorphism. Define $N_G[X, Y]$ to be the kernel of λ_G and set

$$N[X, Y] = \bigcap N_G[X, Y]$$

where the intersection is taken over all finitely generated abelian groups. (We will denote the category of these groups by $\mathscr{G}$.)

The purpose of this paper is to define invariants whose vanishing implies that $N[X, Y] = 0$. For in this case a map is null-homotopic if, and only if, all of its induced cohomology homomorphisms (with coefficients $\in \mathscr{G}$) are zero.

Under the hypotheses of Theorem 1.1 below, the set $[X, Y]$ has a natural group structure (see James [9]), and one then can say more about the subset $N[X, Y]$.

THEOREM 1.1. *Let X be a* CW-*complex and suppose that either X is a suspension or Y is a homotopy-associative H-space whose singular homology*

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groups are of finite type. Then $N[X,Y]$ is a normal subgroup of the group $[X,Y]$, and two classes $[f]$ and $[g]$ belong to the same coset of $N[X,Y]$ if, and only if, they induce identical cohomology homomorphisms, taking coefficients in all groups $G \in \mathcal{G}$.

The proof is given in an appendix (§5).

2. **The invariants.** Let Y be a space which is n-simple for all $n \geqq 1$. In particular, this means that $\pi_1(Y)$ is abelian. Suppose that $0 < n(1) < n(2) < \cdots$ are the dimensions in which Y has nonzero homotopy groups, and set $\pi_i = \pi_{n(i)}(Y)$. Recall that a Postnikov system for Y provides a sequence of fibre spaces (and commutative diagrams)

$$
\begin{array}{ccccccccc}
 & & & l_{n(i)} & K(\pi_i, n(i)) & & & & \\
 & & & \swarrow & & & & & \\
(2.1) \quad \cdots & \longrightarrow & Y^{n(i)} & \longrightarrow & Y^{n(i-1)} & \longrightarrow & \cdots & \longrightarrow & Y^{n(1)} \\
 & & \nwarrow & & \nearrow & & & & \\
 & & g_{n(i)} & Y & g_{n(i-1)} & & & &
\end{array}
$$

such that the map $g_{n(i)}$ induces an isomorphism on homotopy groups in dimensions $\leqq n(i)$. Denote by $k_{n(i)}$ the ith Postnikov invariant of Y: that is,

$$k_{n(i)}\colon Y^{n(i)} \longrightarrow K(\pi_{i+1}, n(i+1)+1).$$

Define a sequence of first order cohomology operations (for each space Y) by

$$\Psi_{n(i)} = -k_{n(i)} \circ l_{n(i)}\colon K(\pi_i, n(i)) \to K(\pi_{i+1}, n(i+1)+1).$$

Set $\Phi_{n(i)-1} = \sigma\Psi_{n(i)}$, where σ denotes the suspension of cohomology operations. (If $n(1) = 1$, then $\Phi_0 = 0$.)

We next define a sequence of non-negative integers $\tau_{n(i)}$ as follows. Suppose first that π_i is a cyclic group, and let $f\colon S^{n(i)} \to Y$ represent a generator. Define $\tau_{n(i)}$ to be the least positive integer such that

$$\tau_{n(i)} s_i \in f^* H^{n(i)}(Y;\ \pi_i),$$

where s_i generates the cyclic group $H^{n(i)}(S^{n(i)};\pi_i)$. If $f^* H^{n(i)}(Y;\pi_i) = 0$, or if π_i is not cyclic(2), set $\tau_{n(i)} = 0$.

Denote by $\tau_{n(i)}{}^*$ the cohomology operation given by multiplying each cohomology class by the integer $\tau_{n(i)}$; we consider this operation only in dimension $n(i)$, with coefficients in π_i. Thus with each space Y we associate three sequences of cohomology operations: $\Psi_{n(i)}, \Phi_{n(i)-1}$, and $\tau_{n(i)}{}^*$.

(2) The definition of the invariants $\tau_{n(i)}$ can be extended to the case that π_i is not cyclic. Then $\tau_{n(i)}$ becomes a set of elements from Hom (π_i, π_i), but the usefulness of the generalization to applications seems limited.

Suppose now that Y is fixed, and let X be any other space. The operations $\Psi_{n(i)}$ and $\tau_{n(i)}{}^*$ then have as domain the group $H^{n(i)}(X;\pi_i)$, while $\Phi_{n(i-1)-1}$ has this group as range. In §3 we show that Image $\Phi_{n(i-1)-1} \subset \text{Kernel}\,\Psi_{n(i)}$ $(i \geqq 1)$. Define, for $i > 1$

$$\mathfrak{N}^{n(i)}(X, Y) = \frac{\text{Kernel}\,\tau_{n(i)}{}^* \cap [\text{Kernel}\,\Psi_{n(i)}]}{\text{Kernel}\,\tau_{n(i)}{}^* \cap \text{Image}\,\Phi_{n(i-1)-1}}.$$

(Since $\Psi_{n(i)}$ is not necessarily additive we denote by $[\text{Kernel}\,\Psi_{n(i)}]$ the least subgroup of $H^{n(i)}(X;\pi_i)$ containing $\text{Kernel}\,\Psi_{n(i)}$.)

As above suppose that Y is n-simple for all $n \geqq 1$ and suppose in addition that $\pi_i \in \mathcal{G}$, $i \geqq 1$. (These conditions are fulfilled, for example, if the singular homology groups of $Y \in \mathcal{G}$ and either Y is simply-connected or Y is an H-space [17, 5].) We shall prove

THEOREM 2.2. *Let X be a finite-dimensional* CW-*complex. If* $\mathfrak{N}^{n(i)}(X, Y) = 0$ *for all* $i \geqq 2$, *then* $N[X, Y] = 0$.

In §4 we compute the invariants Φ, Ψ, and τ for the classifying spaces of the stable classical groups, and then apply Theorem 2.2 to the problem of classifying (stable) vector space bundles over complexes, obtaining as a special case the theorem of Peterson mentioned in §1.

REMARK. Theorem 2.2 is only a first level result, in the sense that the groups $\mathfrak{N}^{n(i)}(X, Y)$ are defined using only primary cohomology operations. It is possible to define, for each $n \geqq 1$, a sequence of nth order cohomology operations associated with Y, and using these to define a sequence of groups with numerator the same as $\mathfrak{N}^{n(i)}(X, Y)$ but with larger denominators. In §3 we sketch the definition of the second order operations, but for simplicity we do the details of the method only using primary operations.

3. **Proof of Theorem 2.2.** Suppose that Y is a simply-connected space, and denote by ΩY the loop space of Y. A Postnikov system for ΩY is obtained by applying the loop functor Ω to all the spaces and maps in diagram (2.1). Denote by $\Omega Y \to PY \to Y$ the path fibre space over Y [17, Chapter 4, §4]. Recall that a Moore-Postnikov [13] decomposition of this fibre space gives a sequence of fibre spaces (and commutative diagrams)

$$\begin{array}{ccccccccc} & & & K(\pi_{i-1}, n(i-1)-1) & & & & & \\ & & \scriptstyle j_{n(i)} \swarrow & & & & & & \\ (3.1) \quad \cdots \longrightarrow & Y_{n(i)} & \xrightarrow{q_{n(i)}} & Y_{n(i-1)} & \longrightarrow & \cdots & \longrightarrow & Y_{n(1)} = Y. \\ & \scriptstyle h_{n(i)} \nwarrow & & \nearrow \scriptstyle h_{n(i-1)} & & & & & \\ & & PY & & & & & & \end{array}$$

Denote by $p_{n(i)}$ the composite map $Y_{n(i)} \to \cdots \to Y$. This is a fibre map and the fibre space $Y_{n(i)} \to Y$ is induced from the fibre space $PY^{n(i-1)} \to Y^{n(i-1)}$ by the map $g_{n(i-1)}$ of Y into $Y^{n(i-1)}$. Thus $p_{n(i)}$ has fibre $\Omega Y^{n(i-1)}$ and $h_{n(i)}$ restricted to ΩY is $\Omega g_{n(i-1)}$. Moreover $Y_{n(i)}$ is $(n(i)-1)$-connected and $p_{n(i)}$ induces an isomorphism on homotopy groups in dimensions $\geqq n(i)$. We can construct such a decomposition by the Cartan-Serre-G. Whitehead method of successively killing homotopy groups [7, Chapter 5, §8]. This method is feasible for making computations, since it does not involve the Postnikov system $\{Y^{n(i)}\}$. (In §4 we give two examples of computations using this construction.) By the construction the fibre space $Y_{n(i+1)} \to Y_{n(i)}$ is induced from the fibre space

$$\Omega K_{n(i)} \to PK_{n(i)} \to K_{n(i)}$$

$(K_{n(i)} = K(\pi_i, n(i)))$ by a map $\gamma_{n(i)} : Y_{n(i)} \to K_{n(i)}$. Moreover, it is clear that $\gamma_{n(i)}$ is the map $g_{n(1)}$ in the Postnikov system for $Y_{n(i)}$ (see 2.1).

Denote by $\varepsilon_{n(i)}$ and $\iota_{n(i)}$ the respective fundamental classes of $Y_{n(i)}$ and $K_{n(i)}$. That is,

$$\varepsilon_{n(i)} \in H^{n(i)}(Y_{n(i)};\pi_i), \iota_{n(i)} \in H^{n(i)}(K_{n(i)};\pi_i) \text{ and } \gamma_{n(i)}{}^* \iota_{n(i)} = \varepsilon_{n(i)}.$$

(We identify π_i and $\pi_{n(i)}(Y_{n(i)})$ by means of $p_{n(i)}{}^*$.) I claim that

$$\Phi_{n(i)-1} = j_{n(i+1)}{}^* \varepsilon_{n(i+1)} \qquad (i \geqq 2). \tag{3.2}$$

To see this notice that the inclusion $K(\pi_i, n(i)-1) \subset^{j} Y_{n(i+1)}$ can be factored into

$$K(\pi_i, n(i)-1) \subset l\, \Omega Y^{n(i)} \subset {}^{u} Y_{n(i+1)}.$$

Let τ denote the transgression in the fibre space $K(\pi_{i+1}, n(i+1)-1) \to Y_{n(i+2)} \to Y_{n(i+1)}$. Then $\tau(\sigma\iota_{n(i+1)}) = \varepsilon_{n(i+1)}$, and hence by §4 of [12], and the naturality of the transgression,

$$u^*\varepsilon_{n(i+1)} = -k_{n(i)-1}(\Omega Y).$$

Recall that by [8],

$$k_{n(i)-1}(\Omega Y) = \sigma k_{n(i)}(Y). \tag{3.3}$$

Thus,

$$j^*\varepsilon_{n(i+1)} = l^*u^*\varepsilon_{n(i+1)} = -l^*k_{n(i)-1}(\Omega Y) = -\sigma l^* k_{n(i)}(Y) = \sigma \Psi_{n(i)} = \Phi_{n(i)-1}$$

as claimed. It follows from (3.2) that the invariant $\Phi_{n(i)-1}$ is the same for the space Y and the space $Y_{n(j)}$ for any $j < i$, and the same is then true of the invariant $\Psi_{n(i)}$, provided σ is an isomorphism.

Let f be a map from X to $Y_{n(i)}$ $(i \geqq 1)$. We say that f *lifts* to $Y_{n(j)}$ $(j > i)$, if there is a map f' from X to $Y_{n(j)}$ such that $q_{n(i+1)} \circ \cdots \circ q_{n(j)} \circ f' \simeq f$. If X is a CW-complex, then $f^*\varepsilon_{n(i)}$ is the (single) obstruction to lifting the map f to $Y_{n(i+1)}$.

Recall [15; 19] that for any space X the following sequence is exact $(i \geqq 2)$.

$$\cdots \to [X, \Omega K_{n(i-1)}] \xrightarrow{j_{n(i)\#}} [X, Y_{n(i)}] \xrightarrow{q_{n(i)\#}} [X, Y_{n(i-1)}] \xrightarrow{\gamma_{n(i-1)\#}} [X, K_{n(i-1)}].$$

Moreover (see [16; 19]) there is a map

$$\mu : \Omega K_{n(i-1)} \times Y_{n(i)} \to Y_{n(i)}$$

such that for classes $u, v \in [X, Y_{n(i)}]$

$$q_{n(i)\#}(u) = q_{n(i)\#}(v)$$

if, and only if, there is a class $w \in [X, \Omega K_{n(i-1)}]$ with $\mu_\#(w, u) = v$. Using this we prove

LEMMA 3.4. *Let f be a map from a* CW-*complex X into $Y_{n(i)}$ $(i \geqq 2)$. The map $q_{n(i)} f$ lifts to $Y_{n(i+1)}$ if, and only if,*

$$f^* \varepsilon_{n(i)} \in \operatorname{Image} \Phi_{n(i-1)-1} \subset H^{n(i)}(X_i; \pi_i).$$

Proof. Let h be any map $X \to Y_{n(i)}$ and let $w \in H^{n(i-1)-1}(X; \pi_i)$. We regard w as an element of $[X, \Omega K_{n(i-1)}]$ and set

$$v = \mu_\#(w, [h]) \in [X, Y_{n(i)}].$$

Then $q_{n(i)\#}(v) = q_{n(i)\#}[h]$, and so $q_{n(i)} \circ h' \simeq q_{n(i)} \circ h$, where h' represents v.

Let Δ denote the diagonal map $X \to X \times X$. Then h' may be taken to be the following composition:

$$X \xrightarrow{\Delta} X \times X \xrightarrow{w \times h} \Omega K_{n(i-1)} \times Y_{n(i)} \xrightarrow{\mu} Y_{n(i)}.$$

Moreover the map μ has the property that if l_1 and l_2 denote the respective inclusions $\Omega K_{n(i-1)}$, $Y_{n(i)} \subset \Omega K_{n(i-1)} \times Y_{n(i)}$, then

$$\mu \circ l_1 \simeq j_{n(i)}, \text{ and } \mu \circ l_2 \simeq \text{identity}.$$

Therefore, since $Y_{n(i)}$ is $(n(i)-1)$-connected, setting $\varepsilon = \varepsilon_{n(i)}$ and $\Phi = \Phi_{n(i-1)-1}$ we obtain

$$\begin{aligned} h'^* \varepsilon &= \Delta^* \circ (w \times h)^* \mu^*(\varepsilon) \\ &= \Delta^* \circ (w \times h^* (j_{n(i)})^* \, \varepsilon \otimes 1 + 1 \otimes \varepsilon) \\ &= \Phi(w) + h^* \varepsilon. \end{aligned}$$

To apply this to the proof of Lemma 3.4, suppose first that there is a class $u \in H^{n(i-1)-1}(X; \pi_i)$ such that $f^* \varepsilon = \Phi(u)$. Take $h = f$, $w = -u$ and construct h' as above. Since Φ is additive (it is a suspension),

$$h'^* \varepsilon = \Phi(-u) + \Phi(u) = -\Phi(u) + \Phi(u) = 0,$$

and so h' lifts to $Y_{n(i+1)}$, which provides a lifting of $q_{n(i)} \circ f$ as required.

Conversely, suppose that g is a map from X to $Y_{n(i+1)}$ which lifts $q_{n(i)} \circ f$. Set

$h = q_{n(i+1)} \circ g$. Since $q_{n(i)} \circ f \simeq q_{n(i)} \circ h$, there is a class $w \in H^{n(i-1)-1}(X;\pi_i)$ such that $[f] = \mu_{\#}(w,[h])$. Take $h' = f$. Since h lifts to $Y_{n(i+1)}, h^*\varepsilon = 0$ and so, by the above equation:

$$f^*\varepsilon = \Phi(w),$$

which completes the proof of the lemma.

REMARK. By introducing higher order cohomology operations, one can give a necessary and sufficient condition that the map $q_{n(i-\gamma)} \circ \cdots \circ q_{n(i)} \circ f$ lifts to $Y_{n(i+1)} (\gamma > 0)$. As an illustration we sketch the case $\gamma = 1$. Consider the fibre space $F \xrightarrow{l} Y_{n(i)} \to Y_{n(i-2)}$, where $p = q_{n(i-1)} \circ q_{n(i)}$. Then, $F = \Omega(Y^{n(i+1)}_{n(i-2)})$; that is, a space with two nonvanishing homotopy groups, π_{i-2} in dimension $n(i-2)-1$ and π_{i-1} in dimension $n(i-1)-1$, and with $\Phi_{n(i-2)-1}$ as k-invariant. Set

$$\Phi'_{n(i-2)-1} = \gamma_{n(i)} \circ l : F \to K_i.$$

This is a secondary cohomology operation, defined on Kernel $\Phi_{n(i-2)-1}$ and taking values in the cosets of Image $\Phi_{n(i-1)-1}$. One then proves an analogous result to Lemma 3.4:

The map $p \circ f$ lifts to $Y_{n(i+1)}$ if, and only if,

$$f^*\varepsilon_{n(i)} \in \text{Image}\,\Phi_{n(i-1)-1} + \text{Image}\,\Phi'_{n(i-2)-1} \subset H^{n(i)}(X;\pi_i).$$

Suppose now that X is a finite-dimensional CW-complex and that Y is a space whose homotopy groups $\in \mathcal{G}$. Furthermore suppose that $\mathfrak{N}^{n(i)}(X,Y) = 0$ for $i \geqq 2$. To prove Theorem 2.2 we must show that if $[f] \in N[X,Y]$, then f is null-homotopic, and to do this it suffices to show that f can be lifted to each $Y_{n(i)} (i > 1)$.

Since $[f] \in N[X,Y], f^*\varepsilon_{n(1)} = 0$ and hence f lifts to $Y_{n(2)}$. Moreover $Y_{n(2)}$ is simply-connected and so Lemma 3.4 can be applied to the spaces $Y_{n(i)}$ $(i > 2)$ lying over $Y_{n(2)}$. Suppose then that for some $i > 2, f$ lifts to a map, f_i, from X to $Y_{n(i)}$. To complete the inductive step we must show that f lifts to $Y_{n(i+1)}$, which will then complete the proof of Theorem 2.2.

Set $u = f_i^*\varepsilon_{n(i)} \in H^{n(i)}(X;\pi_i)$. Now $\Psi_{n(i)}$ is the first k-invariant for the space $Y_{n(i)}$, and therefore the map $\Psi_{n(i)} \circ \gamma_{n(i)}$ is null-homotopic. Thus Image $\Phi_{n(i-1)-1} \subset$ Kernel $\Psi_{n(i)}$, since $\Phi_{n(i-1)-1} = j^*_{n(i)}\gamma_{n(i)}{}^*\iota_{n(i)}$, by 3.2. Similarly,

$$\Psi_{n(i)}(u) = \Psi_{n(i)}f_i^*\varepsilon_{n(i)} = f_i^*\Psi_{n(i)}\gamma_{(ni)}{}^*\iota_{n(i)} = 0,$$

which shows that $u \in$ Kernel $\Psi_{n(i)}$.

Consider now the invariant $\tau_{n(i)}$. If $\tau_{n(i)} = 0$, then trivially $u \in$ Kernel $\tau_{n(i)}{}^*$. If $\tau_{n(i)} \neq 0$, then π_i is cyclic and there is a class $v \in H^{n(i)}(Y;\pi_i)$ such that $g^*v = \tau_{n(i)}s_{n(i)}$, where $g: S^{n(i)} \to Y$ represents a generator for π_i. Let $g_{n(i)}: S^{n(i)} \to Y_{n(i)}$ represent a generator for $\pi_{n(i)}(Y_{n(i)})$. Since $p_{n(i)}{}^*$ is an isomorphism we may take $g = p_{n(i)} \circ g_{n(i)}$, and thus obtain

$$p_{n(i)}{}^*v = \pm\, \tau_{n(i)}\varepsilon_{(ni)},$$

using the fact that $g_{n(i)}{}^*\varepsilon_{n(i)} = \pm\, s_{n(i)}$. Therefore,

$$\tau_{n(i)}{}^*(u) = \tau_{n(i)}(f_i{}^*\varepsilon_{n(i)}) = \pm f_i{}^* p_{n(i)}{}^* v = f^* v = 0,$$

since $f \simeq p_{n(i)} \circ f_i$ and $[f] \in N[X, Y]$. Thus in either case $u \in \text{Kernel}\, \tau_{n(i)}{}^*$. By hypothesis $\mathfrak{N}^{n(i)}(X, Y) = 0$, and therefore $u \in \text{Image}\, \Phi_{n(i-1)-1}$. Thus by Lemma 3.4, $q_{n(i)} \circ f_i$ lifts to $Y_{n(i+1)}$ and therefore so does f. This completes the inductive step for the proof of Theorem 2.2.

4. **Stable vector bundles.** Suppose that Y is a CW-complex such that for some positive integer $N, \Omega_0^N\, Y$ has the homotopy type of Y. (Ω_0 denotes the component of the constant loop.) Then it follows from Theorem 3 of [10] that $\Omega^N Y_{N+n(i)}$ and $Y_{n(i)}$ $(i \geqq 1)$ have the same homotopy type. Applying (3.3) we obtain, for $i \geqq 1$,

$$\text{(4.1)} \qquad \Psi_{n(i)} = \sigma^N \Psi_{N+n(i)}, \quad \Phi_{n(i)-1} = \sigma^N \Phi_{N+n(i)-1},$$

since $\Psi_{N+n(i)}(Y_{N+n(1)}) = \Psi_{N+n(i)}(Y)$, by the remark following the proof of (3.2).

We apply this to the case of the classifying spaces for the stable classical groups—O and U—defined by Bott [3], studying first the space B_O. (In general B_G denotes a classifying space for a group G.) Bott shows that $B_O \equiv \Omega_0^8 B_O$, with the following periodic homotopy groups.

$\gamma \bmod 8$	1	2	3	4	5	6	7	8
$\pi_\gamma(B_O)$	Z_2	Z_2	0	Z	0	0	0	Z

Thus by (4.1) there are four distinct (stable) values of the Ψ invariants, since $\Psi_{n(i)} = \sigma^8 \Psi_{n(i)+8}$. We in fact give the values of $\Phi_{n(i)-1}$, for we then obtain $\Psi_{n(i)}$ by desuspending.

THEOREM 4.2. *For the space B_O the Φ-operations are as follows. For $k \geqq 0$,*

$$\Phi_{8k+1} = \delta_2 Sq^2 : H^{8k+1}(X; Z_2) \to H^{8k+4}(X; Z),$$

$$\Phi_{8k+3} = \pm\ \delta_3 P_3^1 + \delta_2 Sq^4 : H^{8k+3}(X; Z) \to H^{8k+8}(K; Z),$$

$$\Phi_{8k+7} = Sq^2 : H^{8k+7}(X; Z) \to H^{8k+9}(X; Z_2),$$

$$\Phi_{8k+8} = Sq^2 : H^{8k+8}(X; Z_2) \to H^{8k+10}(X; Z_2).$$

The $\tau_{n(i)}$ are given by:

$$\tau_1 = \tau_2 = 1;\ \tau_j = 0 \textit{ for } j > 8 \textit{ and } j \equiv 1, 2 \bmod 8;$$

$$\tau_{8k+4} = 2((4k+1)!), \quad \tau_{8k+8} = (4k+3)!, \ k \geqq 0.$$

Here X is any space, Sq^i and P_3^j denote the respective mod 2 and mod 3 Steenrod operators, and δ_n $(n \geqq 2)$ denotes the Bockstein operator associated with the exact coefficient sequence $0 \to Z \xrightarrow{n} Z \to Z_n \to 0$.

Using the above invariants we can compute the groups $\mathfrak{N}^{n(i)}(K, B_O)$ defined in §2. By (1.1) and (2.2) we obtain

THEOREM 4.3. *Let K be a finite-dimensional* CW-*complex and suppose that $\mathfrak{N}^{n(i)}(K,B_O)=0$ for all $i\geqq 2$. Then two (stable) real vector bundles over K are equivalent if, and only if, they have identical Pontryagin and Stiefel-Whitney characteristic classes.*

Here we have used the fact that the cohomology ring of B_O is determined by the universal Pontryagin and Stiefel-Whitney classes.

The integers $\tau_{n(i)}$ in 4.2 are given by the Bott divisibility conditions for the Pontryagin classes of real vector bundles over $4k$-dimensional spheres [4].

To obtain the Φ-invariants for B_O, take a Moore-Postnikov decomposition for the fibre space $PB_O \to B_O$. Set

$$B(0,n(i)) = Y_{n(i)} \qquad (i \geqq 1),$$

where the Y's are the spaces given in §2. (Thus $B(0,1)=B_O$.) By (4.1) it is sufficient to compute the invariants $\Phi_{n(i)-1}$ for $4\leqq i\leqq 7$, as the remaining Φ's and all the Ψ's, are then given by desuspending. Now Φ_7 and Φ_8 take values with Z_2 as the coefficient group, and

$$\Phi_9 \in H^{12}(Z,9;\ Z) \approx Z_2.$$

Thus these three invariants can be computed by using mod 2 coefficients. Moreover

$$\Phi_{11}\in H^{16}(Z,11;\ Z) \approx Z_2+Z_3,$$

and therefore we can determine the mod 2 (respectively, mod 3) summand of this operation by using mod 2 (respectively, mod 3) coefficients.

We first study the mod 2 cohomology of the spaces(3) $B(0,k)$, applying (3.2) to determine the mod 2 component of the Φ-invariants. We begin with the known result [21],

$$H^*(B(0,4)) = Z_2[W_4,W_6,W_7,\cdots],$$

where the W's are the images of the universal Stiefel-Whitney classes from B_O and where we delete those W_i $(i\geqq 2)$ such that $i=2^r+1$, $r\geqq 0$.

Consider the fibering $K(Z,3)\to B(0,8)\to B(0,4)$. Using the result of Serre [18] for $H^*(Z,3)$, we apply Lemma 2.1 of [21] to show that

$$H^*(B(0,8)) = Z_2[W_8',W_{12}',W_{14}',\cdots],$$

where $W_i'=q_8^*W_i$, and where we delete those $W_i'(i\geqq 8)$ such that $i=2^s+2^r+1$, with either $s\geqq r\geqq 1$ or $s>0$ and $r=0$. Moreover by the Wu formula [23],

$$Sq^iW_8' = W_{8+i}' \qquad (i=4,6,7).$$

(3) In Notices Amer. Math. Soc. **9** (1962), 328–329, R.E. Stong states a complete description of the mod 2 cohomology rings of $B(0,k)$ and $B(U,k)$. *Added in proof.* See Trans. Amer. Math. Soc. **107** (1963), 526–544.

To compute the invariants $\Phi_7, \cdots, \Phi_{11}$ for B_O we need the Serre exact sequence [17, Chapter 3, §4]. That is, suppose that $F \xrightarrow{i} B \xrightarrow{\pi} B$ is a fibre space where F is $(p-1)$-connected and B is $(q-1)$-connected $(p, q \geqq 1)$. Then one has the following exact sequence, using any principal ring for coefficients:

$$(*) \qquad \cdots \to H^{r-1}(F) \xrightarrow{\tau} H^r(B) \xrightarrow{\pi} H^r(E) \xrightarrow{i^*} H^r(F) \to \cdots \to H^{p+q-1}(F).$$

Here τ denotes the transgression homomorphism.

In order to compute Φ_7 we apply (*) to the fibering $K(Z,7) \to B(0,9) \to B(0,8)$. For any integral cohomology class u let $\bar{u}$ denote its mod 2 reduction. Then $\tau(\bar{\iota}_7) = W_8'$ and $\tau(Sq^2\bar{\iota}_7) = 0$, which shows by (*) and (3.2) that

$$Sq^2\bar{\iota}_7 = j_9^*\varepsilon_9 = \Phi_7.$$

Furthermore,

$$j_9^* Sq^2\varepsilon_9 = Sq^2 Sq^2 \bar{\iota}_7 = Sq^3 Sq^1 \bar{\iota}_7 = 0,$$

and hence by (*),

$$(4.4) \qquad Sq^2\varepsilon_9 = 0,$$

since $H^{11}(B(0,8)) = 0$. Notice also that

$$(4.5) \qquad q_{10}^* H^{13}(B(0,9)) = 0,$$

since the classes in dimension 13 come from ε_9 by squaring operations.

To determine Φ_8 we apply (*) to the fibering $K(Z_2, 8) \to B(0,10) \to B(0,9)$. Using 4.4 and 4.5 one has that

$$Sq^2\iota_8 = j_{10}^*\varepsilon_{10} = \Phi_8, \text{ and } Sq^3\,\varepsilon_{10} = 0.$$

(The latter fact uses the Adem relation [1] $Sq^3Sq^2 = 0$.) Finally, by the same type of argument, one shows that

$$Sq^3\iota_9 = j_{12}^*\bar{\varepsilon}_{12} = \Phi_9.$$

We are left with showing that

$$Sq^5\bar{\iota}_{11} = j_{16}^*\bar{\varepsilon}_{16} = \bar{\Phi}_{11}.$$

Suppose to the contrary that $j_{16}^*\bar{\varepsilon}_{16} = 0$. We show that this assumption leads to a contradiction. By (*), applied to the fibering $K(Z,11) \to B(0,16) \to B(0,12)$, we see that $Sq^4\bar{\varepsilon}_{12} \neq 0$ and that there is a class $u \in H^{16}(B(0,12))$ such that $q_{16}^* u = \bar{\varepsilon}_{16}$. Consider the fibering $K(Z_2, 9) \xrightarrow{j_{12}} B(0,12) \xrightarrow{q_{12}} B(0,10)$. By [18], $H^{16}(Z_2, 9)$ has the following basis:

$$Sq^7(\iota_9), Sq^6Sq^1(\iota_9), Sq^5Sq^2(\iota_9), Sq^4Sq^2Sq^1(\iota_9).$$

Now $j_{12}^*\bar{\varepsilon}_{12} = Sq^3\iota_9$, and hence by the Adem relations,

$$j_{12}^*(Sq^4\bar{\varepsilon}_{12}) = Sq^4Sq^3(\iota_9) = Sq^5Sq^2(\iota_9).$$

On the other hand $Sq^7\varepsilon_{10}$, $Sq^6Sq^1\varepsilon_{10}$ and $Sq^4Sq^2Sq^1\varepsilon_{10}$ are linearly independent in $H^{17}(B(0,10))$. (One sees this by applying j_{10}^* to these classes and using the Adem relations in $H^{17}(Z_2,8)$.) Since $\tau(\iota_9) = \varepsilon_{10}$, it follows that τ is a monomorphism on a summand complementary to $Sq^5Sq^2(\iota_9)$ in $H^{16}(Z_2,9)$. Thus by (*), j_{12}^*u must belong to the summand spanned by $Sq^5Sq^2\iota_9$ and hence

$$u = q_{12}^*v + a(Sq^4\varepsilon_{12}) \qquad (a \in Z_2)$$

for some class $v \in H^{16}(B(0,10))$.

By a similar argument one shows that

$$v = q_{10}^*w + (bSq^6 + cSq^4Sq^2)\varepsilon_{10} \qquad (b,c \in Z_2),$$

where $w \in H^{16}(B(0,9))$.

Finally, we consider the fibering $K(Z,7) \to B(0,9) \to B(0,8)$. $H^{16}(Z,7)$ has $Sq^7Sq^2\iota_7$ and $Sq^6Sq^1Sq^2\iota_7$ as basis elements and these classes transgress to zero in $H^{16}(B(0,8))$. Since W_{16}' and $W_8'^2$ generate $H^{16}(B(0,8))$, we have

$$w = q_9^*W_{16}' + (dSq^7 + eSq^6Sq^1)\varepsilon_9 \qquad (d,e \in Z_2).$$

Thus, since $B(0,16)$ is 15–connected,

$$\bar{\varepsilon}_{16} = r^*W_{16}'$$

where $r = q_9 \circ q_{10} \circ q_{12} \circ q_{16}$. But by the Bott divisibility criterion, one has that W_{16}' is zero in $B(0,16)$ (see argument by Milnor in [11]) and hence we have obtained a contradiction. Thus,

$$\Phi_{11} = Sq^5\bar{\iota}_{11},$$

as claimed.

This completes the mod 2 calculations needed in 4.2. The remaining calculation is the mod 3 summand in Φ_{11}. Since $\Phi_3 = \sigma^8\Phi_{11}$, it suffices to show that

$$\pm\, \delta_3\, P_3^1\, \iota_3 = j_8^*\, \varepsilon_8,$$

and this follows at once from the fact that $\delta_3P_3^1\iota_3$ generates $H^8(Z,3;\ Z)$ and that $H^*(B(0,4);\ Z)$ has no 3-torsion. This completes the proof of Theorem 4.2.

We turn now to the classifying space B_U. Bott [3] shows that $B_U \equiv \Omega_0^2 B_U$, and that

$$\pi_{2i}(B_U) = Z, \quad \pi_{2i-1}(B_U) = 0, \qquad i \geqq 1.$$

Thus by 4.1 there is just one (stable) Φ invariant to compute, since $\Phi_{2i-1} = \sigma^2\Phi_{2k+1}$ $(i \geqq 1)$. (See §3 of [14].)

THEOREM 4.6. *For the space* B_U,

$$\Phi_{2i-1} = \delta_2 Sq^2 : H^{2i-1}(X;Z) \to H^{2i+2}(X;Z) \qquad (i \geqq 1).$$

The τ invariants are given by

$$\tau_{2i} = (i-1)!, \qquad i \geqq 1.$$

Again we can use these invariants to compute the groups $\mathfrak{N}^{n(i)}(X, B_U)$ (see §2). Combining (1.1) and (2.1) we obtain

THEOREM 4.7. *Let K be a finite-dimensional complex and suppose that $\mathfrak{N}^{n(i)}(K, B_U) = 0$ for all $i \geqq 2$. Then two (stable) complex vector bundles over K are equivalent if, and only if, they have identical Chern classes.*

This includes Theorem 3.2 of [14]. For if the torsion in $H^{2i}(K;Z)$ is relatively prime to $(i-1)!$, then Kernel $\tau_{2i}{}^* = 0$ and hence $\mathfrak{N}^{2i}(K, B_U) = 0$.

The values of the integers τ_{2i} in 4.6 come from the Bott divisibility theorem for complex bundles over spheres [4]. One evaluates Φ_3 for B_U (hence obtaining all the Φ's and Ψ's) by applying the exact sequence (*) to the fibering $K(Z,3) \to B(U,6) \to B(U,4)$. Here $B(U,2i) = Y_{2i}$, where the Y's are a decomposition for the fibre space $PB_U \to B_U$. One uses the fact that $B(U,2)$ and $B(U,4)$ have nonzero integral cohomology only in even dimensions. We leave the details to the reader.

REMARK. The classes $\Phi_{n(i)}$ have another interpretation from that given here. They in fact occur as the initial differentials in an exact couple whose spectral sequence converges to $[S^rX, Y]$ ($S^rX = r$th suspension of $X, r \geqq 1$). Thus the Φ's which we have computed for B_0 and B_U occur in spectral sequences which converge to the K-functors

$$\tilde{K}_0^{-r}(X), \ \tilde{K}_U^{-r}(X),$$

of Atiyah-Hirzebruch [2]. A brief discussion of this topic is given in [22].

5. **Appendix.** Let X and Y be topological spaces, and for each abelian group G let λ_G denote the set function defined in §1. We define an equivalence relation $\equiv$, between classes in $[X, Y]$ by saying that

$$[f] \equiv [g] \text{ if } \lambda_G[f] = \lambda_G[g],$$

for all $G \in \mathscr{G}$. We prove

LEMMA 5.1. *Let X_i, Y_i $(i=1,2)$ be spaces such that Y_1 and Y_2 have singular homology groups of finite type. Let $[f_i], [g_i] \in [X_i, Y_i]$ $(i=1,2)$. If $[f_i] \equiv [g_i]$, then*

$$[f_1 \times f_2] \equiv [g_1 \times g_2],$$

as classes in $[X_1 \times X_2,\ Y_1 \times Y_2]$.

Proof. Since each space Y_i has homology of finite type, there is a chain complex of finite type, $C_*(Y_i)$, which is a chain equivalent subcomplex of the singular complex $S(Y_i)$. Hence for each $G \in \mathscr{G}$,

$$H^*(Y_i,G) \approx H^*({}^iC(G)),$$

where ${}^iC(G) = \operatorname{Hom}(C_*(Y_i),G)$, and therefore,

$$H^*(Y_1 \times Y_2;G) \approx H^*({}^1C(G) \otimes {}^2C(G)).$$

To show that $[f_1 \times f_2] \equiv [g_1 \times g_2]$ it suffices to show that $\lambda_G[f_1 \times f_2] = \lambda_G[g_1 \times g_2]$, where G is the integers and all cyclic groups Z, Z_{p^r}, $r \geqq 1, p$ a prime. We first show

(5.2) $$\lambda_G[f_1 \times f_2] = \lambda_G[g_1 \times g_2], \text{ for } G = Z_pr,\ r \geqq 1.$$

Suppose that this has been proved for all integers r such that $1 \leqq r < n$. We show that this implies the statement for $r = n$. Let $w \in H^q({}^1C(Z_{p^n}) \otimes^2 C(Z_{p^n}))$, $q > 0$. By taking canonical bases for the respective cochain complexes [6, Chapter 5, §8)], one can show that

$$w = \sum_i u_i \otimes v_i + \sum_{1 \leqq j \leqq n-1} \delta_j'(x_j),$$

where

$$u_i \in H^{a_i}({}^1C(Z_{p^n})),\ v_i \in H^{b_i}({}^2C(Z_{p^n}))\ (a_i + b_i = q),$$

and

$$x_j \in H^{q-1}({}^1C(Z_{p^j}) \otimes {}^2C(Z_{p^j})).$$

(Here δ_j' is the Bockstein coboundary associated with the exact sequence $0 \to Z_{p^n} \to Z_{p^{n+j}} \to Z_{p^j} \to 0$.)

Thus,

$$\begin{aligned}(f_1 \times f_2)^*w &= \sum_i (f_1 \times f_2)^*(u_i \otimes v_i) + \sum_j \delta_j'(f_1 \times f_2)^*x_j \\ &= \sum_i f_1{}^*u_i \otimes f_2{}^*v_i + \sum_j \delta_j'(f_1 \times f_2)^*x_j.\end{aligned}$$

But $[f_i] \equiv [g_i]$ and therefore,

$$f_1{}^*u_i = g_1{}^*u_i, \quad f_2{}^*v_i = g_2{}^*v_j.$$

Moreover by the inductive hypothesis,

$$(f_1 \times f_2)^*x_j = (g_1 \times g_2)^*x_j.$$

Consequently,

$$(f_1 \times f_2)^*w = \sum_i g_1{}^*u_i \otimes g_2{}^*v_i + \sum_j \delta_j'(g_1 \times g_2)^*x_j = (g_1 \times g_2)^*w,$$

and thus $\lambda_G[f_1 \times f_2] = \lambda_G[g_1 \times g_2]$, where $G = Z_{p^n}$. Since

$$H^*({}^1C(Z_p) \otimes {}^2C(Z_p)) = H^*({}^1C(Z_p)) \otimes H^*({}^2C(Z_p)),$$

(5.2) is clearly true for $r = 1$, completing the inductive proof. A similar argument shows that $\lambda_Z[f_1 \times f_2] = \lambda_Z[g_1 \times g_2]$. We leave the details to the reader.

Suppose now that Y is a homotopy-associative H-space with homology of finite type, and let $\mu : Y \times Y \to Y$ denote the multiplication. Let $[f], [g] \in [X, Y]$. We define the product of these classes (which we will write as $[f] + [g]$, even though it is not necessarily abelian), as the homotopy class of the map

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y \xrightarrow{\mu} Y.$$

From Lemma 5.1 and the naturality of the set $N[X, Y]$, we have

LEMMA 5.3. *If* $[f] \equiv [f_1]$ *and* $[g] \equiv [g_1]$, *then* $[f] + [g] \equiv [f_1] + [g_1]$.

Let A be any group and suppose that $\sim$ is an equivalence relation on A such that, if $a \sim a_1$ and $b \sim b_1$ then $a \cdot b \sim a_1 \cdot b_1$. Then the set of equivalence classes, $A/\sim$, has a natural group structure. If K denotes the set of elements in A which are equivalent to the identity of A, then K is a normal subgroup of A, since it is the kernel of the natural map $A \to A/\sim$. Moreover, $a \sim b$ if, and only if, $a \cdot b^{-1} \in K$.

Proof of Theorem 1.1. If X is a suspension the proof of the theorem is immediate, since $(f + g)^* = f^* + g^*$. Suppose then that Y is a homotopy-associative H-space with homology of finite type. In this case, it need not be true that $(f + g)^* = f^* + g^*$. Let $[f], [g] \in N[X, Y]$. But $[f] \in N[X, Y]$ if, and only if, $[f] \equiv 0$. Thus Theorem 1.1 follows from Lemma 5.3 and the above remarks.

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