A CLASS OF NILSTABLE ALGEBRAS[1]

BY

WILLIAM G. WITTHOFT

1. **Introduction.** In what follows we shall consider strictly power-associative algebras over a field of characteristic different from two and three that satisfy the identity

$$(1) \qquad x(xa) + (ax)x = 2(xa)x,$$

whose equivalent linearized form is

$$(2) \qquad x(ya) + y(xa) + (ay)x + (ax)y = 2(ya)x + 2(xa)y.$$

Kosier has shown [10] that every semisimple algebra of this class has an identity and is the direct sum of simple algebras. Moreover, the simple algebras of degree greater than two are Jordan or quasiassociative. Hereafter let A be a central simple degree two algebra of this class. Then A has an identity, $1 = e + f$, that is, the sum of two orthogonal primitive idempotents, e and f, every scalar extension A_K of A is simple, and e and f are primitive in every A_K.

It is known [2; 3] that A can be decomposed relative to e into a vector space direct sum $A = A(1) + A(\frac{1}{2}) + A(0)$, where $e \cdot x = \frac{1}{2}(ex + xe) = \lambda x$ for all x in $A(\lambda)$, $\lambda = 1, \frac{1}{2}, 0$. Moreover $A(1) = eF + N_1$ and $A(0) = fF + N_0$ are vector space decompositions in which N_1^+ and N_0^+ are nilsubalgebras of the algebra A^+. (A^+ is the same linear space as A with the multiplication $x \cdot y = \frac{1}{2}(xy + yx)$.) $A(1)^+$ and $A(0)^+$ are orthogonal subalgebras of A^+, e is a two-sided identity in $A(1)$ and a two-sided annihilator of $A(0)$,

$$xy = yx = 0, \text{ for all } x \text{ in } A(1) \text{ and } y \text{ in } A(0),$$

$$A(1) \cdot A(\tfrac{1}{2}) \subset A(\tfrac{1}{2}) + A(0),$$

$$A(0) \cdot A(\tfrac{1}{2}) \subset A(\tfrac{1}{2}) + A(1),$$

and, for each x and y in $A(\frac{1}{2})$, there is an α in F and n in $N_1 + N_0$ for which $x \cdot y = \alpha 1 + n$.

Without further information about the multiplication of subspaces little can be predicted even for commutative power-associative algebras. L. A. Kokoris has

Received by the editors March 6, 1963.

(1) This paper contains the results of the author's Ph. D. thesis submitted at Illinois Institute of Technology.

shown [5] that simple commutative power-associative algebras of degree two exist that are not Jordan. On the other hand let us define A to be nilstable(2) in case $A_u(\lambda)A_u(\frac{1}{2})$ and $A_u(\frac{1}{2})A_u(\lambda) \subset A_u(\frac{1}{2}) + N_{u(1-\lambda)}, \lambda = 0,1$, where u is any idempotent in A, $A_u(\lambda), \lambda = 1, \frac{1}{2}, 0$, are the subspaces in the decomposition of A relative to u, and $N_{u(1-\lambda)}$ is the nilsubspace of $A_u(1-\lambda)$, $\lambda = 0,1$. Then nilstable simple commutative power-associative algebras were shown by Kokoris [8] to be Jordan. In fact, the added assumption of nilstability has produced the solution of the degree two flexible algebras [9]. We shall therefore assume hereafter that A is nilstable. Lemmas 2.4 and 2.6 will show that A is nilstable if and only if A^+ is nilstable, and Theorem 6 in [6] states that A^+ is nilstable whenever F has characteristic zero, so that we shall also have described all algebras of degree two and characteristic zero. Actually, by these same lemmas it suffices to assume that $A_u(\lambda)A_u(\frac{1}{2})$ and $A_u(\frac{1}{2})A_u(\lambda)$ are contained in $A_u(\frac{1}{2}) + N_{u0} + N_{u1}$ for every idempotent u and $\lambda = 0,1$. The principal result will be that A is a noncommutative Jordan algebra and so possesses a known [9] multiplication table. We shall obtain this result by showing that A^+ is Jordan. Then, by considering a certain trace-like function on A, we shall show that A^+ is simple and that A satisfies the flexible law $x(yx) = (xy)x$. Being Jordan admissible and flexible A is known [12] to satisfy the Jordan identity $x^2(yx) = (x^2y)x$. In case A is not nilstable the algebras constructed by Kokoris provide examples of algebras of our class that do not satisfy the Jordan identity even though they are nilstable with respect to at least one idempotent. Finally, as a consequence of (1) and the flexible law, A satisfies $x(xy) + (yx)x = x(yx) + (xy)x$. Moreover, we can write each x in A as $x = \alpha_1 e + x_{1/2} + \alpha_0 f$ with $x_{1/2}^2 = \beta 1$ and then define $t(x) = \alpha_0 + \alpha_1$ and $n(x) = \alpha_0\alpha_1 - \beta$. Then A is a quadratic algebra and the results of [13] apply.

2. **Construction of an ideal in** A. Let A be decomposed relative to the idempotent e. We shall show that $N_1 + N_0 = 0$ by showing that each N_λ is an ideal of $A(\lambda)$ and then constructing a proper ideal of A containing $N_1 + N_0$. The first few propositions below repeat results found in [10].

LEMMA 2.1. *If x is in $A(\frac{1}{2})$, then ex is in $A(\frac{1}{2})$.*

In case A is commutative $ex = \frac{1}{2}x$. Here let $ex = a_1 + (\frac{1}{2}x + x^*) + a_0$ with a_λ in $A(\lambda)$ and x^* in $A(\frac{1}{2})$, then $xe = -a_1 + (\frac{1}{2}x - x^*) - a_0$, since $ex + xe = x$. Similarly, write $ex^* = b_1 + (\frac{1}{2}x^* + x^{**}) + b_0$. Now (1), with x and a replaced by e and x, respectively, becomes $ex^* = a_1 - a_0 - x^* + 3x^*e$ so that $ex^* = \frac{1}{4}a_1 + \frac{1}{2}x^* - \frac{1}{4}a_0$. Consequently $x^{**} = 0$. Apply (1) again by replacing a and x by x^* and e respectively and obtain $0 = ex^{**} = (1/16)a_1 + (1/16)a_0$. By equating corresponding $A(\lambda)$ components, $a_1 = a_0 = 0$. Thus, for all x in $A(\frac{1}{2})$, we can write

---

(2) The results of this paper for the stable case were announced by Kosier in Abstract 61T-296, Notices Amer, Math. Soc. **8** (1961), 618.

$$ex = \tfrac{1}{2}x + x^*,$$

$$(3) \qquad xe = \tfrac{1}{2}x - x^*,$$

$$ex^* = x^*e = \tfrac{1}{2}x^*.$$

Consequently $(ex)e = \frac{1}{4}x = e(xe)$ so that we have proved

LEMMA 2.2. *Every x in $A(\frac{1}{2})$ is of the form ey and ze for some y and z in $A(\frac{1}{2})$.*

LEMMA 2.3. *If $ex = \alpha x$ for some x in $A(\frac{1}{2})$ and $\alpha \neq \frac{1}{2}$ in F, then $x = 0$.*

For now (1), with a and x replaced by x and e, respectively, becomes $(2\alpha - 1)^2 x = 0$, so that $x = 0$.

THEOREM 2.1. *The subspaces $A(1)$ and $A(0)$ are subalgebras of A.*

Let x and y be in $A(1)$ and write xy as the sum of its $A(\lambda)$ components, $xy = a_1 + a_{1/2} + a_0$, so that $yx = b_1 - a_{1/2} - a_0$. Then (2), with x in the asymmetric position and y and e in the others, becomes $(a_{1/2} + 4a^*) + 2a_0 = 0$, so that $a_0 = 0$ and $ea_{1/2} = \frac{1}{4}a_{1/2}$, implying $a_{1/2} = 0$. Similarly, for x and y in $A(0)$, call $xy = a_1 + a_{1/2} + a_0$ and $yx = -a_1 - a_{1/2} + b_0$. Again giving x the distinguished position in (2) produces $a_1 = a_{1/2} = 0$.

Continuing in this vein, let x be in $A(1)$ and y in $A(\frac{1}{2})$ and write $xy = a_1 + a_{1/2} + a_0$ and $yx = -a_1 + b_{1/2} + b_0$. At the same time call $xy^* = r_1 + r_{1/2} + r_0$ and $y^*x = -r_1 + s_{1/2} + s_0$.

LEMMA 2.4. *For each x in $A(1)$ and y in $A(\frac{1}{2})$, $xy = a_{1/2} + a_0$, $yx = (a_{1/2} - 4a^*) + a_0$, and $xy^* = y^*x = a^*$.*

Replacing x, y, and a in (2) by e, y, and x, respectively, and using (3) results in

$$(4) \qquad \tfrac{1}{2}(b_{1/2} - a_{1/2}) + (b_0 - a_0) = a^* - 3b^*,$$

so that, by equating corresponding components,

$$(5) \qquad a_0 = b_0,$$

and

$$(6) \qquad b_{1/2} - a_{1/2} = 2a^* - 6b^*.$$

Multiplying (4) on the left by e produces

$$(7) \qquad b_{1/2} - a_{1/2} = 4a^* - 8b^*.$$

From (6) and (7) it is apparent that

$$(8) \qquad a^* = b^*.$$

Applying (2) to x, y, and e again, this time with y in the asymmetric position, and using (5) and (8) produces

(9) $$-a_1 + 2a^* = 3y^*x - xy^*,$$

while replacing a with e results in $a_1 + \frac{1}{2}(a_{1/2} - b_{1/2}) = 3y^*x - xy^*$ so that

(10) $$a_1 = 0 \text{ and } b_{1/2} = a_{1/2} - 4a^*.$$

Making the same computations with x, y^*, and e will therefore result in $r_1 = 0$, $r_0 = s_0$, and $s_{1/2} = r_{1/2} - 4r^*$, that is, $xy^* = r_{1/2} + r_0$ and $y^*x = (r_{1/2} - 4r^*) + r_0$. However, if we consider (2) in the light of these results and with y^* in the select position we find $xy^* - y^*x = 4r^* = 0$, so that

(11) $$xy^* = y^*x = r_{1/2} + r_0.$$

But then (9) with the use of (10) and (11) becomes $2a^* = 3y^*x - xy^* = 2(r_{1/2} + r_0)$, and consequently, by equating corresponding components, $r_{1/2} = a^*$ and $r_0 = 0$.

LEMMA 2.5. *For every x in $A(1)$ and y in $A(\frac{1}{2})$, $(xy)^*_{1/2} = (yx)^*_{1/2} = (x \cdot y)^*_{1/2}$ where the subscript $\frac{1}{2}$ indicates the $A(\frac{1}{2})$ component.*

By Lemma 2.4 we have $(xy)_{1/2} = a_{1/2}$ and $(yx)_{1/2} = a_{1/2} - 4a^*$. Therefore $(xy)^*_{1/2} = a^*$, and $e(yx)_{1/2} = \frac{1}{2}a_{1/2} + a^* - 2a^* = \frac{1}{2}(yx)_{1/2} + a^*$ so that $(yx)^*_{1/2} = a^*$.

Analogous methods will produce the corresponding results for products of zero-space elements by half-space elements. Calling $xy = a_1 + a_{1/2} + a_0$ and $yx = b_1 + b_{1/2} - a_0$ for x in $A(0)$ and y in $A(\frac{1}{2})$, we obtain

LEMMA 2.6. *For each x in $A(0)$ and y in $A(\frac{1}{2})$, $xy = a_1 + a_{1/2}$, $yx = a_1 + (a_{1/2} + 4a^*)$, $xy^* = y^*x = a^*$, and $(xy)^*_{1/2} = (yx)^*_{1/2} = (x \cdot y)^*_{1/2}$.*

At this point we can conclude that A is e-nilstable if and only if A^+ is, since Lemmas 2.4 and 2.6 show that the $A(0) + A(1)$ components of xy and yx are the same whenever y is in $A(\frac{1}{2})$ and x is in $A(1) + A(0)$. Now let x and y lie in $A(\frac{1}{2})$ and set $xy = a_1 + a_{1/2} + a_0$ and $yx = b_1 - a_{1/2} + b_0$.

LEMMA 2.7. *For each x and y in $A(\frac{1}{2})$, $xy^* = \frac{1}{4}(b_1 - a_1) - a^* + \frac{1}{4}(a_0 - b_0)$, $y^*x = \frac{1}{4}(b_1 - a_1) + a^* + \frac{1}{4}(a_0 - b_0)$, $xy^* = -yx^*$, $x^*y = -y^*x$, and $x^*y^* = y^*x^* = 0$.*

Substituting x, y, and e in (2) with y in the place of a, we obtain

(12) $$\tfrac{1}{2}(b_1 - a_1) + 4a^* + \tfrac{1}{2}(a_0 - b_0) = 3y^*x - xy^*.$$

Repeating the substitution, this time with e in the place of a, gives

(13) $$yx^* - 3x^*y = 3y^*x - xy^*.$$

Now call $xy^* = r_1 + r_{1/2} + r_0$ and $y^*x = s_1 - r_{1/2} + s_0$. Then, corresponding to (12), we obtain $\frac{1}{2}(r_1 - s_1) + 4r^* + \frac{1}{2}(r_0 - s_0) = 3y^{**}x - xy^{**} = 0$ so that $r_1 = s_1$, $r^* = 0$, and $r_0 = s_0$. Then the right side of (12) becomes $3(r_1 - r_{1/2} + r_0) - (r_1 + r_{1/2} + r_0)$ so that, by equating corresponding components in (12),

$r_1 = \frac{1}{4}(b_1 - a_1)$, $r_{1/2} = -a^*$, and $r_0 = \frac{1}{4}(a_0 - b_0)$. Again, corresponding to (13) is $y^*x^* - 3x^*y^* = 3y^{**}x - xy^{**} = 0$, that is, for all x and y in $A(\frac{1}{2})$, $3x^*y^* = y^*x^*$. But then, by the symmetry of the assumptions for x and y, $3y^*x^* = x^*y^*$, and so $x^*y^* = y^*x^* = 0$. If we continue by calling $yx^* = c_1 + c_{1/2} + c_0$ so that, by what has been shown so far, $x^*y = c_1 - c_{1/2} + c_0$ with $c^* = 0$, then, by (13), $c_1 = \frac{1}{4}(a_1 - b_1) = -r_1$, $c_{1/2} = a^* = -r_{1/2}$, and $c_0 = \frac{1}{4}(b_0 - a_0) = -r_0$. As a corollary to this lemma we might observe that $xx^* = -xx^* = 0$ for all x in $A(\frac{1}{2})$.

To obtain another corollary to Lemma 2.7 write $xy = (\alpha_1 e + n_1) + a_{1/2} + (\alpha_0 f + n_0)$ and $yx = (\beta_1 e + m_1) - a_{1/2} + (\beta_0 f + m_0)$ with α_i and β_i in F and n_i and m_i in N_i, for $i = 1, 0$ then by Lemma 2.7

$$xy^* = \tfrac{1}{4}[(\beta_1 - \alpha_1)e + (m_1 - n_1)] - a^* + \tfrac{1}{4}[(\alpha_0 - \beta_0)f + (n_0 - m_0)]$$

and

$$y^*x = \tfrac{1}{4}[(\beta_1 - \alpha_1)e + (m_1 - n_1)] + a^* + \tfrac{1}{4}[(\alpha_0 - \beta_0)f + (n_0 - m_0)].$$

But $x \cdot y = \gamma 1 + n$ and $x \cdot y^* = \delta 1 + m$ for some γ and δ in F and n and m in $N_1 + N_0$. Consequently $\gamma 1 = \frac{1}{2}(\alpha_1 + \beta_1)e + \frac{1}{2}(\alpha_0 + \beta_0)f$ and $\delta 1 = \frac{1}{4}(\beta_1 - \alpha_1)e + \frac{1}{4}(\alpha_0 - \beta_0)f$, that is, $\alpha_1 + \beta_1 = \alpha_0 + \beta_0$ and $-\alpha_1 + \beta_1 = \alpha_0 - \beta_0$, and so $\beta_1 = \alpha_0$ and $\beta_0 = \alpha_1$, producing the following.

COROLLARY. *For each x and y in $A(\frac{1}{2})$, $xy = (\alpha_1 e + n_1) + a_{1/2} + (\alpha_0 f + n_0)$, $yx = (\alpha_0 e + m_1) - a_{1/2} + (\alpha_1 f + m_0)$.*

By a method similar to that used in [11] we can now show that each N_λ is an ideal of $A(\lambda)$.

LEMMA 2.8. *For every x in $A(1)$ and y in $A(\frac{1}{2})$, $(xy)_{1/2} = 2[e(x \cdot y)]_{1/2} = 2(x \cdot ey)_{1/2}$ and $(yx)_{1/2} = 2[(x \cdot y)e]_{1/2} = 2(x \cdot ye)_{1/2}$. For every x in $A(0)$ and y in $A(\frac{1}{2})$, $(xy)_{1/2} = 2[(x \cdot y)e]_{1/2} = 2(x \cdot ye)_{1/2}$ and $(yx)_{1/2} = 2[e(x \cdot y)]_{1/2} = 2(x \cdot ey)_{1/2}$.*

The first statement is a consequence of Lemma 2.4. Using the notation of the lemma we have $2[e(x \cdot y)]_{1/2} = 2[e(a_{1/2} - 2a^* + a_0)]_{1/2} = 2(\frac{1}{2}a_{1/2} + a^* - a^*) = (xy)_{1/2}$. In the same way $2(x \cdot ey)_{1/2} = (x \cdot y)_{1/2} + 2(x \cdot y^*)_{1/2} = (a_{1/2} - 2a^*) + 2a^* = (xy)_{1/2}$, $2[(x \cdot y)e]_{1/2} = 2[(a_{1/2} - 2a^* + a_0)e]_{1/2} = (yx)_{1/2}$, and $2(x \cdot ye)_{1/2} = (x \cdot y)_{1/2} - 2(x \cdot y^*)_{1/2} = (yx)_{1/2}$. In the same way Lemma 2.6 produces $2[(x \cdot y)e]_{1/2} = 2[(a_1 + a_{1/2} + 2a^*)e]_{1/2} = 2(\frac{1}{2}a_{1/2} - a^* + a^*) = (xy)_{1/2}$, $2(x \cdot ye)_{1/2} = (x \cdot y)_{1/2} - 2(x \cdot y^*)_{1/2} = (xy)_{1/2}$, $2[e(x \cdot y)]_{1/2} = 2[e(a_1 + a_{1/2} + 2a^*)]_{1/2} = (yx)_{1/2}$, and $2(x \cdot ey)_{1/2} = (x \cdot y)_{1/2} + 2(x \cdot y^*)_{1/2} = (yx)_{1/2}$.

THEOREM 2.2. *The nilsubspace N_λ is an ideal of $A(\lambda)$ for $\lambda = 1, 0$.*

Suppose that N_1 is not a subalgebra. Then there are elements x and y of N_1 for which $xy = e + n$, so that $yx = -e + m$, with m and n in N_1. Let a be any element of $A(\frac{1}{2})$ and write, by Lemma 2.4, $xa = b_{1/2} + b_0$ and $ax = (b_{1/2} - 4b^*) + b_0$. Now (2) with x and a interchanged becomes

$$(14) \qquad a + 4a^* = 2ma - am - na + b_{1/2}y - yb_{1/2} + 4yb^* - 8b^*y.$$

The left side of (14) is an element in $A(\frac{1}{2})$ while each term on the right is of the form zw or wz with z in N_1 and w in $A(\frac{1}{2})$. Indeed, since zw and wz are in $A(\frac{1}{2}) + A(0)$ by Lemma 2.4, their zero-space components vanish, and we need only consider their half-space components. But these can all be written, by Lemma 2.8, in the form $(z \cdot we)_{1/2}$ or $(z \cdot ew)_{1/2}$. That is to say, $a + 4a^*$ lies in $[N_1 \cdot A(\frac{1}{2})]_{1/2}$. But $a + 4a^* = 4e(ea)$, and every element in $A(\frac{1}{2})$ is of the form ec for some c in $A(\frac{1}{2})$. Consequently,

$$(15) \qquad A(\tfrac{1}{2}) \subset [N_1 \cdot A(\tfrac{1}{2})]_{1/2}.$$

To state (15) differently, define, for each x in $A(1)$, a linear transformation, S_x, in $A(\frac{1}{2})$ by the formula $S_x(w) = (x \cdot w)_{1/2}$ for all w in $A(\frac{1}{2})$. It has been shown that S_x is a nilpotent transformation whenever x is a nilpotent element [3] and that the associative algebra generated by a set of transformations, S_x, each determined by a nilpotent x, is a nilpotent algebra [1]. Calling this enveloping algebra, S_{N_1} we can rewrite (15) as $A(\frac{1}{2}) \subset A(\frac{1}{2})S_{N_1}$, and therefore $A(\frac{1}{2}) \subset A(\frac{1}{2})S_{N_1} \subset \cdots \subset A(\frac{1}{2})S_{N_1}^n \subset \cdots$ for all positive integers n. Since $S_{N_1}^k = 0$ for some k, $A(\frac{1}{2}) = 0$. But then A is the algebra direct sum $A(1) \oplus A(0)$, contrary to the simplicity of A. Hence N_1 is a subalgebra and therefore an ideal of A_1.

In the same way N_0 is shown to be a subalgebra. Here we use the fact that for every x in $A(0)$ there is a linear transformation, T_x, in $A(\frac{1}{2})$ given by $T_x(w) = (x \cdot w)_{1/2}$ for all w in $A(\frac{1}{2})$. Again, T_x is nilpotent whenever x is in N_0, the enveloping algebra T_{N_0} generated by all T_x with x in N_0 is nilpotent, and $A(\frac{1}{2}) \subset A(\frac{1}{2})T_{N_0}$, producing the same sort of contradiction.

Now call $N = N_0 + N_1$, $B = A(\frac{1}{2}) + N$, and $I = \{x \in A : Ax + xA \subset B\}$. Then, by nilstability and Theorem 2.2, $N \subset I$. In fact $N = I \cap [A(1) + A(0)]$, since $e^2 = e$ is not in B, so that we may write $I = I_{1/2} + N \subset B$. We shall show that I is an ideal(3). To do this we shall show separately that $AN + NA$ and $AI_{1/2} + I_{1/2}A$ are contained in I. First observe that $[A(1) + A(0)]N + N[A(1) + A(0)] \subset N \subset I$. Next, call $^*A = [NA(\frac{1}{2})]_{1/2}$ and $A^* = [A(\frac{1}{2})N]_{1/2}$ so that

$$A(\tfrac{1}{2})N + NA(\tfrac{1}{2}) \subset A^* + {}^*A + N.$$

We shall see that $A^* + {}^*A \subset I$. We begin with methods similar to those used in [9].

LEMMA 2.9. *If $x \cdot y$ and $x \cdot ey$ are in N for some x and y in $A(\frac{1}{2})$ then xy and yx are in B.*

---

(3) The author is grateful to the referee for suggesting I as an ideal easier to establish than that in the original.

Using the notation and results of Lemma 2.7 we have $x \cdot y = \frac{1}{2}(a_1 + b_1) + \frac{1}{2}(a_0 + b_0)$ in N and $x \cdot ey = \frac{1}{2}x \cdot y + x \cdot y^* = \frac{1}{2}b_1 + \frac{1}{2}a_0$ in N. Hence b_1 is in N_1 and a_0 is in N_0 so a_1 and b_0 are in N_1 and N_0, respectively, and xy and yx are in B.

LEMMA 2.10. *The subspace B is not an ideal in A^+.*

For, $N[A(1) + A(0)] = [A(1) + A(0)]N \subset N \subset B$ since N_λ is an ideal of $A(\lambda)$, and $A(\frac{1}{2})[A(1) + A(0)]$ and $[A(1) + A(0)]A(\frac{1}{2}) \subset B$ by nilstability so that, were B an ideal in A^+, $A(\frac{1}{2}) \cdot A(\frac{1}{2})$ would be contained in B, indeed in N. But then $x \cdot y$ and consequently $x \cdot ey$ is in N for all x, y in $A(\frac{1}{2})$ so that $A(\frac{1}{2})A(\frac{1}{2}) \subset B$ and B would be an ideal in A. By simplicity of A, and since e is not in B, $A(\frac{1}{2}) \subset B = 0$, and $A = A(1) \oplus A(0)$ contrary to simplicity.

As a result of Lemma 2.10 we can prove, as Kokoris did in Lemma 7 of [7], that $A(\frac{1}{2})$ has a non-nilpotent element. In fact, all the results of that paper concluding with the statement that the set $C = A(\frac{1}{2}) \cdot N + N$ is an ideal in A^+ are now at our disposal.

LEMMA 2.11. *If x is in $^*A + A^*$ and y is in $A(\frac{1}{2})$ then xy and yx are in B.*

Every such x is a sum of elements of the form $(nz)_{1/2}$ or $(zn)_{1/2}$ for some n in N and z in $A(\frac{1}{2})$. For any such n and z and y in $A(\frac{1}{2})$, $y \cdot (n \cdot z)$ is in C since C is an ideal in A^+ and $n \cdot z$ is in $A(\frac{1}{2}) \cdot N \subset C$. Indeed, $y \cdot (n \cdot z)_1$ and $y \cdot (n \cdot z)_0$ are in C since $(n \cdot z)_1$ and $(n \cdot z_0)$ are in N by nilstability, while $y \cdot (n \cdot z)_{1/2}$ is in C and in $A(1) + A(0)$, and therefore in N. That is, $y \cdot (n \cdot z)_{1/2}$ is in N for every y in $A(\frac{1}{2})$, hence $ey \cdot (n \cdot z)_{1/2}$ is in N for every y in $A(\frac{1}{2})$, and, by Lemma 2.9, $y(n \cdot z)_{1/2}$ and $(n \cdot z)_{1/2}y$ are in B. In case n is in N_1, $(nz)_{1/2} = 2(n \cdot ez)_{1/2} = (n \cdot z)_{1/2} + 2(n \cdot z^*)_{1/2}$ and $(zn)_{1/2} = 2(n \cdot ze)_{1/2} = (n \cdot z)_{1/2} - 2(n \cdot z^*)_{1/2}$ by Lemma 2.8. In case n is in N_0, $(nz)_{1/2} = (n \cdot z)_{1/2} - 2(n \cdot z^*)_{1/2}$ and $(zn)_{1/2} = (n \cdot z)_{1/2} + 2(n \cdot z^*)_{1/2}$ by the same lemma. So in either case $y(zn)_{1/2}$, $y(nz)_{1/2}$, $(zn)_{1/2}y$, and $(nz)_{1/2}y$ will be in B provided $y(n \cdot z^*)_{1/2}$ and $(n \cdot z^*)_{1/2}y$ are in B. But, in fact, for any z in $A(\frac{1}{2})$, z^* is in $A(\frac{1}{2})$ and so $y \cdot (n \cdot z^*)$ is in C and, as in the preceding, $y(n \cdot z^*)_{1/2}$ and $(n \cdot z^*)_{1/2}y$ are in B.

We now remark that, for all x in $^*A + A^*$, $Nx + xN \subset B$ by nilstability, ex and xe and consequently fx and xf are in B by Lemma 2.1, and therefore, by Lemma 2.11, $^*A + A^* \subset I$. So we have shown that $AN + NA \subset I$, and to show that $AI_{1/2} + I_{1/2}A \subset I$ now requires only showing that $eI_{1/2}$ (and consequently $I_{1/2}e, fI_{1/2}$, and $I_{1/2}f$) and $A(\frac{1}{2})I_{1/2} + I_{1/2}A(\frac{1}{2})$ are contained in I. We begin with the former.

LEMMA 2.12. *If x is in $I_{1/2}$ then ex is in I.*

For all y in $A(\frac{1}{2})$, we have, by Lemma 2.7, $x^*y = -y^*x$ is in B and $yx^* = -xy^*$ is in B. Further, $x^*e = ex^* = x^*f = fx^* = \frac{1}{2}x^*$ is in B. Finally, for all n in N,

x^*n and nx^* are in $A^* + {}^*A + N \subset B$. Thus x^*, and consequently $ex = \frac{1}{2}x + x^*$, are in I.

LEMMA 2.13. *If x is in $I_{1/2}$ and y is in $A(\frac{1}{2})$ then xy and yx are in I.*

We have that $(xy)_1$, $(xy)_0$, $(yx)_1$, and $(yx)_0$ are in N since x is in $I_{1/2}$, and $(yx)_{1/2} = -(xy)_{1/2}$ so we need only examine $(xy)_{1/2}$. But, for all n in N, $(xy)_{1/2}\,n$ and $n(xy)_{1/2}$ are in ${}^*A + A^* + N \subset B$. Also $(xy)_{1/2}e$ and $e(xy)_{1/2}$ are in $A(\frac{1}{2}) \subset B$, so we need only examine products $(xy)_{1/2}z$ and $z(xy)_{1/2}$ with z in $A(\frac{1}{2})$. Replace a and y in (2) by y and z, respectively, and obtain $z(xy) + (yx)z - 2(xy)z = 2(zy)x - (yz)x - x(zy)$. The right side here is a member of B since x is is in I. On the left, the terms $z(xy)_1$, $z(xy)_0$, etc., are in B by nilstability, and, therefore, for all z in $A(\frac{1}{2})$, $z(xy)_{1/2} + (yx)_{1/2}z - 2(xy)_{1/2}z$ is in B. Now $(xy)_{1/2} = -(yx)_{1/2}$ so that

$$z(xy)_{1/2} - 3(xy)_{1/2}z \text{ is in } B. \tag{16}$$

By Lemma 2.7 we can call $(xy)_{1/2}z = a_1 + a_{1/2} + a_0$ and $z(xy)_{1/2} = (a_1 + 4r_1) - a_{1/2} + (a_0 - 4r_0)$ where $r_1 = [(xy)_{1/2}z^*]_1 = [z^*(xy)_{1/2}]_1$ and $r_0 = [(xy)_{1/2}z^*]_0 = [z^*(xy)_{1/2}]_0$. Then (16) becomes: $-2a_1 + 4r_1$ is in N_1 and $-2a_0 - 4r_0$ is in N_0. But, by repeating the preceding argument with z^* in the place of z we obtain in place of (16): $z^*(xy)_{1/2} - 3(xy)_{1/2}z^*$ is in B for all z in $A(\frac{1}{2})$. That is, r_1 is in N_1 and r_0 is in N_0 and therefore a_1 is in N_1 and a_0 is in N_0 so that $(xy)_{1/2}z$ and $z(xy)_{1/2}$ are in B. This completes the proof of Lemma 2.13 and at the same time produces the following desired result.

THEOREM 2.3. *The set I is an ideal and $N \subset I \subset B$.*

3. **Classification of A.**

THEOREM 3.1. *A is Jordan admissible.*

Since A is simple and e is not in I, $N \subset I = 0$ and $A = eF + A(\frac{1}{2}) + fF$. By Lemma 10 of [3], $A(\frac{1}{2}) \cdot A(\frac{1}{2}) \subset 1F$. From this fact follows the argument on p. 331 of [4] by which it is shown that A^+ is Jordan.

Now let $\delta(x) = \alpha_1 + \alpha_0$ for each $x = \alpha_1 e + x_{1/2} + \alpha_0 f$ in A. Then δ is a linear functional. We shall show that δ satisfies, for all x, y, and z in A,

(a) $\delta(xy) = \delta(yx)$,

(b) $\delta[(xy)z] = \delta[x(yz)]$, and

(c) $\delta(x) = 0$, whenever x is nilpotent.

Then it is known that the set N_δ of all x in A for which $\delta(xy) = 0$ for every y in A is an ideal of A containing the nilradical. Now $\delta(ee) = \delta(e) = 1$ so, as before, $N_\delta = 0$. In fact we shall show that δ satisfies the three properties in the attached algebra A^+; hence N_δ^+ is an ideal of A^+ containing the nilradical of A^+. But N_δ^+ and N_δ are the same subspace since $\delta(x \cdot y) = \frac{1}{2}\delta(xy + yx) = \frac{1}{2}[\delta(xy) + \delta(yx)] = \delta(xy)$. Hence A^+ is semisimple and consequently is either simple or else the direct sum

$A(1) \oplus A(0)$. The latter contradicts the simplicity of A, so that, pending proofs of (a), (b), and (c), we have the following.

THEOREM 3.2. *A is J-simple.*

To prove (a) write $x = \alpha_1 e + x_{1/2} + \alpha_0 f$ and $y = \beta_1 e + y_{1/2} + \beta_0 f$ and, using the corollary to Lemma 2.7., $x_{1/2}y_{1/2} = \gamma_1 e + a_{1/2} + \gamma_0 f$ and $y_{1/2}x_{1/2} = \gamma_0 e - a_{1/2} + \gamma_1 f$. Then $xy = (\alpha_1\beta_1 + \gamma_1)e + (xy)_{1/2} + (\alpha_0\beta_0 + \gamma_0)f$ and $yx = (\alpha_1\beta_1 + \gamma_0)e + (yx)_{1/2} + (\alpha_0\beta_0 + \gamma_1)f$ so that $\delta(xy) = \alpha_1\beta_1 + \gamma_1 + \alpha_0\beta_0 + \gamma_0 = \delta(yx)$. Of course $\delta(x \cdot y) = \delta(y \cdot x)$ since $x \cdot y = y \cdot x$.

To verify (b) observe first of all that $\delta[(xy)z - x(yz)] = \delta[(x \cdot y) \cdot z - x \cdot (y \cdot z)]$ so that δ satisfies (b) in A if and only if it does in A^+. By (2) and the linearity of δ, $\delta[x(yz)] + \delta[y(xz)] + \delta[(zy)x] + \delta[(zx)y] = 2\delta[(xz)y] + 2\delta[(yz)x]$, which, by application of (a) to its right member, becomes $-\delta[x(yz)] - \delta[y(xz)] + \delta[(zy)x] + \delta[(zx)y] = 0$. Applying (a) again produces $-\delta[(yz)x] - \delta[(xz)y] + \delta[x(zy)] + \delta[y(zx)] = 0$ or $\delta[x(zy) - (xz)y] = \delta[(yz)x - y(zx)]$. Consequently

$$\begin{aligned} &4\delta[x \cdot (y \cdot z) - (x \cdot y) \cdot z] \\ &\quad = \delta[x(yz) + x(zy) + (yz)x + (zy)x - (xy)z - (yx)z - z(xy) - z(yx)] \\ &\quad = 2\delta[x(yz) - (xy)z + (zy)x - z(yx)] = 4\delta[x(yz) - (xy)z]. \end{aligned}$$

Now δ can be shown to satisfy (b) in A^+ by a direct computation. Let $x = \alpha_1 e + x_{1/2} + \alpha_0 f$, $y = \beta_1 e + y_{1/2} + \beta_0 f$, and $z = \gamma_1 e + z_{1/2} + \gamma_0 f$. Then $x \cdot (y \cdot z) = \alpha_1\beta_1\gamma_1 e + \alpha_1(y_{1/2} \cdot z_{1/2})_1 + \frac{1}{2}(\beta_1 + \beta_0)(x_{1/2} \cdot z_{1/2})_1 + \frac{1}{2}(\gamma_1 + \gamma_0)(x_{1/2} \cdot y_{1/2})_1 + [x \cdot (y \cdot z)]_{1/2} + \alpha_0\beta_0\gamma_0 f + \alpha_0(y_{1/2} \cdot z_{1/2})_0 + \frac{1}{2}(\beta_1 + \beta_0)(x_{1/2} \cdot z_{1/2})_0 + \frac{1}{2}(\gamma_1 + \gamma_0)(x_{1/2} \cdot y_{1/2})_0$. If we call $(x_{1/2} \cdot y_{1/2})_1 = \phi e$, then, by the corollary to Lemma 2.7, $(x_{1/2} \cdot y_{1/2})_0 = \phi f$. Similarly, let $(x_{1/2} \cdot z_{1/2})_1 = \psi e$ and $(y_{1/2} \cdot z_{1/2})_1 = \omega e$. Then $\delta[x \cdot (y \cdot z)] = \alpha_1\beta_1\gamma_1 + \alpha_0\beta_0\gamma_0 + (\alpha_1 + \alpha_0)\omega + (\beta_1 + \beta_0)\psi + (\gamma_1 + \gamma_0)\phi$. In the same way we find the value of $\delta[(x \cdot y) \cdot z]$ and so verify (b) in A^+.

Property (c) can be obtained by observing that A has the subspace decomposition $A = 1F + uF + A(\frac{1}{2})$ where $u = e - f$ and so $u^2 = 1$ and $u \cdot x = 0$ for all x in $A(\frac{1}{2})$. Let $x = \alpha_1 + \beta u + x_{1/2}$, with $x_{1/2}^2 = \gamma 1$, be any element in A, then $\delta(x) = 2\alpha$ and $x^2 = (\alpha^2 + \beta^2 + \gamma)1 + 2\alpha\beta u + 2\alpha x_{1/2}$ so that

$$x^2 - 2\alpha x + (\alpha^2 - \beta^2 - \gamma)1 = 0. \tag{17}$$

Whenever x is nilpotent, the minimal polynomial for x is of the form x^k for some $k > 1$ and divides the left member of (17), so that $k = 2$ and $2\alpha = \delta(x) = 0$. By the agreement of powers in A and A^+ δ satisfies (c) in A^+ as well.

Moreover, it is shown in [10] that A satisfies the flexible law by showing that, as a consequence of (2) and properties (a) and (b) of δ, $\delta[((xy)x - x(yx))z] = 0$ for all x, y, and z in A. Thus $(xy)x - x(yx)$ is in N_δ and so vanishes for all x and y in A.

Being flexible and J-admissible, A is known [12] to satisfy the Jordan identity. We can summarize the results of this section in the following principal result.

THEOREM 3.3. *If A is a nilstable central simple strictly power-associative algebra of degree two satisfying* (1) *over a field of characteristic* $\neq 2, 3$ *then A is a noncommutative Jordan algebra that is J-simple.*

The multiplication table for every such algebra has been described [9]. On the other hand, when F has a characteristic different from zero and the assumption of nilstability is dropped, examples of commutative (hence satisfying(1)), power-associative, central simple, degree two algebras are known [5] that fail to satisfy the Jordan identity because they are not u-stable with respect to every idempotent u they possess.

BIBLIOGRAPHY

1. A. A. Albert, *On Jordan algebras of linear transformations*, Trans. Amer. Math. Soc. **59** (1946), 524–555.

2. ———, *Power-associative rings*, Trans. Amer. Math. Soc. **64** (1948), 552–593.

3. ———, *A theory of commutative power-associative algebras*, Trans. Amer. Math. Soc. **69** (1950), 503–527.

4. ———, *On commutative power-associative algebras of degree two*, Trans. Amer. Math. Soc. **74** (1953), 323–343.

5. L. A. Kokoris, *Power-associative commutative algebras of degree two*, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 534–537.

6. ———, *New results on power-associative algebras*, Trans. Amer. Math. Soc. **77** (1954), 363–373.

7. ———, *Simple power-associative algebras of degree two*, Ann. of Math. (2) **64** (1956), 544–550.

8. ———, *On nilstable algebras*, Proc. Amer. Math. Soc. **9** (1958), 697–701.

9. ———, *Flexible nilstable algebras*, Proc. Amer. Math. Soc. **13** (1962), 335–340.

10. Frank Kosier, *On a class of nonflexible algebras*, Trans. Amer. Math. Soc. **102** (1962), 299–318.

11. R. H. Oehmke, *On flexible power-associative algebras of degree two*, Proc. Amer. Math. Soc. **12** (1961), 151–158.

12. R. D. Schafer, *Noncommutative Jordan algebras of characteristic* 0, Proc. Amer. Math. Soc. **6** (1955), 472–475.

13. ———, *On a class of quadratic algebras*, Proc. Amer. Math. Soc. **13** (1962), 187–191.

ILLINOIS INSTITUTE OF TECHNOLOGY,
CHICAGO, ILLINOIS