IDEALS AND INVARIANT SUBSPACES OF ANALYTIC FUNCTIONS(1)

BY

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1. **Introduction.** Let K be the open unit disk and Λ, the unit circle. It is well known that if f is a bounded analytic function on K then it has nontangential boundary values $f^*(e^{i\theta})$ a.e. (almost everywhere) on Λ. We say f is an *inner function* if $|f^*| = 1$ a.e. on Λ. Inner functions have been used to advantage in answering questions about certain spaces of analytic functions on K. In this paper corresponding results are obtained for more general regions using a generalization of inner function.

First let us state one result for the disk. Let $A(K)$ be the class of functions continuous on $\bar{K}$ and analytic on K. Under pointwise addition and multiplication $A(K)$ is a Banach algebra where for $f \in A(K)$ the norm of f is defined by

$$\|f\| = \max_{\bar{K}} |f|.$$

The following theorem describing the closed ideals of $A(K)$ was proved independently by A. Beurling (unpublished) and W. Rudin [8].

(1.1) THEOREM. *If I is a closed ideal of $A(K)$ then there is a closed set E on Λ and an inner function ϕ such that $I = \{f \in A(K) \mid f = 0$ on E; f/ϕ is bounded on $K\}$.*

A direct translation of this theorem for other regions is not true in general. For let R be an annulus $\{z \mid r_1 < |z| < r_2\}$ and Γ the boundary of R. A bounded analytic function F on R will have nontangential boundary values $F^*(r_j e^{i\theta}), j = 1,2,$ a.e. on Γ. Let $A(R)$ be the Banach algebra of functions continuous on $\bar{R}$ and analytic on R. For ζ fixed in R, $I = \{F \in A(R) \mid F(\zeta) = 0\}$ is a closed ideal of $A(R)$. Suppose (1.1) is true for $A(R)$. Then there is a closed set E on Γ and a bounded single-valued analytic function Φ on R with $|\Phi^*| = 1$ a.e. on Γ such that $I = \{F \in A(R) \mid F = 0$ on E; F/Φ is bounded on $R\}$. Since $z - \zeta$ is in I it follows that E is the empty set. It also follows that Φ must be bounded away from zero near Γ. This in turn means that Φ can be extended to be analytic in a neigh-

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borhood of $\bar{R}$. Then $|\Phi| = 1$ continuously on Γ. Also Φ has a simple zero at ζ and vanishes nowhere else on $\bar{R}$. Thus $\log|\Phi(z)| = -G(z,\zeta)$ where $G(z,\zeta)$ is the Green's function of R with singularity at ζ. Then $\Phi(z) = \exp[-(G(z,\zeta) + iH(z))]$ where $H(z)$ is the harmonic conjugate of $G(z,\zeta)$ on $\bar{R} - \{\zeta\}$. But the period of $H(z)$ along $r_1 e^{i\theta}$ is not a multiple of 2π and thus Φ is not single-valued, contradicting our assumption. However, this example does suggest how we should generalize the notion of inner function: we must permit an inner function to be multiple-valued.

In this paper we will be considering a region R on a Riemann surface which satisfies the following conditions:

(a) $\bar{R}$, the closure of R, is compact.

(b) Γ, the boundary of R, is union of a finite number of disjoint simple closed analytic curves $\Gamma_1, \Gamma_2, \cdots, \Gamma_N$.

(c) R lies on one side of Γ.

$A(R)$ will denote the Banach algebra of functions continuous on $\bar{R}$ and analytic on R. One of our main results is the determination of the closed ideals of $A(R)$.

For $j = 1,2,\cdots,N$ let Φ_j be a 1-1 analytic map from an annulus $R_0 = \{z \mid r_1 < |z| < r_2\}$, where $r_1 < 1 < r_2$, onto a neighborhood of Γ_j such that $\Phi_j(\Lambda) = \Gamma_j$ and $\Phi_j(R_0 \cap K) \subset R$. Let ν be the measure induced on Γ by the measure $d\theta$ on $\Lambda = \{e^{i\theta} \mid 0 \leqq \theta < 2\pi\}$ and the homeomorphisms Φ_j restricted to Λ, $j = 1,2,\cdots,N$.

A *multiplicative function* F on R is a multiple-valued analytic function on R such that $|F|$ is single-valued. In §2 we show that if F is a bounded multiplicative function on R, then $|F|$ possesses nontangential boundary values a.e.-ν (almost everywhere with respect to ν) on Γ. A bounded multiplicative function Φ is an *inner function* if $|\Phi| = 1$ a.e.-ν on Γ. For Φ an inner function and E a closed set on Γ we define $I(\Phi) = \{F \in A(R) \mid |F|/|\Phi| \text{ is bounded on } R\}$ and $I(E) = \{F \in A(R) \mid F = 0 \text{ on } E\}$. It is easy to see that $I(E)$ is a closed ideal of $A(R)$. In §4 we show that $I(\Phi)$ is also a closed ideal of $A(R)$. In §7 we prove the following generalization of (1.1).

THEOREM 1. *If I is a closed ideal of $A(R)$, there is an inner function Φ and a closed set E on Γ such that $I = I(\Phi) \cap I(E)$.*

Before stating our second main result we need some more definitions. $H_\infty(R)$ is the Banach algebra of bounded analytic functions on R with the norm $\|F\| = \sup_R |F|$. $H_p(R)$ for $1 \leqq p < \infty$ is the class of analytic functions F on R such that $|F|^p$ has a harmonic majorant on R. In §7 we show that $H_p(R)$ is a Banach space where $\|F\|_p = (H_F(t_0))^{1/p}$ for t_0 a fixed point on R and H_F the least harmonic majorant of $|F|^p$ on R. It turns out that $H_2(R)$ is a Hilbert space with this norm. A closed subspace C of $H_2(R)$ is said to be *invariant* if $FG \in C$ for all $F \in A(R)$ and all $G \in C$. For Φ an inner function we define $(C\Phi) = \{F \in H_2(R) \mid |F|^2/|\Phi|^2 \text{ has a harmonic majorant on } R\}$. In §8 we show

$C(\Phi)$ is a closed invariant subspace of $H_2(R)$. The following theorem is proved in §9.

THEOREM 2. *If C is a closed invariant subspace of $H_2(R)$ then there is an inner function Φ such that $C = C(\Phi)$.*

This theorem reduces to the known result for the case $R = K$ due to A. Beurling [1]. Beurling's theorem has been generalized in many ways. (See [3, Chapter 7].)

2. $H_p(K)$ **and boundary values on** Γ. We will first review some of the properties of functions in $H_p(K)$. Where other references are not given we refer the reader to [3].

It is almost immediate from definition that the classes $H_p(K)$ are invariant under conformal transformations of K. For suppose s maps K conformally onto K. Let $f \in H_p(K)$ where $1 \leqq p < \infty$ and let h be a harmonic majorant of $|f|^p$. Then $h \cdot s$ is a harmonic majorant of $|f \cdot s|^p$. Thus $f \cdot s \in H_p(K)$.

$H_p(K)$ is usually defined as the class of analytic functions f on K such that $\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta$ is bounded for $0 < r < 1$. The previous definition is of course equivalent to this one. (See [7].)

If $f \in H_p(K)$, $1 \leqq p \leqq \infty$, then f has nontangential boundary values $f^*(e^{i\theta})$ a.e. on Λ, $f^* \in L_p(\Lambda, d\theta)$, and if $f \not\equiv 0$ then $\log |f^*| \in L_1(\Lambda, d\theta)$. $H_p(K)$ is a Banach space with the norm $\|f\|_p = ((1/2\pi)\int_0^{2\pi} |f^*(e^{i\theta})|^p d\theta)^{1/p}$ for $1 \leqq p < \infty$ and $\|f\|_\infty = \|f^*\|_{L_\infty} = \sup_K |f|$ for $p = \infty$.

If k is a non-negative function in $L_p([0,2\pi], d\theta)$, $1 \leqq p \leqq \infty$, such that $\log k(\theta) \in L_1([0,2\pi], d\theta)$, then

$$(2.1) \qquad h(z) = \exp\,(1/2\pi) \int_0^{2\pi} (\log k(\theta)) \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta$$

is in $H_p(K)$ and $|h^*(e^{i\theta})| = k(\theta)$ a.e. A function which can be represented as in (2.1) is called an *outer function*.

The next result follows almost from definition.

(2.2) LEMMA. *If f and g are outer functions and $f^*/g^* \in L_p(\Lambda, d\theta)$ then f/g is an outer function in $H_p(K)$.*

If in formula (2.1) we let $k = |f^*|$ for $f \in H_p(K)$ and $f \not\equiv 0$, the resulting function is denoted by f_1 and is called the *outer factor* of f. Note that $|f_1^*| = |f^*|$ a.e. on Λ. If $f \in H_p(K)$ then $f_0 = f/f_1$ is an inner function and is called the *inner factor* of f. (This terminology is due to A. Beurling.) We have then for $f \in H_p(K)$ that $f = f_0 f_1$. The representation of f as the product of an inner function and an outer function is unique. For if $f = gh$ where g is an inner function and h is an outer function then $|f_1^*| = |f^*| = |g^*|\,|h^*| = |h^*|$ a.e.. Thus $f_1 = h$ and $f_0 = h$.

For $f, g \in H_\infty(K)$ we say f *divides* g (written $f|g$) if $g/f \in H_\infty(K)$. Inner functions on K display an important property with respect to this relation:

(2.3) LEMMA. *If $\mathscr{F}$ is a collection of inner functions on K then there is an inner function ϕ such that*

(1) *$\phi|f$ for all $f \in \mathscr{F}$.*

(2) *If α is an inner function such that $\alpha|f$ for all $f \in \mathscr{F}$ then $\alpha|\phi$.*

For ϕ and $\mathscr{F}$ as above ϕ is called a *greatest common divisor* (written g.c.d.) of $\mathscr{F}$. If ψ is also a g.c.d. of $\mathscr{F}$ then $\psi = \lambda\phi$ for some constant λ with $|\lambda| = 1$.

We will need the following lemma later.

(2.4) LEMMA. *Let $\mathscr{F}$ be a collection of inner functions on K and ϕ a g.c.d. of $\mathscr{F}$. If h and α are inner functions on K such that $\alpha|hf$ for all $f \in \mathscr{F}$, then $\alpha|h\phi$.*

Proof. Let δ be a g.c.d. of the collection of inner functions $\mathscr{G} = \{hf | f \in \mathscr{F}\}$. Then $h\phi|\delta$ since $h\phi|hf$ for all $f \in \mathscr{F}$. Then $\beta = \delta/(h\phi)$ is an inner function and $\delta = \beta h\phi$. Thus $\beta h\phi|hf$ for all $f \in \mathscr{F}$ and hence $\beta\phi|f$ for all $f \in \mathscr{F}$. Thus $\beta\phi|\phi$. This implies β is constant. Hence $h\phi$ is a g.c.d. of $\mathscr{G}$. Therefore $\alpha|h\phi$ since $\alpha|g$ for all $g \in \mathscr{G}$.

The existence of boundary values on Γ for functions in $H_p(R)$ is deduced quickly from the existence of boundary values of functions in $H_p(K)$. Let $F \in H_p(R)$ and α be a simple arc on Γ. Let β be a simple arc with its interior in R connecting the end points of α such that $\beta \cup \alpha$ bounds a simply connected region $\mathscr{N}$. Let P be a 1-1 analytic map of K onto $\mathscr{N}$. Then P can be extended to be a homeomorphism of $\bar{K}$ onto $\bar{\mathscr{N}} = \mathscr{N} \cup \alpha \cup \beta$ and to be 1-1 analytic in the neighborhood of the inverse image of the interior of α. Then $F \cdot P \in H_p(K)$. This is clear for $p = \infty$. For $1 \leqq p < \infty$ let H be a harmonic majorant $|F|^p$ on R. Then $H \cdot P$ is a harmonic majorant of $|F \cdot P|^p$ on K and thus $F \cdot P \in H_p(K)$. Thus $F \cdot P$ has nontangential boundary values a.e.-$d\theta$ on Λ which implies F has nontangential boundary values a.e.-ν on α. Since the boundary function of $F \cdot P$ is in $L_p(\Lambda, d\theta)$ it follows that the boundary function of F is in $L_p(\alpha', \nu)$ for any proper subarc α' of α. These comments yield the following theorem.

(2.5) THEOREM. *If $F \in H_p(R)$, $1 \leqq p \leqq \infty$, then $F(t)$ approaches a limit $F^*(\tau)$ as t approaches $\tau \in \Gamma$ nontangentially for a.a.-ν (almost all with respect to ν) $\tau \in \Gamma$; and $F^* \in L_p(\Gamma, \nu)$.*

The obvious adjustments in the above remarks give us:

(2.6) THEOREM. *If F is a bounded multiplicative function on R, then $|F(t)|$ approaches a limit $|F^*(\tau)|$ as t approaches $\tau \in \Gamma$ nontangentially for a.a.-ν $\tau \in \Gamma$; and $|F^*| \in L_\infty(\Gamma, \nu)$.*

3. **Modulus invariant analytic functions on K and multiplicative functions on R.** In this section we investigate a correspondence between certain analytic functions

on K and the multiplicative functions on R which arises by viewing K as the universal covering surface of R.

(3.1) DEFINITION. T is a map from K onto R which has the following properties:

(a) T is analytic and locally 1-1.

(b) Given a path(3) α on R and a point $z \in K$ with $T(z) = \alpha(0)$ there is a unique path $\tilde{\alpha}$ on K such that $\alpha = T \cdot \tilde{\alpha}$ and $\tilde{\alpha}(0) = z$.

That such a map T exists is well known. (See [9, Chapter 4].) It is easy to see that T is 1-1 if and only if R is conformally equivalent to K.

(3.2) DEFINITION. S is the set of all fractional transformations $s(z) = \lambda(z-a)/(\bar{a}z-1)$, where λ and a are constants with $|\lambda| = 1$ and $|a| < 1$, mapping K conformally onto itself such that $T \cdot s = T$.

S is then a Fuchsian group without fixed points. That is, S is a group under function composition; if $s \in S$ and s is not the identity then $s(z) \neq z$ for all $z \in K$; and given any $z \in K$ the set $\{s(z) \mid s \in S\}$ has no accumulation point in K. (See [9, Chapter 9].) Also S is *transitive* in the sense that if $T(z_1) = T(z_2)$ for $z_1, z_2 \in K$, then there is a unique $s \in S$ such that $s(z_1) = z_2$. (See [9, Chapter 4].) Of course when T is 1-1 S contains just the identity.

An analytic function f on K is said to be *invariant* if $f \cdot s = f$ for all $s \in S$. A measurable function m on Λ is said to be *invariant* if $m \cdot s = m$ a.e. on Λ for all $s \in S$. Since

$$\|f \cdot s - f\|_p = \left((1/2\pi)\int_0^{2\pi} |f^* \cdot s - f^*|^p d\theta\right)^{1/p}$$

for $1 \leqq p < \infty$ and

$$\|f \cdot s - f\|_\infty = \|f^* \cdot s - f^*\|_{L_\infty}$$

we have:

(3.3) LEMMA. *If $f \in H_p(K)$, $1 \leqq p \leqq \infty$, then f is invariant if and only if f^* is invariant.*

An analytic function f on K is said to be *modulus invariant* (written m.i.) if $|f \cdot s| = |f|$ for all $s \in S$. Note that f is m.i. if and only if for each $s \in S$ there is a constant λ_s, $|\lambda_s| = 1$, such that $f \cdot s = \lambda_s f$.

We deal next with the correspondence induced by T between the modulus invariant analytic functions on K and the multiplicative functions on R.

(3.4) DEFINITION. By T_z for $z \in K$ we mean the map T restricted to a neighborhood of z where T is 1-1. For f an m.i. analytic function on K we define $f \cdot T^{-1}$ as the set of function elements $f \cdot (T_z)^{-1}$ centered at $T(z)$ for all $z \in K$.

(3.5) LEMMA. *If f is an m.i. analytic function on K then $f \cdot T^{-1}$ is a multiplicative function on R.*

(3) By a path we mean a continuous map from the closed interval [0, 1].

Proof. We will show first that $f \cdot T^{-1}$ is a (multiple-valued) analytic function on R. Consider function elements $f \cdot (T_{z_1})^{-1}$ and $f \cdot (T_{z_2})^{-1}$ in $f \cdot T^{-1}$. Let $t_j = T(z_j)$, $j = 1, 2$. Let $\tilde{\alpha}$ be a path in K from z_1 to z_2 and let $\alpha = T \cdot \tilde{\alpha}$. Then for $F_x = f \cdot (T_{\tilde{\alpha}(x)})^{-1}$, $0 \leqq x \leqq 1$, we see that $F_1 = f \cdot (T_{z_2})^{-1}$ is the analytic continuation of $F_0 = f \cdot (T_{z_1})^{-1}$ along α. On the other hand let t be any point on R and β a path from t_1 to t. Then there is a path $\tilde{\beta}$ in K such that $\beta = T \cdot \tilde{\beta}$ and $\tilde{\beta}(0) = z_1$. Then $f \cdot (T_{\tilde{\beta}(1)})^{-1}$ is the analytic continuation of $f \cdot (T_{z_1})^{-1}$ along β. We have then shown that $F \cdot T^{-1}$ is a multiple-valued analytic function on R.

It remains to show that $F \cdot T^{-1}$ is multiplicative. Let $f \cdot (T_z)^{-1}$ and $f \cdot (T_{z'})^{-1}$ be function elements of $f \cdot T^{-1}$ at a point $t = T(z) = T(z')$. By the transitivity of S there is an $s \in S$ such that $s(z) = z'$. Since f is m.i. we have $|f(z')| = |f(s(z))| = |f(z)|$. Thus $|f \cdot (T_{z'})^{-1}(t)| = |f(z')| = |f(z)| = |f \cdot (T_z)^{-1}(t)|$. Hence $f \cdot T^{-1}$ is multiplicative. This completes the proof.

As is easy to see $f \cdot T^{-1}$ is single-valued if and only if f is invariant. Also, if F is a single-valued analytic function on R, then $f = F \cdot T$ is an invariant analytic function on K and $F = f \cdot T^{-1}$.

(3.6) LEMMA. *If F is a multiplicative function on R then there is an m.i. analytic function f on K such that $F = f \cdot T^{-1}$.*

Proof. Let F_{t_0} be a function element of F at $t_0 = T(z_0)$. Consider the function element $F_{t_0} \cdot T_{z_0}$ at z_0. Let $\tilde{\alpha}$ be a path starting at z_0 and let $\alpha = T \cdot \tilde{\alpha}$. Let $F_{\alpha(1)}$ be the analytic continuation of F_{t_0} along α. Then $F_{\alpha(1)} \cdot T_{\tilde{\alpha}(1)}$ is the analytic continuation of $F_{t_0} \cdot T_{z_0}$ along $\tilde{\alpha}$. We have then that $F_{t_0} \cdot T_{z_0}$ generates an analytic function f on K.

We show next that f is m.i. Let $s \in S$ and $z_1 = s(z_0)$. Let $\tilde{\beta}$ be a path in K from z_0 to z_1. Then $f \cdot (T_{z_1})^{-1}$ at t_0 is the analytic continuation of F_{t_0} along $\beta = T \cdot \tilde{\beta}$. Thus $|F_{t_0}| = |f \cdot (T_{z_1})^{-1}|$ in the neighborhood of t_0. Now since $T = T \cdot s$ we have that $(T_{z_1})^{-1} \cdot T_{z_0} = s$ in the neighborhood of z_0. Hence $|f(z)| = |f \cdot (T_{z_1})^{-1}(T_{z_0}(z))| = |f \cdot s(z)|$ in the neighborhood of z_0. Thus $|f| = |f \cdot s|$ on all of K. Thus f is m.i.

Now $F_{t_0} (= f \cdot (T_{z_0})^{-1})$ is a function element of both F and $f \cdot T^{-1}$. Thus $F = f \cdot T^{-1}$.

(3.7) LEMMA. *If f and g are m.i. analytic functions, then $f \cdot T^{-1} = g \cdot T^{-1}$ if and only if there exists $s \in S$ such that $f = g \cdot s$.*

Proof. Assume $f \cdot T^{-1} = g \cdot T^{-1}$. Let F_{t_0} be a function element of $F = f \cdot T^{-1}$ at a point t_0 on R. Then there exist $z_0, z_1 \in K$ such that $T(z_0) = T(z_1) = t_0$ and $f \cdot (T_{z_0})^{-1} = F_{t_0} = g \cdot (T_{z_1})^{-1}$, and there exists $s \in S$ such that $s^{-1}(z_0) = z_1$. Now $T = T \cdot s^{-1}$; thus $(T_{z_0})^{-1} = s \cdot (T_{z_1})^{-1}$ in the neighborhood of t_0. Then

$g \cdot (T_{z_1})^{-1} = f \cdot (T_{z_0})^{-1} = f \cdot s\,(T_{z_1})^{-1}$ there. Therefore $g = f \cdot s$ in the neighborhood of z_1, and $g = f \cdot s$ on K. This proves half of the lemma.

Assume now that there exists $s \in S$ such that $f = g \cdot s$. For z_0 fixed in K let $z_1 = s^{-1}(z_0)$. Then $f \cdot (T_{z_1})^{-1} = f \cdot s^{-1} \cdot (T_{z_0})^{-1} = g \cdot (T_{z_0})^{-1}$. Since $f \cdot (T_{z_1})^{-1}$ and $g \cdot (T_{z_0})^{-1}$ are function elements in $f \cdot T^{-1}$ and $g \cdot T^{-1}$ respectively, it follows that $f \cdot T^{-1} = g \cdot T^{-1}$.

A proof of the following lemma is in [7] for the case when R is a plane region which will carry over for more general R.

(3.8) LEMMA. *If F is a (single-valued) analytic function on R and $f = F \cdot T$, then $F \in H_p(R)$, $1 \leqq p \leqq \infty$, if and only if $f \in H_p(K)$.*

4. **Modulus invariant inner functions on K and inner functions on R.** We will show here that under the correspondence $f \to f \cdot T^{-1}$ between m. i. analytic functions on K and multiplicative functions on R, m.i. inner functions on K correspond to inner functions on R. This will enable us to prove that $I(\Phi)$ is a closed ideal of $A(R)$ for Φ an inner function on R.

We say two points z_1 and z_2 in K are *equivalent* if there exists s in S such that $s(z_1) = z_2$. Two subsets K_1 and K_2 of $\bar{K}$ are said to be equivalent if there exists s in S such that $s(K_1) = K_2$.

(4.1) DEFINITION. Δ is a simply connected subset of $\bar{K}$ which has the following properties:

(a) For Δ_0 the interior of Δ, $0 \in \Delta_0$ and no two points of Δ_0 are equivalent.

(b) Any point in K has an equivalent point in $\Delta \cap K$.

(c) The boundary of Δ is a simple closed piecewise analytic curve.

(d) The closure of that part of the boundary of Δ which is in K consists of a finite number of analytic arcs which are pairwise equivalent and such that any two meet at no more than one point. These are called the *inner sides* of Δ.

(e) That part of the boundary of Δ which is on Λ is nonempty and consists of a finite number of disjoint subarcs of Λ. These are called the *free sides* of Δ. We further stipulate that no two points which lie on disjoint free sides are equivalent.

For the existence of Δ see [4, Chapter 7] and [10, pp. 512–514, 525].

Since Δ has a finite number of sides we have:

(4.2) LEMMA. *There is a finite subset S' of S such that $\bigcup_{s \in S'} s(\Delta)$ is a neighborhood of Δ in $\bar{K}$.*

By the reflection principle T can be extended to be analytic and locally 1-1 in the neighborhood of the free sides of Δ; and disjoint free sides will map onto disjoint boundary curves of R. Thus there are N free sides of Δ, $\gamma_1, \cdots, \gamma_N$ where $T(\gamma_j) = \Gamma_j$. We let $\gamma = \bigcup \gamma_j$ and $\Omega = \bigcup_{s \in S} s(\gamma)$. Then Ω is an open subset of Λ and T can be extended to be analytic and locally 1-1 in the neighborhood of $K \cup \Omega$.

A proof of the following lemma can be found in [10, p. 525].

(4.3) LEMMA. $\int_\Omega d\theta = 2\pi$.

By virtue of this lemma we have that for T^* the restriction of T to Ω, T^* is defined a.e. on Λ. It follows that if $F \in H_p(R)$ and $f = F \cdot T$, then $f^* = F^* \cdot T^*$ a.e. on Λ. Also we have:

(4.4) LEMMA. *If F is a multiplicative function on R and f is an m.i. analytic function on K such that $F = f \cdot T^{-1}$, then F is an inner function if and only if f is an inner function.*

We can now prove:

(4.5) THEOREM. *If Φ is an inner function on R then $I(\Phi)$ is a closed ideal of $A(R)$.*

Proof. It is immediate that $I(\Phi)$ is an ideal. We must show that $I(\Phi)$ is closed. By (3.6) and (4.4) there is an m.i. analytic inner function ϕ on K such that $\Phi = \phi \cdot T^{-1}$. Now $|\phi| = |\Phi| \cdot T$. Hence if $F \in H_\infty(R)$ and $f = F \cdot T$, then $|F|/|\Phi|$ is bounded on R if and only if $\phi | f$. Suppose $G_n \to G$ in $A(R)$ where $G_n \in I(\Phi)$. Let $g = G \cdot T$. We will show that $\phi | g$ which implies $G \in I(\Phi)$. This in turn shows that $I(\Phi)$ is closed. Let $g_n = G_n \cdot T$. Then $g_n \to g$ in $H_\infty(K)$. Also $\phi | g_n$. That is, $h_n = g_n/\phi \in H_\infty(K)$. Now $\| h_n - h_m \|_\infty = \| h_n^* - h_m^* \|_{L\infty} = \| \phi^* \|_{L\infty} \| h_n^* - h_m^* \|_{L\infty} = \| \phi^* h_n^* - \phi^* h_m^* \|_{L\infty} = \| g_n^* - g_m^* \|_{L\infty} \to 0$ as $n, m \to \infty$. Hence there exists $h \in H_\infty(K)$ such that $\| h_n - h \|_\infty \to 0$ as $n \to \infty$. Thus $\| g_n - \phi h \|_\infty = \| \phi h_n - \phi h \|_\infty = \| \phi^* h_n^* - \phi^* h^* \|_{L\infty} = \| h_n^* - h^* \|_{L\infty} = \| h_n - h \|_\infty \to 0$ as $n \to \infty$. Therefore $g = \phi h$ on K. That is, $\phi | g$ which is what we wanted to show.

We conclude this section with two central facts about m.i. inner functions.

(4.6) LEMMA. *If $f \in H_p(K)$ and f_0, f_1 are its inner and outer factors respectively, then f_0 and f_1 are m.i. if f is m.i.*

Proof. Let $s \in S$. Then since f is m.i. there is a constant $a \in \Lambda$ such that $(f_0 \cdot s)(f_1 \cdot s) = f \cdot s = af$. Note that $f_0 \cdot s$ is an inner function and $f_1 \cdot s \in H_p(K)$. Let h_0 and h_1 be the inner and outer factors of $f_1 \cdot s$ respectively. Then $f_1 = (h_0 \cdot s^{-1})(h_1 \cdot s^{-1})$. Since $h_0 \cdot s^{-1}$ is an inner function and f_1 is an outer function it follows that $h_0 \cdot s^{-1} = b$ for some constant $b \in \Lambda$. Therefore $f_1 \cdot s = b \cdot h_1$ and thus $af_0 f_1 = b(f_0 \cdot s)h_1$. Then $f_1 = h_1$ and $af_0 = b(f_0 \cdot s)$. Then $f_1 \cdot s = bf_1$ and $f_0 \cdot s = (a/b)f_0$. Since s was picked arbitrarily from S the lemma is proved.

(4.7) LEMMA. *Let $\mathscr{F}$ be a collection of inner functions on K which are m.i. If ϕ is a g.c.d. of $\mathscr{F}$, then ϕ is m.i.*

Proof. Let $\mathscr{F} = \{f_i \mid i \in I\}$ for some index set I. For each $i \in I$ let $h_i = f_i/\phi$. Then h_i is an inner function and $f_i = \phi h_i$. Fix $s \in S$. Since each f_i is m.i. there are constants $b_i \in \Lambda$ such that $f_i = b_i(f_i \cdot s) = (\phi \cdot s)(b_i(h_i \cdot s))$ and

$f_i = \bar{b}_i(f_i \cdot s^{-1}) = (\phi \cdot s^{-1})(\bar{b}_i(h_i \cdot s^{-1}))$. Note that $\phi \cdot s$, $h_i \cdot s$, $\phi \cdot s^{-1}$ and $h_i \cdot s^{-1}$ are inner functions. Thus $\phi \cdot s \mid f_i$ and $\phi \cdot s^{-1} \mid f_i$ for all $f_i \in \mathscr{F}$. Since ϕ is a g.c.d. of $\mathscr{F}$ it follows that $\phi \cdot s \mid \phi$ and $\phi \cdot s^{-1} \mid \phi$. The latter relation implies $\phi \mid \phi \cdot s$. Thus ϕ and $\phi \cdot s$ divide each other. This implies $\phi \cdot s = b\phi$ for some constant $b \in \Lambda$. Since s was picked arbitrarily from S the lemma is proved.

5. **A generalization of the F. and M. Riesz Theorem on measures.** We denote the space of continuous complex-valued functions on Γ by $C(\Gamma)$ and the class of continuous first order differentials on Γ by $D(\Gamma)$. To α, a continuous first order differential in the neighborhood of Γ, we associate $\alpha^* \in D(\Gamma)$ as follows. In terms of the uniformizer $re^{i\theta} = \Phi_k^{-1}(t)$ in the neighborhood of Γ_k, α has the form $a_k(re^{i\theta})dr + b_k(re^{i\theta})d\theta$ where a_k and b_k are continuous. We define α^* as $b_k(e^{i\theta})d\theta$ on Γ_k.

A proof of the following lemma is in [5, p. 8].

(5.1) Lemma. *If P and Z are disjoint finite collections of points on $\bar{R}$, then there exists a function F meromorphic on R with zeros at points of Z and poles at points of P of prescribed orders and no other zeros or poles on $\bar{R}$.*

Suppose d is a divisor on $\bar{R}$. Then by (5.1) there is a function F meromorphic on $\bar{R}$ which has d as its divisor on $\bar{R}$. Also there is a meromorphic differential on $\bar{R}$ with d as its divisor. For let α_1 be a meromorphic differential on $\bar{R}$ and let d_1 be its divisor on $\bar{R}$. Let F be a meromorphic function on $\bar{R}$ with divisor d/d_1. Then $\alpha = F_1\alpha_1$ has d as its divisor on $\bar{R}$.

In particular we have that there exists a nonvanishing analytic differential on $\bar{R}$. We fix one and denote it by ω. Then for $\alpha \in D(\Gamma)$, $\alpha/\omega^* \in C(\Gamma)$ and we define $\|\alpha\| = \max_\Gamma |\alpha/\omega^*|$. Then $D(\Gamma)$ is a Banach space isomorphically isometric to $C(\Gamma)$ by the map $\alpha \to \alpha/\omega^*$.

Propositions 1, 4 and 7 and Theorem 2 in [6] yield the following theorem.

(5.2) Theorem. *If L is a continuous linear functional on $D(\Gamma)$, then there exists a function $F \in H_1(R)$ such that $L(\alpha) = \int_\Gamma F^*\alpha$ for all $\alpha \in D(\Gamma)$ if and only if $L(\beta^*) = 0$ for all analytic differentials β on $\bar{R}$.*

(5.3) Corollary. *If μ is a Borel measure on Γ such that $\int_\Gamma W d\mu = 0$ for all $W \in A(R)$ then there exists $F \in H_1(R)$ such that $F^*\omega^* = d\mu$ as a measure on Γ.*

Proof. We define L, a continuous linear functional on $D(\Gamma)$, by $L(\alpha) = \int_\Gamma (\alpha/\omega^*)d\mu$. If β is an analytic differential on $\bar{R}$ then β/ω is an analytic function on $\bar{R}$ and thus $L(\beta^*) = 0$. By (5.2) there exists $F \in H_1(R)$ such that $L(\alpha) = \int_\Gamma F^*\alpha = \int_\Gamma (\alpha/\omega^*)F^*\omega^*$ for all $\alpha \in D(\Gamma)$. This says $\int_\Gamma GF^*\omega^* = \int_\Gamma G d\mu$ for all $G \in C(\Gamma)$. It follows $F^*\omega^* = d\mu$.

Corollary (5.3) is a generalization of the well-known result of F. and M. Riesz for measures on Λ. (See [3, pp. 47, 51] and [6, §3].)

In terms of the boundary uniformizer $\Phi_k^{-1}(t) = re^{i\theta}$, $\omega^* = b_k(e^{i\theta})d\theta$ on Γ_k where b_k is continuous and nonvanishing on Γ_k, $k = 1,2,\cdots,N$. Hence as a measure $\omega^* = Vdv$ where V is continuous and nonvanishing on Γ. Now for μ a Borel measure on Γ there exists a unique function $P \in L_1(\Gamma, dv)$ and a unique measure σ which is singular with respect to v such that $d\mu = Pdv + d\sigma$. For $M = P/V$ we get $d\mu = M\omega^* + d\sigma$. We call $M\omega^*$ the *absolutely continuous part of* $d\mu$ and $d\sigma$ the *singular part of* $d\mu$. This decomposition is, of course, unique.

6. **Some lemmas.** We will need the following lemma in the proof of Theorem 1.

(6.1) LEMMA. *If* $M \in L_1(\Gamma, v)$ *then* $m = M \cdot T^* \in L_1(\Lambda, d\theta)$.

To prove this we will need:

(6.2) LEMMA.

$$\sum_{s \in S} \max_{e^{i\theta} \in \gamma} |ds(e^{i\theta})/d\theta| < \infty.$$

Proof. When S contains just the identity the result is trivial. If S contains more than one element it contains denumerably infinite elements $s_1, s_2, \cdots$. (See [2, p. 70].)

Let $s_j(z) = \lambda_j(z - a_j)/(\bar{a}_j z - 1)$, where λ_j and a_j are constants, $|\lambda_j| = 1, |a_j| < 1$. Direct calculation shows $|ds(e^{i\theta})/d\theta| = (1 - |a_j|^2)/|\bar{a}_j e^{i\theta} - 1|^2$. Now $a_j \in \Delta_j = s_j^{-1}(\Delta)$ since $s_j(a_j) = 0 \in \Delta$. By (4.2) there is $J > 0$ such that $\bigcup_{j=1}^{J} \Delta_j$ is a neighborhood of Δ in $\bar{K}$. Since all the a_j are in K, we have then, that for some positive constant, b, $|e^{i\theta} - a_j| > b$ uniformly for $j = 1,2,\cdots$ and $e^{i\theta} \in \gamma$. Thus we have

$$\begin{aligned} |ds_j(e^{i\theta})/d\theta| &= (1 - |a_j|^2)/|\bar{a}_j e^{i\theta} - 1|^2 \\ &\leqq 1 - |a_j|^2/b^2 \\ &= ((1 + |a_j|)/b^2)(1 - |a_j|) \\ &\leqq (2/b^2)(1 - |a_j|) \end{aligned}$$

for $j = 1,2,\cdots$ and $e^{i\theta} \in \gamma$.

Hence it remains to show $\sum 1 - |a_j| < \infty$. To this end consider a function F analytic and not identically zero on $\bar{R}$ with a zero at $t_0 = T(0)$. Such a function exists by virtue of (5.1). Then $f = F \cdot T$ is a bounded analytic function on K, not identically zero and $f(a_j) = F(T \cdot s_j^{-1}(0)) = F(T(0)) = F(t_0) = 0$. This implies $\sum 1 - |a_j| < \infty$. (See [3, p. 63].) This proves (6.2).

Proof of (6.1). First note that $M \in L_1(\Gamma, v)$ means that $M \cdot \Phi_k \in L_1(\Lambda, d\theta)$ for $k = 1,2,\cdots,N$.

If S contains just the identity, then $N = 1$, T is 1-1 analytic on $\bar{R}$ and thus $M \cdot T^* = M \cdot \Phi_1 \cdot \Phi_1^{-1} \cdot T^* \in L_1(\Lambda, d\theta)$ which proves the lemma for this case.

Assume S has infinitely many elements $s_1, s_2, \cdots$. We let T_k^* be the restriction of T^* to γ_k. Note that T_k^* is 1-1 on the interior of γ_k. Since T is analytic and locally 1-1 on $K \cup \Omega$ there is a $P > 0$ such that $|d(T_k^*)^{-1} \cdot \Phi_k(e^{i\phi})/d\phi| < P$ for all $e^{i\phi} \in \Lambda$, $k = 1, 2, \cdots, N$. If we let $Q = \max_k P \int_\Lambda | M \cdot \Phi_k(e^{i\phi}) | d\phi$ and $e^{i\phi} = \Phi_k^{-1} \cdot T_k^*(e^{i\alpha})$ we have for $k = 1, 2, \cdots, N$

$$\begin{aligned}\int_{\gamma_k} |m(e^{i\alpha})|\, d\alpha &= \int_{\gamma_k} |M \cdot \Phi_k \cdot \Phi_k^{-1} \cdot T_k^*(e^{i\alpha})|\, d\alpha \\ &= \int_\Lambda |M \cdot \Phi_k(e^{i\phi})| \left(\frac{-i}{(T_k^*)^{-1} \cdot \Phi_k(e^{i\phi})}\right) \\ &\qquad\qquad \cdot \left(\frac{d(T_k^*)^{-1} \cdot \Phi_k(e^{i\phi})}{d\phi}\right) d\phi \\ &\leqq P \int_\Lambda |M \cdot \Phi(e^{i\phi})| d\phi \\ &\leqq Q.\end{aligned}$$

Thus for $j = 1,2,\cdots$ and $k = 1,2,\cdots,N$

$$\begin{aligned}\int_{s_j(\gamma_k)} |m(e^{i\theta})| d\theta &= \int_{\gamma_k} |m(e^{i\alpha})| \left(\frac{i}{s_j(e^{i\alpha})}\right) \left(\frac{ds_j(e^{i\alpha})}{d\alpha}\right) d\alpha \\ &\leqq Q \max_{e^{i\alpha} \in \gamma_k} |ds_j(e^{i\alpha})/d\alpha|.\end{aligned}$$

Using (6.2) we get

$$\begin{aligned}\int_\Lambda |m(e^{i\theta})|\, d\theta &= \int_\Omega |m(e^{i\theta})|\, d\theta \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{N} \int_{s_j(\gamma_k)} |m(e^{i\theta})|\, d\theta \\ &= NQ \sum_{j=1}^{\infty} \max_{e^{i\alpha} \in \gamma} |ds_j(e^{i\alpha})/d\alpha| \\ &< \infty.\end{aligned}$$

In the proof of Theorem 1 we will also use

(6.3) Lemma. *Let C_1 and C_2 be closed subspaces of $A(R)$. Then $C_1 \supset C_2$ if for any Borel measure μ on Γ such that $\int_\Gamma F d\mu = 0$ for all $F \in C_1$ it follows $\int_\Gamma F d\mu = 0$ for all $F \in C_2$.*

Proof. For $F \in A(R)$ we let F^* be the restriction of F to Γ. Then $A^* = \{F^* | F \in A(R)\}$ is a closed subspace of $C(\Gamma)$ and the map $F \to F^*$ is an isometric isomorphism of A onto A^*. Then for $C_j^* = \{F^* | F \in C_j\}$, $j = 1,2$, C_1^*

and C_2^* are closed subspaces of $C(\Gamma)$. Assume $C_1 \not\supset C_2$. Then there is a function $G^* \in C_2^* - C_1^*$. Hence there exists a continuous linear functional L on $C(\Gamma)$ such that $L(F^*)=0$ for all $F^* \in C_1^*$ and $L(G^*) \neq 0$. There exists a Borel measure μ on Γ such that $\int_\Gamma W d\mu = L(W)$ for all $W \in C(\Gamma)$. Thus $\int_\Gamma F d\mu = \int_\Gamma F^* d\mu = L(F^*) = 0$ for all $F \in C_1$, and $\int_\Gamma G d\mu = \int_\Gamma G^* d\mu = L(G^*) \neq 0$ for some $G \in C_2$. This proves the lemma.

7. **Proof of Theorem 1.** Assume I is a closed ideal of $A(R)$. We will find an inner function Φ and a closed set E on Γ such that $I = I(\Phi) \cap I(E)$.

Let $J = \{f \in H_\infty(K) \mid f = F \cdot T \text{ for some } F \in I\}$ and $\mathscr{F}$ be the class of inner factors of functions in J. Let ϕ be a g.c.d. of $\mathscr{F}$. Then ϕ is m.i. since each function in $\mathscr{F}$ is m.i. (see (4.6) and (4.7)). Let $\Phi = \phi \cdot T^{-1}$. Then Φ is an inner function by (4.4). We put $E = \{t \in \Gamma \mid F(t) = 0 \text{ for all } F \in I\}$. Clearly E is a closed set on Γ.

We show first that $I \subset I(\Phi) \cap I(E)$. Consider $F \in I$. Then certainly $F \in I(E)$. We must show $F \in I(\Phi)$. Let $f = F \cdot T$ and f_0 be the inner factor of f. Then $\phi \mid f_0$ since $f_0 \in \mathscr{F}$. Thus $\phi \mid f$, and hence $|F|/|\Phi|$ is bounded on R. Thus $F \in I(\Phi)$. Hence $F \in I(\Phi) \cap I(E)$ and $I \subset I(\Phi) \cap I(E)$. To complete the proof we must show that $I \supset I(\Phi) \cap I(E)$. To this end let μ be any Borel measure on Γ such that $\int_\Gamma F d\mu = 0$ for all $F \in I$. By (6.3) we need only show that $\int_\Gamma G d\mu = 0$ for all $G \in I(\Phi) \cap I(E)$.

If $F \in I$ then $WF \in I$ for all $W \in A(R)$ and thus $\int_\Gamma WF d\mu = 0$ for all $W \in A(R)$. By (5.3) there exists $B_F \in H_1(R)$ such that $F d\mu = B_F^* \omega^*$ on Γ. Let $M\omega^*$ and $d\sigma$ be the absolutely continuous and singular parts of $d\mu$ respectively. Then $B_F^* \omega^* = F d\mu = FM\omega^* + F d\sigma$ on Γ. It follows that $B_F^*\omega^* = FM\omega^*$ and that $F d\sigma$ is the zero measure. The first identity implies $B_F^* = FM$ a.e.-ν on Γ. Since $F d\sigma$ is the zero measure and F was picked in I arbitrarily, it follows that E includes the carrier of σ.

We let $f = F \cdot T$, $b_F = B_F \cdot T$ and $m = M \cdot T^*$. Then $f \in H_\infty(K)$ and $b_F \in H_1(K)$, and both are invariant. By (6.1) $m \in L_1(\Lambda, d\theta)$. Clearly m is invariant. Let f_0 and f_1 be the inner and outer factors of f respectively, and b_{F0} and b_{F1} the inner and outer factors of b_F respectively. Now since $B_F^* = FM$ a.e.-ν on Γ it follows

$$f_0^* f_1^* m = f^* m = b_F^* = b_{F0}^* b_{F1}^* \quad \text{a.e.-}d\theta \text{ on } \Lambda \tag{7.1}$$

Since $|f_0^*| = |b_{F0}^*| = 1$ a.e.-$d\theta$, (7.1) gives us $|b_{F1}^*|/|f_1^*| = |m|$ a.e.-$d\theta$ on Λ and thus $b_{F1}^*/f_1^* \in L_1(\Lambda, d\theta)$. By (2.2) $w_F = b_{F1}/f_1$ is an outer function in $H_1(K)$. By (6.1) we have

$$f_0^* m = b_{F0}^*(b_{F1}^*/f_1^*) = b_{F0}^* w_F^* \quad \text{a.e.-}d\theta \text{ on } \Lambda. \tag{7.2}$$

We now fix $P \in I$. Rewriting (7.2) with $F = P$ we have

$$p_0^* m = b_{P0}^* w_P^* \quad \text{a.e.-}d\theta \text{ on } \Lambda. \tag{7.3}$$

Multiplying (7.2) by p_0^* and (7.3) by f_0^* we get

$$b_{F0}^* p_0^* w_F^* = f_0^* p_0^* m = f_0^* b_{P0}^* w_P^* \qquad \text{a.e.-}d\theta \text{ on } \Lambda. \tag{7.4}$$

Thus we have $b_{F0}p_0w_F = f_0b_{P0}w_P$ on K. Since w_F and w_P are outer functions and $b_{F0}p_0$ and $b_{P0}f_0$ are inner functions, it follows $b_{F0}p_0 = b_{P0}f_0$. Thus $p_0 \mid b_{P0}f_0$ for all $f_0 \in \mathscr{F}$. By (2.4) $p_0 \mid \phi b_{P0}$. Then $\alpha = \phi b_{P0}/p_0$ is an inner function. Multiplying (7.3) by ϕ^*/p_0^* we get

$$\phi^* m = (\phi^* b_{P0}^*/p_0^*)w_P^* = \alpha^* w_P^* \qquad \text{a.e.-}d\theta \text{ on } \Lambda. \tag{7.5}$$

Let $G \in I(\Phi) \cap I(E)$ and let $g = G \cdot T$. Since $G \in I(\Phi)$, $|G|/|\Phi|$ is bounded on R and thus $\phi \mid g$. That is, $h = g/\phi \in H_\infty(K)$. Note $h^*\phi^* = g^*$ a.e.-$d\theta$ on Λ. Multiplying (7.5) by h^* we have

$$g^* m = h^*\phi^* m = h^*\alpha^* w_P^* \qquad \text{a.e.-}d\theta \text{ on } \Lambda. \tag{7.6}$$

Since g is invariant, g^* is invariant; and thus g^*m is invariant. Now for $d = \alpha h w_P$, (7.6) shows $d^* = g^*m$ a.e.-$d\theta$ on Λ; thus d^* is invariant. This implies d is invariant. Note $d \in H_1(K)$. Then $D = d \cdot T^{-1}$ is single-valued and in $H_1(R)$ by (3.8). Now $GM = D^*$ a.e.-ν on Γ since $g^*m = d^*$ a.e.-$d\theta$ on Λ. By (5.2) we have $\int_\Gamma GM\omega^* = \int_\Gamma D^*\omega^* = 0$.

We showed earlier that E includes the support of σ. Since $G \in I(E)$, $G = 0$ on E and thus $\int_\Gamma G d\sigma = 0$. Hence we have $\int_\Gamma G d\mu = \int_\Gamma GM\omega^* + \int_\Gamma G d\sigma = 0$. Since G was picked arbitrarily from $I(\Phi) \cap I(E)$ we have $I \supset I(\Phi) \cap I(E)$. This proves the theorem.

8. $H_p(R)$. For $G(t,\tau)$ the Green's function of R with singularity at t, and f a continuous function on Γ, it is well known that

$$F(t) = -(1/2\pi)\int_\Gamma f(\tau) * d_\tau G(t,\tau) \tag{8.1}$$

is harmonic on R, with continuous boundary values $f(t)$(4).

Let $H(t,\tau)$ be the harmonic conjugate of $G(t,\tau)$ on $\bar{R} - t$ and $W(t,\tau) = G(t,\tau) + iH(t,\tau)$. W is an additive analytic function on $\bar{R} - t$; and $d_\tau W$ is an analytic differential on $\bar{R} - t$ which at t is of the form

$$-(1/(z(\tau) - z(t)))dz + (\text{regular terms}).$$

Since $G(t,\tau) = 0$ for $\tau \in \Gamma$ it follows $* d_\tau G(t,\tau) = -i d_\tau W(t,\tau)$ along Γ. We can rewrite (8.1) as

$$F(t) = -(1/2\pi i)\int_\Gamma f(\tau) d_\tau W(t,\tau). \tag{8.2}$$

Note that $-(1/2\pi i)d_\tau W(t,\tau) = -(1/2\pi) * d_\tau G(t,\tau)$ is a positive measure on Γ.

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(4) The symbol $* \, d_\tau G(t,\tau)$ denotes the conjugate differential of $d_\tau G(t,\tau)$.

(8.3) THEOREM. *If* $F \in H_p(R)$, $1 \leqq p \leqq \infty$, *then* $F(t) = -(1/2\pi i)\int_\Gamma F^*(\tau)d_\tau W(t,\tau)$.

Proof. We can assume that R_0 as defined in §1 was chosen such that $\Phi_j(R_0) \cap \Phi_k(R_0)$ is empty for $j \neq k$. For $r_1 < \rho < 1$, let

$$\Omega_\rho = \bigcup_{j=1}^{N} \{\Phi_j(\rho e^{i\theta}) \,|\, 0 \leqq \theta \leqq 2\pi\}$$

and let D_ρ be the region interior to D_ρ. Then, by the residue theorem, for $t \in D_\rho$

$$\begin{aligned} F(t) &= -(1/2\pi i)\int_{\Omega_\cdot} F(\tau)d_\tau W(t,\tau) \\ &= -(1/2\pi i)\sum_{j=1}^{N}\int_{|z|=1} F(\Phi_j(z))\frac{dW(t,\Phi_j(z))}{dz}dz. \end{aligned}$$

Now $F\cdot\Phi_j \in H_p(R_0) \subset H_1(R_0)$. Thus

$$\lim_{\rho\to 1}\int_0^{2\pi} |F\cdot\Phi_j(\rho e^{i\theta}) - F^*\cdot\Phi_j\cdot(e^{i\theta})|\,d\theta = 0.$$

(See [6, p. 20].) Also $dW(t,\Phi_j(z))/dz$ is continuous on $\{z \,|\, \rho \leqq |z| \leqq 1\}$ for $t \in D_\rho$. Thus

$$\begin{aligned} F(t) &= -(1/2\pi i)\sum_{j=1}^{\infty}\lim_{\rho\to 1}\int_{|z|=\rho} F(\Phi_j(z))\frac{dW(t,\phi_j(z))}{dz}dz \\ &= -(1/2\pi i)\sum_{j=1}^{N}\int_{|z|=1} F^*(\Phi_j(z))\frac{dW(t,\Phi_j(z))}{dz}dz \\ &= -(1/2\pi i)\int_\Gamma F^*(\tau)d_\tau W(t,\tau) \end{aligned}$$

which is what we wanted to prove.

Since $C(\Gamma)$ is dense in $L_1(\Gamma, \nu)$ it follows from (8.2) that for $f \in L_1(\Gamma, \nu)$

$$F(t) = -(1/2\pi i)\int_\Gamma f(t)d_\tau W(t,\tau) \tag{8.4}$$

is harmonic on R.

(8.5) THEOREM. *If* $F \in H_p(R)$, $1 \leqq p < \infty$, *then*

$$H_F(t) = -(1/2\pi i)\int_\Gamma |F^*(\tau)|^p\, d_\tau W(t,\tau)$$

is the least harmonic majorant of $|F|^p$ *on* R.

Proof. For $r_1 < \rho < 1$ let Ω_ρ and D_ρ be as in the proof of (8.3). Let G_ρ be the Green's function of D_ρ and

$$H_\rho(t) = -(1/2\pi)\int_{\Omega\rho} |F(\tau)|^p * d_\tau G_\rho(t,\tau).$$

If U is a harmonic majorant of $|F(t)|^p$ on $\bar{D}_\rho$, then $H_\rho \leqq U$ on Ω_ρ and thus $H_\rho \leqq U$ on $\bar{D}_\rho$ since H_ρ is harmonic on D_ρ. In particular $H_\rho \leqq H_{\rho'}$ on $\bar{D}_\rho$ for $\rho \leqq \rho'$. Since $F \in H_p(R)$, there is a harmonic majorant U_0 of $|F|^p$ on R. Thus $H_\rho \leqq U_0$ on D_ρ for $r_1 \leqq \rho \leqq 1$. By Harnack's principle there is a harmonic function H_1 such that on compact subsets of R, H_ρ converges uniformly to H_1 as $\rho \to 1$. Clearly H_1 is the least harmonic majorant of $|F|^p$ on R.

Now in terms of the boundary uniformizer Φ_j

$$* d_\tau G_\rho(t,\tau) = \rho \frac{\partial G_\rho(t, \Phi_j(\rho e^{i\phi}))}{\partial n} d\theta$$

along $\{\Phi_j(\rho e^{i\theta}) \,|\, 0 \leqq \theta \leqq 2\pi\}$, where $\partial/\partial n$ denotes the outward normal derivative. Since $G(t,\tau) - G_\rho(t,\tau) \to 0$ as $\rho \to 1$ uniformly for $\tau \in \bar{R}$

$$\frac{\partial G_\rho(t,\Phi_j(\rho e^{i\theta}))}{\partial n} \to \frac{\partial G(t,\Phi_j(e^{i\theta}))}{\partial n}$$

uniformly for $0 \leqq \theta \leqq 2\pi$ as $\rho \to 1$. Now

$$|F(\Phi_k(\rho e^{i\theta}))|^p \to |F^*(\Phi_k(e^{i\theta})|^p \text{ as } \rho \to 1 \text{ for a. a.-}d\theta\ \theta \in [0,2\pi].$$

Thus it follows by Fatou's theorem that

$$\begin{aligned} H_F(t) &= -(1/2\pi)\int_\Gamma |F^*(\tau)|^p * d_\tau G(t,\tau) \\ &\leqq \lim_{\rho \to 1} -(1/2\pi)\int_{\Omega_\rho} |F(\tau)|^p * d_\tau G_\rho(t,\tau) \\ &= \lim_{\rho \to 1} H_\rho(t) \\ &= H_1(t). \end{aligned}$$

Using Theorem (8.3) and Hölder's inequality we also have

$$\begin{aligned} |F(t)|^p &= |-(1/2\pi)\int_\Gamma F^* * d_\tau G(t,\tau)|^p \\ &\leqq -(1/2\pi)\int_\Gamma |F^*|^p * d_\tau G(t,\tau) \\ &= H_F(t). \end{aligned}$$

That is, H_F is a harmonic majorant of $|F|^p$. Hence $H_1 \leqq H_F$ on R. Thus $H_1 = H_F$, which is what we wanted to show.

Observe that by (8.5) $(H_F(t_0))^{1/p}$ defines a norm on $H_p(R)$ for t_0 fixed on R. Moreover, we have an inner product on $H_2(R)$:

$$(F_1,F_2) = -(1/2\pi)\int_\Gamma F_1^* \overline{F_2^*} * d_\tau G(t_0,\tau).$$

For convenience we will let $t_0 = T(0)$ although our results do not depend on this choice.

(8.6) DEFINITION. For $1 \leqq p \leqq \infty$ we set $I_p = \{f \in H_p(K) \mid f \text{ is invariant}\}$.

(8.7) LEMMA. *I_p is a closed subspace of $H_p(K)$.*

Proof. It is clear that I_p is a subspace of $H_p(K)$. Suppose $f_n \to f$ in $H_p(K)$. Then $f_n(z) \to f(z)$ for all $z \in K$ and thus $f_n(z) = f_n(s(z)) \to f(s(z))$ for all $s \in S$ and $z \in K$. Hence $f(z) = f(s(z))$ and $f \in I_p$. Hence I_p is closed.

(8.8) LEMMA. *The map $F \to F \cdot T$ is an isometric isomorphism from $H_p(R)$ onto I_p.*

Proof. Clearly the map $F \to F \cdot T$ is an isomorphism from $H_p(R)$ onto I_p. (See Lemma (3.8).) For $p = \infty$ it is clear that this map is isometric. Therefore we consider $1 \leqq p < \infty$. Suppose $F \in H_p(R)$. Let $f = F \cdot T$. Then

$$h_f(re^{i\phi}) = (1/2\pi)\int_0^{2\pi} |f^*(e^{i\theta})|^p \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}d\theta$$

is the least harmonic majorant of $|f|^p$ on K, and $(h_f(0))^{1/p} = \|f\|_p$ as defined in §2. Now $h_f \cdot s$ is the least harmonic majorant of $|f \cdot s|^p = |f|^p$; thus $h_f \cdot s = h_f$. Thus $h_f \cdot T^{-1}$ is a well-defined harmonic function on R and $|F|^p \leqq h_f \cdot T^{-1}$ on R. Hence $H_F \leqq h_f \cdot T^{-1}$ on R and thus $H_F \cdot T \leqq h_f$ on K. But $H_F \cdot T$ is a harmonic majorant of $|f|^p$; so $H_F \cdot T = h_f$. Therefore $\|F \cdot T\|_p = \|f\|_p = (h_f(0))^{1/p} = (H_F(t_0))^{1/p} = \|F\|_p$. Hence the map $F \to F \cdot T$ is an isometry.

By the preceding two lemmas we have that $H_p(R)$ is complete. That is,

(8.9) THEOREM. *$H_p(R)$ is a Banach space.*

(8.10) COROLLARY. *$H_2(R)$ is a Hilbert space.*

(8.11) THEOREM. *If Φ is an inner function on Rt hen $C(\Phi)$ is a closed invariant subspace of $H_2(R)$.*

Proof Suppose $F_1, F_2 \in C(\Phi)$. Let H_j be a harmonic majorant of $|F_j|^2/|\Phi|^2$ on R, $j = 1,2$. Since $|F_1 + F_2|^2/|\Phi|^2 \leqq 4(|F_1|^2/|\Phi|^2 + |F_2|^2/|\Phi|^2)$ it follows that $4(H_1 + H_2)$ is a harmonic majorant of $|F_1 + F_2|^2/|\Phi|^2$. Thus $C(\Phi)$ is a subspace of $H_2(R)$. It is clear that $C(\Phi)$ is invariant. It remains to show $C(\Phi)$ is closed.

Let ϕ be an m.i. inner function on K such that $\Phi = \phi \cdot T^{-1}$ (see Lemmas (3.6) and (4.4)). Let $F \in C(\Phi)$ and H be a harmonic majorant of $|F|^2/|\Phi|^2$. Then $H \cdot T$ is a harmonic majorant of $|F \cdot T|^2/|\Phi|^2$ and thus $F \cdot T/\phi \in H_2(K)$. Conversely given $f \in I_2$ such that $f/\phi \in H_2(K)$, $F = f \cdot T^{-1} \in C(\Phi)$. We let $C(\phi) = \{f \in H_2(K) \mid f/\phi \in H_2(K)\}$. It is known that $C(\phi)$ is a closed subspace of $H_2(K)$. (See [1].) Then $C(\phi) \cap I_2$ is a closed subspace of $H_2(K)$. It follows by our previous comments and (8.8) that $C(\Phi)$ is isomorphically isometric to $C(\phi) \cap I_2$. Thus $C(\Phi)$ is a closed subspace of $H_2(R)$.

9. **Proof of Theorem 2.** Assume C is a closed invariant subspace of $H_2(R)$. We consider two cases.

Case I. Not all functions in C vanish at t_0.

We let $C' = \{f \mid f = F \cdot T, F \in C\}$. Then $C' \subset H_2(K)$. We let $\mathcal{F}$ be the class of inner factors of functions in C' and let ϕ be a g.c.d. of $\mathcal{F}$. Then ϕ is an m.i. inner function on K. (See Lemmas (4.6) and (4.7).) We let $\Phi = \phi \cdot T^{-1}$. Suppose $F \in C$. Let f_0, f_1 be the inner and outer factors of $f = F \cdot T$ respectively. Then $\phi \mid f_0$ and thus $f_0/\phi \in H_\infty(K)$. Now $f_1 \in H_2(K)$ and thus $f_0 f_1/\phi \in H_2(K)$. That is, $|f|^2/|\phi|^2$ has a harmonic majorant. Note that f/ϕ is m.i. This implies that the least harmonic majorant h of $|f/\phi|^2$ is invariant with respect to S. Thus $h \cdot T^{-1}$ is a well-defined harmonic function and is a majorant of $|F|^2/|\Phi|^2$. That is, $F \in C(\Phi)$. Thus we have $C(\Phi) \supset C$.

We prove next that $C(\Phi) \subset C$ and thus $C(\Phi) = C$. It is sufficient to show that if $B \in H_2(R)$ and $(F, B) = 0$ for all $F \in C$, then $(M, B) = 0$ for all $M \in C(\Phi)$.

Assume $B \in H_2(R)$ and $(F, B) = 0$ for all $F \in C$. If $F \in C$ then $PF \in C$ for all $P \in A(R)$, since C is invariant. Thus

$$0 = (PF, B) = -(1/2\pi i)\int_\Gamma PF^* \bar{B}^* d_\tau W(t_0, \tau) \tag{9.1}$$

for all $P \in A(R)$.

Hence by (5.3) there exists $A_F \in H_1(R)$ such that

$$F^* \bar{B}^* (d_\tau W(t_0, \tau))^* = A_F^* \omega^* \text{ on } \Gamma. \tag{9.2}$$

Now $d_\tau W(t_0, \tau)$ is a meromorphic differential on $\bar{R}$ with a simple pole at t_0 and with no other poles on $\bar{R}$. Hence $d_\tau W(t_0, \tau) = D\omega$ on $\bar{R}$ where D is an analytic function on $\bar{R} - t_0$ with a simple pole at t_0. By (9.2) we have

$$F^* \bar{B}^* D \omega^* = F^* \bar{B}^* (d_\tau W(t_0, \tau))^* = A_F^* \omega^*$$

on Γ. Thus

$$F^* \bar{B}^* D = A_F^* \qquad \text{a.e.-}\nu \text{ on } \Gamma. \tag{9.3}$$

Let V be an analytic function on R with a simple zero at t_0 and with no other zeros on $\bar{R}$. Then $E = DV$ is analytic on $\bar{R}$ and $W_F = A_F V \in H_1(R)$. Multiplying (9.3) by V we get

$$F^* \bar{B}^* E = F^* \bar{B}^* DV = A_F^* V = W_F^* \qquad \text{a.e.-}\nu \text{ on } \Gamma. \tag{9.4}$$

We let $f = F \cdot T$, $b = B \cdot T$, $e = E \cdot T$ and $w_F = W_F \cdot T$. Then f and b are in $H_2(K)$, $e \in H_\infty(K)$ and $w_F \in H_1(K)$. By (9.4) we have

$$f^* \bar{b}^* e^* = w_F^* \qquad \text{a.e.-}d\theta \text{ on } \Lambda. \tag{9.5}$$

Let f_0 and w_{F0} be the inner factors of f and w_F respectively, and f_1 and w_{F1} the outer factors of f and w_F respectively. Then (9.5) yields $|w_{F1}^*/f_1^*| = |w_F^*/f^*| = |e^* b^*|$ a.e.-$d\theta$ on Λ. Thus $|w_{F1}^*/f_1^*| = |e^* b^*| \in L_2(\Lambda, d\theta)$. Thus by (2.2) $v_F = w_{F1}/f_1$ is an outer function in $H_2(K)$. Note $v_F^* = w_{F1}^*/f_1^*$ a.e.-$d\theta$ on Λ. Multiplying (9.5) by $1/f_1^*$ we get

(9.6) $$f_0^*\bar{b}^*e^* = (f^*\bar{b}^*e^*)/f_1^* = w_F^*/f_1^* = w_{F0}^*v_F^* \quad \text{a.e.-}d\theta \text{ on } \Lambda.$$

We now fix $Q \in C$ with $Q(t_0) \neq 0$. With $q = Q \cdot T$ we get from (9.6)

(9.7) $$q_0^*\bar{b}^*e^* = w_{Q0}^*v_Q^* \quad \text{a.e.-}d\theta \text{ on } \Lambda.$$

Multiplying (9.6) by q_0^* and (9.7) by f_0^* we get

(9.8) $$q_0^*w_{F0}^*v_F^* = q_0^*f_0^*\bar{b}^*e^* = f_0^*w_{Q0}^*v_0^* \quad \text{a.e.-}d\theta \text{ on } \Lambda.$$

This implies $q_0w_{F0}v_F = f_0w_{Q0}v_Q$ on K. Since v_F and v_Q are outer functions and q_0w_{F0} and f_0w_{Q0} are inner functions, it follows that $q_0w_{F0} = f_0w_{Q0}$ on K. Hence $q_0 \mid f_0w_{Q0}$ for all $f_0 \in \mathscr{F}$. By (2.4) $q_0 \mid \phi w_{Q0}$. Thus $a = \phi w_{Q0}/q_0$ is an inner function. Multiplying (9.7) by $\phi^*/q_0{}^*$ we get

(9.9) $$\phi^*\bar{b}^*e^* = (\phi^*w_{Q0}^*/q_0^*)v_Q^* = a^*v_Q^* \quad \text{a.e.-}d\theta \text{ on } \Lambda.$$

We now observe that $a(0) = 0$ by the following argument. First, $W_Q(t_0) = A_Q(t_0)V(t_0) = 0$ since $V(t_0) = 0$. Thus $w_{Q0}(0)w_{Q1}(0) = w_Q(0) = W_Q(t_0) = 0$. Hence $w_{Q0}(0) = 0$ since $w_{Q1}(0) \neq 0$. On the other hand $q_0(0) \neq 0$ since $q(0) = Q(t_0) \neq 0$. Hence $a(0) = \phi(0)w_{Q0}(0)/q_0(0) = 0$. We should also note that $av_Q \in H_2(K)$.

We now consider $M \in C(\Phi)$. Recall that we want to show $(M,B) = 0$. Let $m = M \cdot T$. Then $h = m/\phi \in H_2(K)$, $\phi h = m$ on K and $\phi^*h^* = m^*$ a.e.-$d\theta$ on Λ. Multiplying (9.9) by h^* we get

(9.10) $$m^*\bar{b}^*e^* = h^*\phi^*\bar{b}^*e^* = a^*h^*v_Q^* \quad \text{a.e.-}d\theta \text{ on } \Lambda.$$

Now since av_Q and h are in $H_2(K)$ it follows $u = ahv_Q \in H_1(K)$. Note that $u(0) = 0$ since $a(0) = 0$. Now since m, b, and e are invariant on K it follows that $m^*\bar{b}^*e^*$ is invariant on Λ. Then u is invariant on K since by (9.10) $m^*\bar{b}^*e^*$ is its boundary function.

Set $U = u \cdot T^{-1}$. Then $U \in H_1(R)$. Since $u^* = m^*\bar{b}^*e^*$ a.e.-$d\theta$ on Λ we have $U^* = M^*\bar{B}^*E = M^*\bar{B}^*DV$ a.e.-ν on Γ. Now $U(t_0) = u(0) = 0$; hence $U/V \in H_1(R)$ since V is analytic on $\bar{R}$ with a simple zero at t_0 and no other zeros on $\bar{R}$. We have then $M^*\bar{B}^*D = U^*/V = (U/V)^*$ a.e.-ν on Γ. Using Theorem (5.2) we conclude

$$\begin{aligned}(M,B) &= -(1/2\pi i)\int_\Gamma M^*\bar{B}^*d_\tau W(t_0,\tau)\\ &= -(1/2\pi i)\int_\Gamma M^*\bar{B}^*D\omega^*\\ &= -(1/2\pi i)\int_\Gamma (U/V)^*\omega^*\\ &= 0.\end{aligned}$$

Since M was chosen arbitrarily from $C(\Phi)$ Case I is proved.

Case II. All functions in C vanish at t_0.

Let n be the minimum of the orders of the zero at t_0 of the functions in C. Let V be an analytic function on $\bar{R}$ with a zero of order n at t_0 and vanishing nowhere else on $\bar{R}$. Let $C_1 = \{F/V \mid F \in C\}$. Clearly C_1 is an invariant subspace of $H_2(R)$. Also C_1 is closed. For suppose $\|(F_k/V) - (F_j/V)\|_2 \to 0$ as $k,j \to \infty$. Let a be a positive constant such that $|V|^2 < a$ on Γ. Then

$$\begin{aligned} \|F_k - F_j\|_2^2 &= -(1/2\pi i) \int_\Gamma |F_k^* - F_j^*|^2 d_\tau W(t_0,\tau) \\ &\leqq -a(1/2\pi i) \int |(F_k^*/V) - (F_j^*/V)|^2 d_\tau W(t_0,\tau) \\ &= a\|(F_k/V) - (F_j/V)\|_2^2 \\ &\to 0 \qquad\qquad \text{as } j,k \to \infty. \end{aligned}$$

Hence there exists $F \in H_2(R)$ such that $F_k \to F$. Since $F_k \in C$, $F \in C$. Thus $F/V \in C_1$ and $F_k/V \to F/V$. Thus C_1 is closed. Now by our choice of n and V not all functions in C_1 vanish at t_0. Thus by Case I, $C_1 = C(\Phi_1)$ for some inner function Φ_1.

Now $\Phi_2(t) = \exp(-nW(t_0,t))$ is an inner function on R which has a zero at t_0 of order n. Note that $\Phi_2(t)$ is analytic on $\bar{R}$ and vanishes nowhere on $\bar{R} - t_0$. It follows that both $|\Phi_2|^2/|V|^2$ and $|V|^2/|\Phi_2|^2$ are bounded on $\bar{R}$. Let ϕ_1 and ϕ_2 be m.i. inner functions on K such that $\Phi_j = \phi_j \cdot T^{-1}, j = 1,2$; and let $\phi = \phi_1\phi_2$. Then ϕ is an m.i. inner function on K. For $\Phi = \phi \cdot T^{-1}$, $|\Phi| = |\Phi_1|\,|\Phi_2|$ on R. We will show $C = C(\Phi)$. Suppose $F \in C$. Then $F/V \in C_1 = C(\Phi_1)$ and thus $|F|^2/|V|^2|\Phi_1|^2$ has a harmonic majorant on R. Since $|V|^2/|\Phi_2|^2$ is bounded it follows that

$$|F|^2/|\Phi|^2 = |F|^2/|\Phi_1|^2|\Phi_2|^2 = (|V|^2/|\Phi_2|^2)(|F|^2/|V|^2|\Phi_1|^2)$$

has a harmonic majorant on R. Thus $F \in C(\Phi)$ and $C \subset C(\Phi)$. On the other hand suppose $F \in C(\Phi)$. Then $|F|^2/|\Phi_1|^2|\Phi_2|^2 = |F|^2/|\Phi|^2$ has a harmonic majorant on R. Since $|\Phi_2|^2/|V|^2$ is bounded it follows that

$$|F|^2/|V|^2|\Phi_1|^2 = (|\Phi_2|^2/|V|^2)(|F|^2/|\Phi|^2)$$

has a harmonic majorant on R. Thus $F/V \in C(\Phi_1)$ and $F \in C$. Hence $C(\Phi) \subset C$. Thus $C(\Phi) = C$ which proves Case II.

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