DISTINGUISHED RINGS OF LINEAR TRANSFORMATIONS(1)

BY

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A ring R of linear transformations of a vector space M over a division ring D is called distinguished iff (1) the lattice J of all R-submodules of M is a distributive sublattice of the lattice L of all subspaces of M, and (2) the set of all linear transformations of M leaving J invariant is R. The study of such rings is motivated by a paper of Wolfson [2] in which J is a chain and by several recent papers of Behrens [4]. Behrens studies Artinian rings R with unity having faithful R-modules M such that the lattice of R-submodules of M is distributive.

Our primary interest is with distinguished rings R for which J is finite as well as distributive. Such a condition does not force the ring R to be Artinian. A basic tool in our study is a lattice theorem (1.1) stating that every element of J is a direct sum of elements of L associated with the irreducible elements of J. It is shown that every finite distributive sublattice of L containing 0 and M is the lattice of submodules of a distinguished ring R. If D has characteristic 0, then a finite sublattice of L must be distributive in order to be the lattice of submodules of a ring of linear transformations of M.

A subspace N of M is called J-distributive iff $N \cap (A \cup B) = (N \cap A) \cup (N \cap B)$ for all $A, B \in J$. It is shown that N is J-distributive iff $N = Me$ for some idempotent $e \in R$. All subspaces of M are J-distributive iff J is a chain. Wolfson proved that R is a Baer ring (i.e., every annihilating right or left ideal of R is generated by an idempotent) if J is a chain. We show that this is almost the only case in which a distinguished ring is a Baer ring.

Every distinguished ring R is a direct sum of subrings of the form e_iRe_j, when $1 = e_1 + \cdots + e_n$ (direct sum), each e_iRe_j is a full ring of linear transformations, $e_iRe_j = 0$ if $i < j$, and $\sum_{i \neq j} e_iRe_j$ is the radical of R. Two distinguished rings are shown to be isomorphic iff their vector spaces are related in an obvious way.

1. **Introduction.** A module ${}_DM$ over a ring D has associated with it a lattice $L({}_DM)$ of all submodules and a ring $E({}_DM)$ of all endomorphisms. For each subring R of $E({}_DM)$, we shall always consider M to be a bimodule ${}_DM_R$. Associated with each sublattice J of $L({}_DM)$ is a subring $R = R(J) = \{r \in E({}_DM) \mid Nr \subset N$ for every $N \in J\}$ of $E({}_DM)$. Clearly $J \subset L(M_R)$. Let us call a sublattice J of ${}_DM$

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a *distinguished lattice* (*d-lattice*) of ${}_DM$ iff $J = L(M_R)$. Also, let us call a subring R of $E({}_DM)$ a *distinguished ring* (*d-ring*) iff $R = R(J)$ where $J = L(M_R)$.

The ring D is restricted in this paper to be a division ring and the module ${}_DM$ to be a unital module over D (i.e., a vector space over D). Thus, every d-ring is a ring of linear transformations of some vector space over a division ring. If $C(D)$ denotes the center of D, then we may consider $C(D) \subset R(J)$ for every $J \subset L({}_DM)$ if we define $xa = ax$ for all $a \in C(D)$ and $x \in M$. It is well known that there is associated with each basis B of ${}_DM$ a division ring $D_B \subset E({}_DM)$ isomorphic to D. Thus, for each $d \in D$ we define $d' \in D_B$ by $xd' = dx$, $x \in B$.

It is easily shown that $R(L({}_DM)) = C(D)$. Hence, $L({}_DM)$ is a d-lattice of ${}_DM$ iff D is a field. If B is a basis of ${}_DM$ and J is the sublattice of $L({}_DM)$ generated by all atoms of the form Dx and $D(x + y)$, $x, y \in B$, then it can be shown that $R(J) = D_B$. On the other hand, if J is generated by only the atoms Dx, $x \in B$, then $R(J) \cong \prod_{i \in \Delta} D_i$ where $\Delta = \operatorname{card} B$ and $D_i = D_B$ for each $i \in \Delta$. At the other extreme, $\{0, M\}$ is a lattice of ${}_DM$ having $E({}_DM)$ as its d-ring.

Our primary interest in this paper is with finite d-lattices of ${}_DM$ and their associated d-rings. Clearly each d-lattice of ${}_DM$ is a complete, modular lattice containing 0 and M, and $R(J)$ is a ring with unity. For each $N \in J$, we denote the lattice dimension of N by $\dim N$ ($\dim N$ is the length of the longest chain in the interval $[0, N]$ of J) and let $\dim J = \dim M$. If $J = L({}_DM)$ then $\dim N$ is the usual vector space dimension of a subspace N of M.

If J is any finite-dimensional modular lattice, then $N \in J$ is called *irreducible* iff $N \neq 0$ and N is not a union of lower-dimensional elements of J. Each irreducible $P \in J$ covers a unique element of J which we will always designate by P^0. Clearly $P^0 = 0$ iff P is an atom of J. The set of all irreducible elements of J is designated by $I(J)$. Given a lattice J and $N \in J$, we shall also let $I(N) = \{A \in I(J) \mid A \leqq N\}$.

Given a finite-dimensional distributive lattice J, each nonzero $N \in J$ may be uniquely represented as an irreducible union of irreducible elements of J, namely as the union of the maximal elements of $I(N)$ [1, p. 142]. It is easily verified in this case that $I(A \cup B) = I(A) \cup I(B)$ and $I(A \cap B) = I(A) \cap I(B)$ for all A, $B \in J$. Incidentally, we shall use the notation $\dot{\cup}$ for *direct union* in a lattice.

The following theorem is of basic importance to the rest of this paper.

1.1 Theorem. *Let L be a complete, complemented, modular lattice with identities* 0 *and I and J be a finite-dimensional sublattice of L containing* 0 *and I. For each irreducible $P \in J$, let us select $\bar{P} \in L$ so that $P = \bar{P} \dot{\cup} P^0$. Then each nonzero $K \in J$ may be represented in the form $K = \dot{\bigcup}_{i=1}^{m} \bar{P}_i$ for some subset $\{P_1, \cdots, P_m\}$ of $I(K)$. If J is distributive, then $K = \dot{\bigcup}_{i=1}^{n} \bar{P}_i$ where $I(K) = \{P_1, \cdots, P_n\}$.*

Proof. The theorem is trivially true if $\dim K = 1$. Let us assume that the conclusion holds for every $K \in J$ of dimension $\leqq n$ and let $K \in J$, $\dim K = n + 1$. If K is irreducible, then $K = K^0 \dot{\cup} \bar{K}$ and K is a direct union of elements of the form $\bar{P}$, P irreducible, since K^0 is. If J is distributive, then $I(K) = \{K\} \cup I(K^0)$ so that $K = \bigcup_{P \in I(K)} \bar{P}$ by the inductive assumption.

If K is reducible, then $K = A \cup B$ where A, $B \in J$, A irreducible, and $\dim B \leqq n$. We know that $A = A^0 \dot{\cup} \bar{A}$ and $A \cap B \leqq A^0$. Since $A \cap (A^0 \cup B) = A^0 \cup (A \cap B) = A^0$, clearly $\bar{A} \cap (A^0 \cup B) \leqq \bar{A} \cap A^0 = 0$. Hence, $K = \bar{A} \dot{\cup} (A^0 \cup B)$ and the first conclusion follows from the inductive assumption once we observe that $\dim(A^0 \cup B) \leqq n$. If J is distributive, the desired conclusion follows from the observation that $I(K) = \{A\} \cup I(A^0 \cup B)$. This proves 1.1.

2. **Distributive lattices of $_DM$.** Every finite-dimensional distributive lattice is actually finite [1, p. 139]. A finite, distributive sublattice of $L(_DM)$ containing 0 and M will be called a *FD-lattice of* $_DM$ henceforth. If J is a FD-lattice of $_DM$ and $I(J) = \{P_1, \cdots, P_n\}$, then by 1.1, $M = \bar{P}_1 \dot{\cup} \cdots \dot{\cup} \bar{P}_n$ where $\bar{P}_i$ is any relative complement (in $L(_DM)$) of P_i^0 in P_i, $i = 1, \cdots, n$.

2.1 Theorem. *If J is a FD-lattice of $_DM$ and, for each $P \in I(J)$, $\bar{P} \in L(_DM)$ is chosen so that $P = P^0 \dot{\cup} \bar{P}$, then $R(J) = \{a \in E(_DM) \mid \bar{P}a \subset P$ for every $P \in I(J)\}$.*

Proof. If $a \in R(J)$ then clearly $\bar{P}a \subset Pa \subset P$ for every $P \in I(J)$. Conversely, if $a \in E(_DM)$ and $\bar{P}a \subset P$ for every $P \in I(J)$, then $K = \bigcup_{P \in I(K)} \bar{P}$ for every $K \in J$ and $Ka \subset \bigcup_{P \in I(K)} P = K$. Hence, $a \in R(J)$. This proves 2.1.

It is clear from 2.1 that if J is a FD-lattice of $_DM$, $I(J) = \{P_1, \cdots, P_n\}$, and $P_i = \bar{P}_i \dot{\cup} P_i^0$ for some $\bar{P}_i \in L(_DM)$, $i = 1, \cdots, n$, then for any $a_i \in \mathrm{Hom}(_D\bar{P}_i, {_DP_i})$, $i = 1, \cdots, n$, there exists a unique $a \in R(J)$ such that $a \mid \bar{P}_i = a_i$, $i = 1, \cdots, n$. In particular, we have the following result (if we select $\bar{P}_i$ to contain each x_i).

2.2 Corollary. *If J is a FD-lattice of $_DM$, $I(J) = \{P_1, \cdots, P_n\}$, and $x_i \in P_i$, $x_i \notin P_i^0$, $i = 1, \cdots, n$, then for any $y_i \in P_i$, $i = 1, \cdots, n$, there exists some $a \in R(J)$ such that $x_i a = y_i$, $i = 1, \cdots, n$.*

2.3 Corollary. *If J is a FD-lattice of $_DM$, then each $K \in J$ is a cyclic $R(J)$ module.*

Proof. If $K = P_1 \cup \cdots \cup P_m$, each P_i irreducible, then select $x_i \in P_i$, $x_i \notin P_i^0$, $i = 1, \cdots, m$, and let $x = x_1 + \cdots + x_m$. If $y \in K$, say $y = y_1 + \cdots + y_m$, $y_i \in P_i$, then by 2.2 $x_i a = y_i$, $i = 1, \cdots, m$, for some $a \in R(J)$ and $xa = y$. Hence, $xR(J) = K$ and 2.3 is proved.

2.4 Theorem. *Every FD-lattice of $_DM$ is a d-lattice of $_DM$.*

Proof. Let J be a FD-lattice of $_DM$ and $R = R(J)$. To prove that J is a d-lattice of $_DM$, we need only show that $L(M_R) \subset J$. To this end, let $x \in M$, $x \neq 0$, $K = xR$, and N be the least element of J containing x. Clearly $K \subset N$. Let $\{P_1, \cdots, P_m\}$ be the set of maximal elements of $I(N)$, so that $N = P_1 \cup \cdots \cup P_m$, and let $x = x_1 + \cdots + x_m$, $x_i \in P_i$. If any $x_i \in P_i^0$, say $x_1 \in P_1^0$, then

$$x \in P_1^0 \cup P_2 \cup \cdots \cup P_m < N$$

since $P_1 \not\subset P_1^0 \cup P_2 \cup \cdots \cup P_m$. This contradicts the choice of N. Hence $x_i \notin P_i^0$, $i = 1, \cdots, m$. It follows as in the proof of 2.3 that $K = xR = N$. Since each cyclic submodule of $L(M_R)$ is contained in J and J is complete, $L(M_R) \subset J$. This proves 2.4.

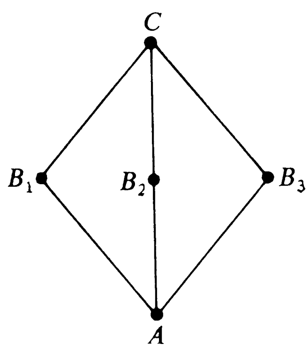
If J is a distributive lattice of finite dimension $n > 1$, then it is well known that $I(J)$ has n elements and that it may be imbedded in a Boolean algebra B of dimension n [1, pp. 139, 140]. Given an n-dimensional vector space ${}_DM$ and an atomic basis $\{A_1, \cdots, A_n\}$ of $L({}_DM)$, the sublattice B' of $L({}_DM)$ generated by this basis is an n-dimensional Boolean algebra. Since $B \cong B'$, J is isomorphic to a sublattice J' of $L({}_DM)$. Clearly $0, M \in J'$ and J' is a FD-lattice of ${}_DM$. This proves the following result.

2.5 Theorem. *Every distributive lattice J of finite dimension $n > 1$ is a FD-lattice of each n-dimensional vector space ${}_DM$.*

Not every finite d-lattice of a vector space ${}_DM$ need be distributive. For example, if D is a finite field and ${}_DM$ is finite-dimensional, then $L({}_DM)$ is a finite lattice that is not distributive. However, many finite d-lattices of a vector space are distributive according to the next result.

2.6 Theorem. *If the division ring D has characteristic zero, then every finite d-lattice of ${}_DM$ is a FD-lattice of ${}_DM$.*

Proof. Let us assume that J is a finite d-lattice of ${}_DM$ that is not distributive. Then J must contain a sublattice of the type shown in the figure [1, p. 134]. Let



us select $B_3' \in L({}_DM)$ such that $B_3 = A \dot{\cup} B_3'$ and let $\{z_i \mid i \in \Delta\}$ be a basis of B_3' in ${}_DM$. Since $B_3 \subset B_1 + B_2$, $z_i = x_i + y_i$ for some $x_i \in B_1$ and $y_i \in B_2$, $i \in \Delta$. If B_1' and B_2' are the subspaces of B_1 and B_2, respectively, generated by $\{x_i \mid i \in \Delta\}$ and $\{y_i \mid i \in \Delta\}$, then $B_1 = A \dot{\cup} B_1'$, $B_2 = A \dot{\cup} B_2'$, $\{x_i \mid i \in \Delta\}$ is a basis of B_1', and $\{y_i \mid i \in \Delta\}$ is a basis of B_2'. For if $\sum d_i x_i \in A$ for some $d_i \in D$, then $\sum d_i z_i \in B_2 \cap B_3 = A$, $\sum d_i z_i = 0$, and each $d_i = 0$. Hence, $A \cap B_1' = 0$ and $\{x_i \mid i \in \Delta\}$ is a basis of B_1'. Similarly, $A \cap B_2' = 0$ and $\{y_i \mid i \in \Delta\}$ is a basis of B_2'.

Finally, each $x \in B_1$ has the form $x = y + z$ for some $y \in B_2$ and $z \in B_3$. In turn, $z = z' + \sum d_i z_i$ for some $z' \in A$ and $d_i \in D$, so that $x = y + z' + \sum d_i x_i + \sum d_i y_i$. Hence, $x - \sum d_i x_i \in B_1 \cap B_2 = A$. Consequently, $B_1 = A \dot{\cup} B_1'$ and, similarly, $B_2 = A \dot{\cup} B_2'$.

For each $i \in \Delta$, let us select a nonzero $\gamma_i = (b_i, c_i) \in C(D) \times C(D)$, and let $B'(\gamma)$ be the subspace of ${}_DM$ generated by $\{b_i x_i + c_i y_i \mid i \in \Delta\}$. If $\sum d_i(b_i x_i + c_i y_i) \in A$ for some $d_i \in D$, then $\sum d_i b_i x_i \in B_2$ and $\sum d_i c_i y_i \in B_1$. It follows that $\sum d_i b_i x_i = \sum d_i c_i y_i = 0$ and $d_i b_i = d_i c_i = 0$ for each i. Since $\gamma_i \neq 0$, $d_i = 0$ for each i. We conclude that $B'(\gamma) \cap A = 0$.

We shall next prove that for each $\gamma = \{\gamma_i \mid i \in \Delta\}$, the subspace $B(\gamma) = A \dot{\cup} B'(\gamma)$ of ${}_DM$ is in J. First, we observe that $B_3 a \subset B_3$ for each $a \in R = R(J)$, so that $(x_i + y_i)a = u_i + d_i(x_i + y_i)$ for some $u_i \in A$ and $d_i \in D$, $i \in \Delta$. Since $x_i a \in B_1$ and $y_i a \in B_2$ for each $a \in R$, we must have $x_i a = v_i + d_i x_i$ and $y_i a = w_i + d_i y_i$ for some $v_i, w_i \in A$, $i \in \Delta$. Hence, $(b_i x_i + c_i y_i)a = b_i v_i + c_i w_i + d_i(b_i x_i + c_i y_i)$, $i \in \Delta$, and $B(\gamma)a \subset B(\gamma)$ for each $a \in R$. Therefore, $B(\gamma) \in J$.

We easily see that there are an infinite number of distinct subspaces of ${}_DM$ of the form $B(\gamma)$. For example, let B_n' be the subspace generated by $\{(n-2)x_i + y_i \mid i \in \Delta\}$ and let $B_n = A \dot{\cup} B_n'$, $n = 2, 3, \cdots$. If $n \neq m$ and $u \in B_n \cap B_m$, then $u = v + \sum d_i[(n-2)x_i + y_i] = w + \sum d_i'[(m-2)x_i + y_i]$ for some $v, w \in A$ and $d_i, d_i' \in D$, $i \in \Delta$. Hence, $\sum [d_i(n-2) - d_i'(m-2)]x_i \in B_1 \cap B_2 = A$ and, similarly, $\sum (d_i - d_i')y_i \in A$. Consequently,

$$\sum [d_i(n-2) - d_i'(m-2)]x_i = 0$$

and $\sum (d_i - d_i')y_i = 0$. Therefore, $d_i(n-2) - d_i'(m-2) = 0$ and $d_i - d_i' = 0$ for each i. From these equations, we easily see that $d_i = d_i' = 0$ for each i. Hence, $B_n \cap B_m = A$ for all m and n, $m \neq n$, and J contains the infinite set $\{B_2, B_3, \cdots\}$ of subspaces. This is contrary to assumption, and proves 2.6.

If J is a FD-lattice of ${}_DM$, then J has a center $C(J)$ consisting of all complemented elements of J. If $\{M_1, \cdots, M_k\}$ is the set of atoms of the Boolean algebra $C(J)$ and J_i denotes the interval $[0, M_i]$ of $J, i = 1, \cdots, k$, then $M = M_1 \dot{\cup} M_2 \dot{\cup} \cdots \dot{\cup} M_k$ and $J \cong J_1 \times J_2 \times \cdots \times J_k$. The d-rings $R = R(J)$ and $R_i = R(J_i)$, $i = 1, \cdots, k$, are easily seen to be related by an isomorphism, $R \cong R_1 \times R_2 \times \cdots \times R_k$, under the correspondence $a \leftrightarrow (a_1, a_2, \cdots, a_k)$, $a \in R$, where $a_i = a \mid M_i$, $i = 1, \cdots, k$.

Let us call the d-ring $R = R(J)$ *indecomposable* iff $C(J) = \{0, M\}$. It is evident that R is indecomposable iff R is not a direct union of two nonzero ideals of R. For if $R = A \dot{\cup} B$, A and B nonzero ideals, then $M = MA \dot{\cup} MB$ and MA, $MB \in C(J)$. Conversely, if $M = M' \dot{\cup} M''$ for some nonzero $M', M'' \in J$ then $R = A \dot{\cup} B$ where $A = \{r \in R \mid Mr \subset M'\}$ and $B = \{r \in R \mid Mr \subset M''\}$ are nonzero ideals of R. Clearly each ring R_i above is indecomposable. We state our remarks above in the following form.

2.7 THEOREM. *Every d-ring is isomorphic to a finite direct product of indecomposable d-rings.*

3. **J-distributivity.** If J is a FD-lattice of $_DM$ and $N \in L(_DM)$, then N is called *J-distributive* iff $N \cap (A \cup B) = (N \cap A) \cup (N \cap B)$ for all $A, B \in J$. It is easily checked that if N is J-distributive and $J_N = \{N \cap K \mid K \in J\}$, then J_N is a FD-lattice of $_DN$. Each element of J is J-distributive since J is a distributive lattice. It is equally clear that every subspace N of $_DM$ is J-distributive if J is a chain. A useful characterization of the J-distributive elements of $L(_DM)$ is given below.

3.1 THEOREM. *If J is a FD-lattice of $_DM$ and $N \in L(_DM)$, then the following statements are equivalent:*

(a) *N is J-distributive.*

(b) *For each $P \in I(J)$ there exists $P' \in L(_DM)$ such that $P' \subset P$, $P' \cap P^0 = 0$, and $N = \bigcup_{P \in I(J)} P'$.*

(c) *$N = Me$ for some idempotent $e \in R(J)$.*

Proof. Let N be J-distributive and for each $P \in I(J)$ let $P' \in L(_DM)$ be selected so that $(N \cap P^0) \dot{\cup} P' = N \cap P$. Clearly $P' \cap P^0 = 0$, since

$$P' \cap P^0 \subset (N \cap P^0) \cap P' = 0.$$

We shall prove that

(1) $N \cap K = \bigcup_{P \in I(K)} P'$ for every nonzero $K \in J$.

To prove (1), we note first that it holds if K is an atom of J, since then $K' = N \cap K$. Let us assume that (1) holds for every element of J of dimension n or smaller, and let $K \in J$ be of dimension $n + 1$. Since N is J-distributive, $N \cap K = \bigcup_{P \in I(K)} N \cap P = \bigcup_{P \in I(K)} [(N \cap P^0) \cup P']$. Now $\dim P^0 \leqq n$ for every $P \in I(K)$, and therefore $N \cap P^0 = \bigcup_{Q \in I(P^0)} Q'$. Hence,

$$N \cap K = \bigcup_{P \in I(K)} \left[\bigcup_{Q \in I(P^0)} Q' \cup P' \right] = \bigcup_{P \in I(K)} P'$$

since $I(P^0) \subset I(K)$ for every $P \in I(K)$. This proves (1).

If we let $K = M$ in (1), we obtain (b).

If (b) holds, then for each $P \in I(J)$ we may choose $\bar{P}$, $P'' \in L(_DM)$ such that $P = P^0 \dot{\cup} \bar{P}$ and $\bar{P} = P' \dot{\cup} P''$. Since $M = \bigcup_{P \in I(J)} \bar{P}$ by Theorem 1.1, there is a well-defined idempotent $e \in E(_DM)$ such that $e \mid P' = \iota$ and $e \mid P'' = 0$ for each $P \in I(J)$, where ι designates the identity mapping. Clearly $e \in R(J)$ since $Pe \subset P$ for every $P \in I(J)$. Since $N = \bigcup_{P \in I(J)} P'$, $Me = N$ by the very definition of e. Thus, (b) implies (c).

Finally, let $e \in R(J)$ be an idempotent and $N = Me$. It is easily shown that $N \cap K = Ke$ for every $K \in J$. Hence, $N \cap (A \cup B) = (A + B)e = Ae + Be = (N \cap A) \cup (N \cap B)$ for all $A, B \in J$, and N is J-distributive. Thus, (c) implies (a). This proves 3.1.

3.2 Corollary. *If J is a FD-lattice of $_DM$ and $x \in M$, $x \neq 0$, then Dx is J-distributive iff $x \in P$, $x \notin P^0$, for some $P \in I(J)$.*

This follows directly from 3.1, part(b).

For each idempotent $e \in R(J)$, $1 - e$ also is an idempotent in $R(J)$ and $M = Me \dot{\cup} M(1-e)$. By Theorem 3.1, $Me = \bigcup_{P \in I(J)} P'$ and $M(1-e) = \bigcup_{P \in I(J)} P''$ for some $P', P'' \in L(_DM)$ such that $Me \cap P = (Me \cap P^0) \dot{\cup} P'$ and $M(1-e) \cap P = [M(1-e) \cap P^0] \dot{\cup} P''$ for every $P \in I(J)$. It is easy to show that $P = P^0 \dot{\cup} (P' \dot{\cup} P'')$ for each $P \in I(J)$ and hence that $M = \bigcup_{P \in I(J)} (P' + P'')$. Since $e | P' = \iota$ and $e | P'' = 0$ for every $P \in I(J)$, we have shown that every idempotent $e \in R(J)$ is arrived at in the way shown in the proof of 3.1.

3.3 Theorem. *If J is a FD-lattice of $_DM$ and S is the set of all J-distributive elements of $L(_DM)$, then $S = L(_DM)$ iff J is a chain.*

Proof. It is evident that if J is a chain then $S = L(_DM)$. If J is not a chain, then there exists a chain $0 = K_0 < K_1 < \cdots < K_r$ contained in J such that (1) if $K \in J$ and $K \neq K_i$ for $i = 0, 1, \cdots, r$, then $K > K_r$, and (2) K_r is covered by at least two distinct elements P and P' of J. Evidently P, $P' \in I(J)$ and $P^0 = P'^0 = P \cap P' = K_r$. Let us select nonzero $x \in P$ and $x' \in P'$ such that $x, x' \notin K_r$, and let $N = D(x + x')$. Clearly N is not J-distributive, since $N \cap (P \cup P') = N$ whereas $N \cap P = N \cap P' = 0$. This proves 3.3.

4. **Projectors.** Let J be a FD-lattice of $_DM$ and $R = R(J)$. For all $K, N \in J$, define the *projector of K into N*, $K^{-1}N$, to be the largest subset of R such that $K(K^{-1}N) \subset N$. Clearly $K^{-1}N$ is an ideal of R. Since $KR \subset K$, evidently $K^{-1}N = K^{-1}(K \cap N)$. Thus, we might as well assume that $K \supset N$ in discussing the projector of K into N. If A is an ideal of R and $N \in J$, then the *projector of A into N*, NA^{-1}, is the largest subset of M such that $(NA^{-1})A \subset N$. It is clear that $NA^{-1} \in J$ and $NA^{-1} \supset N$. Finally, if A and B are ideals of R, we can define the right and left *projectors of B into A*, $B^{-1}A$ and AB^{-1}, as the largest subsets of R such that $B(B^{-1}A) \subset A$ and $(AB^{-1})B \subset A$, respectively. Evidently $B^{-1}A$ and AB^{-1} are ideals of R containing A. Since $AB^{-1} = (A \cap B)B^{-1}$ and $B^{-1}A = B^{-1}(A \cap B)$, we might as well assume that $B \supset A$ in discussing the projectors $B^{-1}A$ and AB^{-1}. The properties of projectors we shall need are contained in the following theorem.

4.1 Theorem. *If J is a FD-lattice of $_DM$ and $R = R(J)$, then:*

(1) $K(K^{-1}N) = N$ and $N(K^{-1}N)^{-1} = K$ if $K \supset N$, $K, N \in J$.

(2) $N_1^{-1}N_2 = (N_1^{-1}N_3)(N_2^{-1}N_3)^{-1}$ and $N_2^{-1}N_3 = (N_1^{-1}N_2)^{-1}(N_1^{-1}N_3)$ if $N_1 \supset N_2 \supset N_3$, $N_i \in J$.

Proof. (1) There exists an idempotent $e \in R$ such that $Me = N$. Hence, $K \cap Me = Ke = N$, $e \in K^{-1}N$, and $K(K^{-1}N) = N$. To prove the second part, we first observe that $N(K^{-1}N)^{-1} \supset K$. Let us assume that $N(K^{-1}N)^{-1} = K' \neq K$.

Then $K' \in J$ and there exists some $P \in I(K')$ such that $P \not\subset K$. Hence, we can select $P' \in L({}_DM)$ so that $P' \dot{\cup} P^0 = P$. Therefore $P' \cap K = 0$ and we can select $e \in R$ so that $e \mid K = 0$ and $e \mid P' = \iota$. Clearly $e \in K^{-1}N$ and $K'(K^{-1}N) \not\subset N$ since $K'e \supset P'$. However, $[N(K^{-1}N)^{-1}](K^{-1}N) \subset N$ according to the definition of $N(K^{-1}N)^{-1}$. This contradiction proves that $N(K^{-1}N)^{-1} = K$.

(2) Clearly

$$(N_1^{-1}N_2)\,(N_1^{-1}N_3) \subset N_1^{-1}N_3,$$

so that $N_1^{-1}N_2 \subset (N_1^{-1}N_3)(N_2^{-1}N_3)^{-1}$ and $N_2^{-1}N_3 \subset (N_1^{-1}N_2)^{-1}\,(N_1^{-1}N_3)$. On the other hand, $[(N_1^{-1}N_3)\,(N_2^{-1}N_3)^{-1}]\,(N_2^{-1}N_3) \subset N_1^{-1}N_3$ so that $N_1[(N_1^{-1}N_3)\,(N_2^{-1}N_3)^{-1}]\,(N_2^{-1}N_3) \subset N_3$. Hence, $N_1[(N_1^{-1}N_3)\,(N_2^{-1}N_3)^{-1}] \subset N_3(N_2^{-1}N_3)^{-1} = N_2$ and $(N_1^{-1}N_3)\,(N_2^{-1}N_3)^{-1} \subset N_1^{-1}N_2$. Therefore, $(N_1^{-1}N_3)\,(N_2^{-1}N_3)^{-1} = N_1^{-1}N_2$. Similarly, $(N_1^{-1}N_2)\,[(N_1^{-1}N_2)^{-1}(N_1^{-1}N_3)] \subset N_1^{-1}N_3$, $N_1(N_1^{-1}N_2)\,[(N_1^{-1}N_2)^{-1}(N_1^{-1}N_3)] \subset N_3$, $N_2[(N_1^{-1}N_2)^{-1}(N_1^{-1}N_3)] \subset N_3$, and $(N_1^{-1}N_2)^{-1}(N_1^{-1}N_3) \subset N_2^{-1}N_3$, Therefore, $(N_1^{-1}N_2)^{-1}(N_1^{-1}N_3) = N_2^{-1}N_3$, and 4.1 is proved.

If J is a FD-lattice of ${}_DM$ and $R = R(J)$, then an ideal of R of the form $A^{-1}0$, resp. $0A^{-1}$, for some ideal A of R is called a *right*, resp. *left*, *annihilating ideal* of R. Let us designate by $L_r(R)$, resp. $L_l(R)$, the set of all right, resp. left, annihilating ideals of R.

4.2 Theorem. *If J is a FD-lattice of ${}_DM$ and $R = R(J)$, then $L_r(R) = \{K^{-1}0 \mid K \in J\}$ and $L_l(R) = \{M^{-1}K \mid K \in J\}$. Thus, the mapping $J \xrightarrow{\alpha} L_r(R)$ defined by $\alpha K = K^{-1}0$, $K \in J$, is a dual isomorphism and the mapping $J \xrightarrow{\beta} L_l(R)$ defined by $\beta K = M^{-1}K$, $K \in J$ is an isomorphism.*

Proof. By 4.1, (2), $K^{-1}0 = (M^{-1}K)^{-1}(M^{-1}0) = (M^{-1}K)^{-1}0$ and therefore $K^{-1}0 \in L_r(R)$ for each $K \in J$. Similarly, $M^{-1}K = (M^{-1}0)(K^{-1}0)^{-1} = 0(K^{-1}0)^{-1}$ and $M^{-1}K \in L_l(R)$ for each $K \in J$. If $A \in L_r(R)$ and $B = 0A^{-1}$, then $B \in L_l(R)$ and $A = B^{-1}0$. If we let $K = MB$, then $K \in J$ and $K^{-1}0 = B^{-1}0 = A$. Thus, A has the desired form. Also, $M(M^{-1}K)A = 0$, $(M^{-1}K)A = 0$, and $B \subset M^{-1}K \subset 0A^{-1} = B$. Hence, $B = M^{-1}K$ and B has the desired form. If $K^{-1}0 = N^{-1}0$ for some $K, N \in J$, then $K = 0(K^{-1}0)^{-1} = 0(N^{-1}0)^{-1} = N$ by 4.1, (1). If $M^{-1}K = M^{-1}N$ for some K, $N \in J$, then $K = M(M^{-1}K) = M(M^{-1}N) = N$ by 4.1, (1). Thus, α and β are bijections. This proves 4.2.

If J is a FD-lattice of ${}_DM$ and $R = R(J)$, and if K, $N \in J$ with $K \supset N$, then $K^{-1}N = (K^{-1}0)\,(N^{-1}0)^{-1} = (M^{-1}K)^{-1}\,(M^{-1}N)$ by 4.1, (2). That is, $K^{-1}N = (\alpha K)(\alpha N)^{-1} = (\beta K)^{-1}(\beta N)$ if we use the mappings α and β of 4.2. This shows that the ideals of the form $K^{-1}N$, where K, $N \in J$, may be defined intrinsically in the ring R itself. We shall use this fact later on in the paper.

If J is a FD-lattice of ${}_DM$ and K, $N \in J$ with $K \supset N$, then evidently $J' = \{N' - N \mid N' \in [N, K]\}$ is a FD-lattice of ${}_D(K - N)$. There is a natural mapping $R \xrightarrow{\phi} R'$ of $R = R(J)$ onto $R' = R(J')$ defined by: $(x + N)(c\phi) = xc + N$

for all $c \in R$ and $x + N \in K - N$. It is easily seen that ϕ is an epimorphism and $\ker \phi = K^{-1}N$. We state this result as follows.

4.3 THEOREM. *If J is a FD-lattice of ${}_DM$ and $K, N \in J$ with $K \supset N$, and if $J' = \{N' - N \mid N' \in [N, K]\}$, then J' is a FD-lattice of ${}_D(K - N)$ and $R(J') \cong R(J)/(K^{-1}N)$.*

5. **Baer rings.** While we defined the projector $K^{-1}N$ above only for $K, N \in J$, it is clear that $K^{-1}N$ may be defined in the same way for any subsets K and N of M or R. In particular, if J is a FD-lattice of ${}_DM$ and $R = R(J)$, then we can easily describe $K^{-1}N$ for any J-distributive subspaces K and N of M. Thus, $K = Me$ and $N = Mf$ for some idempotents $e, f \in R$, and $K^{-1}N \supset (1 - e)R + Rf$. On the other hand, if $a \in R$ then $a \in K^{-1}N$ iff $Mea \subset N$, or iff $eaf = ea$. That is, $a \in K^{-1}N$ iff $a - af \in (1 - e)R$, or iff $a \in (1 - e)R + Rf$. Hence, $K^{-1}N = (1-e)R + Rf$. It follows that $M^{-1}K = Re$ and $K^{-1}0 = (1-e)R$ if $K = Me$. Evidently $M^{-1}K = 0(K^{-1}0)^{-1}$ and $K^{-1}0 = (M^{-1}K)^{-1}0$; that is, $M^{-1}K$ is an annihilating left ideal and $K^{-1}0$ is an annihilating right ideal of R. These are the only annihilating right or left ideals of R generated by idempotents according to the next result.

5.1 THEOREM. *Let J be a FD-lattice of ${}_DM$ and $R = R(J)$. If A is a right ideal and B is a left ideal of R such that $A = B^{-1}0$ and $B = 0A^{-1}$, then A and B are generated by idempotents iff either $0A^{-1}$ or MB is a J-distributive subspace of M.*

Proof. If $A = eR$ for some idempotent $e \in R$, then $B = R(1 - e)$ and $0A^{-1} = MB = M(1 - e)$. Hence, $0A^{-1}(= MB)$ is J-distributive by 3.1. Conversely, if either $0A^{-1}$ or MB is J-distributive, then either $0A^{-1} = Mf$ or $MB = Me$ for some idempotents $e, f \in R$. If $0A^{-1} = Mf$, then $A \subset (1 - f)R$ and $B \subset Rf$ since $MB \subset Mf$. Hence, $A = B^{-1}0 \supset (Rf)^{-1}0 = (1 - f)R \supset A$ and $A = (1 - f)R$, $B = Rf$. If $MB = Me$, then $(MB)^{-1}0 = B^{-1}0 = (1 - e)R = A$ and $B = 0A^{-1} = Re$. This proves 5.1.

A ring R is called a *Baer ring* iff every annihilating right or left ideal of R is generated by an idempotent. In a recent paper [2], Wolfson proved the following result.

5.2 THEOREM. *If J is a finite chain in $L({}_DM)$ containing 0 and M, then $R(J)$ is a Baer ring.*

We point out that this theorem follows directly from 3.3 and 5.1. Thus, for every right ideal A and left ideal B of $R(J)$ such that $A = B^{-1}0$ and $B = 0A^{-1}$, MB is J-distributive by 3.3 and therefore A and B are generated by idempotents according to 5.1.

It is evident that if the FD-lattice J of ${}_DM$ is isomorphic to a direct product of chains, then $R(J)$ is isomorphic to a direct product of Baer rings and hence is a

Baer ring. However, these are not the only distinguished rings that are Baer rings. We shall not give the details, but it may be shown that the ring R of all 3×3 matrices over a division ring D of the form $d_{11}e_{11} + d_{22}e_{22} + d_{31}e_{31} + d_{32}e_{32} + d_{33}e_{33}$, where e_{ij} are the usual matrix units, is a Baer ring. In this case, we may also represent R as $R(J)$, where $J = \{0, Dx_1, Dx_2, Dx_1 + Dx_2, M\}$ and $M = Dx \ + Dx_2 + Dx_3$, a 3-dimensional vector space over D. If the lattice J is slightly more complicated, then $R(J)$ is not a Baer ring as the following theorem shows.

5.3 THEOREM. *If the FD-lattice J of ${}_DM$ contains three irreducible elements P_1, P_2, P_3 such that $P_1 \cup P_2 \subset P_3$ and $P_1 \cap P_2$ is different from* 0, *P_1, and P_2, then $R = R(J)$ is not a Baer ring.*

Proof. Let $\{P_1, \cdots, P_n\} = I(J)$ and let $P_i' \in L({}_DM)$ be selected so that $P_i^0 \dot{\cup} P_i' = P_i, i = 1, \cdots, n$. Select $x_i \in P_i', x_i \notin P_i^0, i = 1, 2, 3$, and let $N = D(x_1 + x_2)$. Since $N \cap (P_1 \cup P_2) = N$ whereas $(N \cap P_1) \cup (N \cap P_2) = 0$, N is not J-distributive. If we imagine that $x_3 \in B_3$, a basis of P_3', then there exists $b \in R$ such that $x_3 b = x_1 + x_2$, $xb = 0$ for $x \in B_3$, $x \neq x_3$, $P_i'b = 0$ if $i \neq 3$. Clearly $Mb = N$, and therefore $N^{-1}0 = b^{-1}0 = A$ is an annihilating right ideal of R. If A were generated by an idempotent, then $K = 0A^{-1}$ would be a J-distributive subspace of M, $K \supset N$. Now for any nonzero $y \in P_1 \cap P_2$, there exists some $a \in R$ such that $x_1 a = y, x_2 a = -y$, and $a \mid P_i' = \iota$ for $i = 3, \cdots, n$. Since $(x_1 + x_2)a = 0$, evidently $a \in A$. However, $P_i a \neq 0$ for $i = 1, \cdots, n$, and therefore $P_i \not\subset K$, $i = 1, \cdots, n$. It follows that $K = 0$, contrary to the fact that $K \supset N$. Consequently, A is not generated by an idempotent. This proves 5.3.

For example, if $M = Dx_1 + Dx_2 + Dx_3 + Dx_4$ is a 4-dimensional vector space over D and $J = \{0, Dx_1, Dx_1 + Dx_2, Dx_1 + Dx_3, Dx_1 + Dx_2 + Dx_3, M\}$, then $R = R(J)$ is the ring of all 4×4 matrices over D of the form

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

In the notations of 5.3, $P_1 = Dx_1 + Dx_2$, $P_2 = Dx_1 + Dx_3$, and $P_3 = M$. Letting $N = D(x_2 + x_3)$ as we did in the proof of 5.3, $N^{-1}0 = A = e_{11}R + (e_{21} - e_{31})R + e_{44}R$ and $B = 0A^{-1} = R(e_{42} + e_{43})$. Since $B^2 = 0$, evidently B, and therefore also A, is not generated by an idempotent.

6. **Subrings of distinguished rings.** Let J be a FD-lattice of ${}_DM$ and $R = R(J)$. An idempotent $e \in R$ is called *simple* iff $e \neq 0$ and there exists an irreducible $P \in J$ such that $Me \subset P$ and $Me \cap P^0 = 0$. A sum $e = e_1 + \cdots + e_m$ of idempotents

$e_i \in R$ is called *direct* iff $e_ie_j = \delta_{ij}e_j$, $i, j = 1, \cdots, m$. Evidently a direct sum of nonzero idempotents of R is a nonzero idempotent of R.

We have seen that if $\{P_1, \cdots, P_n\} = I(J)$, then associated with each nonzero idempotent $e \in R$ is a splitting $P_i = P_i^0 \dot{\cup} P_i' \dot{\cup} P_i''$, $i = 1, \cdots, n$, such that $Me = \bigcup_{i=1}^n P_i'$ and $M(1-e) = \bigcup_{i=1}^n P_i''$. If we define $e_i \in R$ as follows: $e_i \mid P_i' = \iota$, $e_i \mid P_j' = e_i \mid P_k'' = 0$, $i \neq j$, $i, j, k = 1, \cdots, n$, then clearly $e_ie_j = \delta_{ij}e_j$, $i, j = 1, \cdots, n$, and $e = e_1 + \cdots + e_n$. Since each nonzero e_i is simple, we have proved the following result.

6.1 Theorem. *If J is a FD-lattice of ${}_DM$, then each nonzero idempotent in $R(J)$ is a direct sum of simple idempotents.*

If $e = e_1 + \cdots + e_m$ is a direct sum of simple idempotents of $R = R(J)$, then the subring eRe of R is a direct sum of the subrings e_iRe_j. Let us now describe the structure of each subring e_iRe_j.

6.2 Theorem. *If J is a FD-lattice of ${}_DM$ and e, f are nonzero idempotents of $R = R(J)$, then $eRf \cong S = \{a \in \mathrm{Hom}({}_D(Me), {}_D(Mf)) \mid (Ke)a \subset Kf$ for every $K \in J\}$.*

Proof. We can consider $S \subset E({}_DM)$ by letting $a \mid M(1-e) = 0$ for each $a \in S$. Since $(Ke)eRf \subset Kf$ for each $K \in J$, evidently $eRf \subset S$. Now $KS = [Ke + K(1-e)]S = (Ke)S \subset Kf \subset K$ for each $K \in J$, and therefore $S \subset R$. Since $xa = xeaf$ for all $x \in M$ and $a \in S$, $S = eSf \subset eRf$. Thus, $S = eRf$ and 6.2 is proved.

6.3 Corollary. *If J is a FD-lattice of ${}_DM$, e is a nonzero idempotent of $R = R(J)$, and $N = Me$, then $eRl \cong R(J_N)$.*

We recall from §3 that $J_N = \{N \cap K \mid K \in J\}$. It follows from 6.3 that every subring of a d-ring R of the form eRe, e an idempotent, is also a d-ring.

6.4 Theorem. *Let J be a FD-lattice of ${}_DM$ and e, f be simple idempotents of $R = R(J)$ such that $Me \subset P$, $Me \cap P^0 = 0$, and $Mf \subset Q$, $Mf \cap Q^0 = 0$, for given irreducible $P, Q \in J$. Then $eRf \neq 0$ iff $Q \subset P$. If $eRf \neq 0$, then $eRf \cong$ $\mathrm{Hom}({}_D(Me), {}_D(Mf))$.*

Proof. If $Q \subset P$ then $MeRf = Pf \neq 0$ and $eRf \neq 0$. If $Q \not\subset P$ then $Pf \subset Mf \cap Q \cap P \subset Mf \cap Q^0 = 0$, $MeRf = 0$, and $eRf = 0$. In case $Q \subset P$ and $K \in J$, we have $Ke \subset K \cap P \neq 0$ iff $K \supset P$. Hence, $(Ke)a \subset Kf$ for all $a \in \mathrm{Hom}({}_D(Me), {}_D(Mf))$ and $K \in J$, and $eRf \cong \mathrm{Hom}({}_D(Me), {}_D(Mf))$ in view of 6.2. This proves 6.4.

Let us call ring R a *full ring* iff $R = E({}_DN)$ for some vector space ${}_DN$. If J is a FD-lattice of ${}_DM$ and $R = R(J)$, then R is a full ring iff $J = \{0, M\}$. Hence, the result below follows directly from 6.3.

6.5 THEOREM. *If J is a FD-lattice of ${}_DM$ and e is a nonzero idempotent of $R = R(J)$, then eRe is a full ring iff e is simple.*

6.6 COROLLARY. *The ring eRe of 6.5 is a division ring iff* $\dim(Me) = 1$ *in $L({}_DM)$. If eRe is a division ring, then $eRe \cong D$.*

That $eRe \cong D$ is seen as follows. Since $\dim(Me) = 1$, $Me = Dx$ for some $x \in Me$. Now $x(eRe) \in L({}_DM)$ and therefore $x(eRe) = Dx$. Hence, the mapping $eRe \overset{\sigma}{\to} D$ defined by: $xere = (ere)^\sigma x$, $r \in R$, is an isomorphism.

If J is a FD-lattice of ${}_DM$ and $\{P_1, \cdots, P_n\} = I(J)$, then we can order this set so that $P_j \not\subset P_i$ if $j > i$, i, $j = 1, \cdots, n$ (for example, order $I(J)$ so that $\dim(P_i) \leqq \dim(P_{i+1})$ in J). Let $1 = e_1 + \cdots + e_n$ be a direct sum of simple idempotents of $R = R(J)$, with $Me_i \dot\cup P_i^0 = P_i$, $i = 1, \cdots, n$. Then $e_iRe_j = 0$ if $j > i$ and $R = \sum_{j \leqq i=1}^n e_iRe_j$. Thinking in terms of a matrix representation of R, this says that the representation can be broken up into n^2 blocks with the blocks down the main diagonal full matrix rings and the blocks above the main diagonal all zero. The subring

$$A = \sum_{j<i=1}^{n} e_iRe_j$$

clearly is an ideal of R. Since $A^n = 0$ and $R/A \cong e_1Re_1 + \cdots + e_nRe_n$, a direct sum of primitive rings, evidently A is the Jacobson radical of R. We have proved part of the following theorem.

6.7 THEOREM. *Let J be a FD-lattice of ${}_DM$ of dimension n and $R = R(J)$. Then 1 can be expressed as a direct sum of n simple idempotents of R, $1 = e_1 + \cdots + e_n$, in such a way that R is a direct sum of subrings*

$$R = \sum_{j \leqq i=1}^{n} e_iRe_j.$$

The subring

$$A = \sum_{j<i=1}^{n} e_iRe_j$$

is the Jacobson radical of R, and

$$R/A \cong e_1Re_1 + \cdots + e_nRe_n,$$

a direct sum of full rings. The ideal A is nilpotent with index of nilpotency one less than the length of the longest chain in $I(J)$.

Proof. We need only find the index of nilpotency of A to complete the proof. Let $I(J) = \{P_1, \cdots, P_n\}$, with $\dim P_i \leqq \dim P_{i+1}$, $i = 1, \cdots, n-1$. By 6.4, $e_iRe_j \neq 0$ iff $P_i \supset P_j$; and if $P_i \supset P_j$ then $e_iRe_j \cong \mathrm{Hom}({}_D(Me_i), {}_D(Me_j))$. If $e_iRe_j \neq 0$ and $e_jRe_k \neq 0$, then for every nonzero $a \in e_iRe_j$ there exists some $b \in e_jRe_k$ such that $ab \neq 0$. It follows that $A^{m-1} \neq 0$ iff there exists a chain

$P_{i_1} \supset P_{i_2} \supset \cdots \supset P_{i_m}$ in $I(J)$. Hence, the index of nilpotency of A is the length of the longest chain in $I(J)$.

7. **Isomorphic theorems.** Distinguished rings are isomorphic iff the obvious isomorphisms hold between their underlying vector spaces, as we shall show below. Our results are extensions and generalizations of those of Wolfson [2, Theorem 7].

7.1 THEOREM. *Let J and J' be FD-lattices of ${}_DM$ and ${}_{D'}M'$, respectively, and let $R = R(J)$ and $R' = R(J')$. Then $R \cong R'$ iff there exist isomorphisms $D \overset{\sigma}{\to} D'$ and $J \overset{\alpha}{\to} J'$ such that $\dim_D(P - P^0) = \dim_{D'}(\alpha P - \alpha P^0)$ for every $P \in I(J)$.*

Proof. Let $R \overset{\phi}{\to} R'$ be an isomorphism. Then the mapping $J \overset{\alpha}{\to} J'$ defined by: $\alpha K = M'[(M^{-1}K)\phi]$, $K \in J$, is an isomorphism by 4.2. If $P \in I(J)$ and $P' = \alpha P$, then $E({}_D(P - P^0)) \cong R/(P^{-1}P^0)$ and $E({}_{D'}(P' - P'^0)) \cong R'/(P'^{-1}P'^0)$ by 4.3. Since $(P^{-1}P^0)\phi = [(M^{-1}P)^{-1}(M^{-1}P^0)]\phi = (M'^{-1}P')^{-1}(M'^{-1}P'^0) = (P'^{-1}P'^0)$, we conclude that $E({}_D(P - P^0)) \cong E({}_{D'}(P' - P'^0))$ for every $P \in I(J)$. By a classical result of ring theory [3, p. 79], we have that $D \cong D'$ and $\dim_D(P - P^0) = \dim_{D'}(P' - P'^0)$ for every $P \in I(J)$.

Conversely, let us assume that $D \overset{\sigma}{\to} D'$ and $J \overset{\alpha}{\to} J'$ are isomorphisms such that $\dim_D(P_i - P_i^0) = \dim_{D'}(\alpha P_i - \alpha P_i^0)$, $i = 1, \cdots, n$, where $I(J) = \{P_1, \cdots, P_n\}$. We may select $\bar{P}_i \in L({}_DM)$ and $\bar{P}_i' \in L({}_{D'}M')$ such that $P_i = \bar{P}_i \dot{\cup} P_i^0$ and $\alpha P_i = P_i' = \bar{P}_i' \dot{\cup} P_i'^0$, $i = 1, \cdots, n$. By assumption, there exist bases $\{x_{ij} \mid j \in \Delta_i\}$ of $\bar{P}_i$ and $\{x_{ij}' \mid j \in \Delta_i\}$ of $\bar{P}_i'$, $i = 1, \cdots, n$. Each element of M is uniquely represented as a finite sum of the form $\sum_{i,j} d_{ij}x_{ij}$, $d_{ij} \in D$, by 1.1. Thus, there is a unique mapping $M \overset{\theta}{\to} M'$ defined by: $(\sum d_{ij}x_{ij})\theta = \sum d_{ij}^{\sigma}x_{ij}'$. It is clear that θ is a 1-1 semilinear mapping of M onto M' that maps J onto J'. Hence the mapping $E({}_DM) \overset{\psi}{\to} E({}_{D'}M')$ defined by: $a\psi = \theta^{-1}a\theta$, $a \in E({}_DM)$, is an isomorphism [3, p. 45]. If $r \in R$, so that $P_ir \subset P_i$, $i = 1, \cdots, n$, then $P_i'(r\psi) = P_i'(\theta^{-1}r\theta) = P_i(r\theta) \subset P_i\theta \subset P_i'$, $i = 1, \cdots, n$, and $r\psi \in R'$. Conversely, if $r\psi \in R'$ then $P_i'(r\psi) = P_i(r\theta) \subset P_i'$, $P_ir \subset P_i'\theta^{-1} = P_i$, $i = 1, \cdots, n$, and $r \in R$. Thus, if we let $\phi = \psi \mid R$, the mapping $R \overset{\phi}{\to} R'$ is an isomorphism. This proves 7.1.

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