FURTHER GEOMETRIC CONSEQUENCES OF CONFORMAL STRUCTURE(1)

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1. Consider a pair of oriented surfaces S and $\hat{S}$ immersed smoothly in E^3. Suppose that some geometrically significant conformal structure is introduced on each of these surfaces, so that Riemann surfaces R and $\hat{R}$ are defined on S and $\hat{S}$ respectively. (By a geometrically significant conformal structure we mean one which requires for its definition some knowledge of at least one of the two fundamental forms which describe a surface's immersion in E^3.) It is then natural to seek geometric characterizations of cases in which standard differential geometric correspondences between S and $\hat{S}$ yield Teichmüller mappings between R and $\hat{R}$.

In recent papers [4; 6], we have offered such characterizations using two different methods for the determination of conformal structure(2). The first method is the familiar one. It uses the ordinary metric tensor I to impose the customary conformal structure on a surface. The second method applies only to surfaces on which mean and Gaussian curvatures $\mathscr{H}$ and $\mathscr{K}$ are positive. It uses the positive definite second fundamental form II to determine a nonstandard conformal structure on such surfaces.

In this paper we continue these investigations. Together with the procedures already described for obtaining R or $\hat{R}$, we consider a third method which applies only to surfaces on which $\mathscr{K} < 0$. (See [7].) The process uses the positive definite form II' defined by

$$\mathscr{H}'II' = \mathscr{K}I - \mathscr{H}II \tag{1}$$

where

$$\mathscr{H}' = -\sqrt{(\mathscr{H}^2 - \mathscr{K})}$$

to yield a nonstandard conformal structure on such surfaces.

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(2) Please note two corrections on p. 135 of [4]. On line 8, insert "corresponding" between "of" and "nonplanar." And on line 8 from the bottom, replace "removable" by "nonplanar." Note also that the third sentence on p. 135 forces S and $\hat{S}$ in Theorem 1 of [4] to be umbilic free unless f is conformal. Thus the corollary to Theorem 1 in [4] can be strengthened to read as follows. *The standard mapping f between compact parallel surface S and $\hat{S}$ of genus $g \neq 1$ is never a nonconformal Teichmüller mapping $f: R_1 \to \hat{R}_1$.*

The results which follow parallel closely those previously obtained. Lemmas and theorems below have been numbered so as to best indicate their relation to items in [4; 6]. Normally, this paper would begin with a list of all facts about Teichmüller mappings which are used in subsequent sections. However, §2 of [6] does this job adequately. Those interested in further descriptions of Teichmüller mappings, or of the role they play in the study of Riemann surfaces, should consult [1] or [2].

2. This section is devoted to a discussion of two lemmas. Let S be an oriented surface which is C^3 immersed in E^3. By R_1 we denote the usual Riemann surface determined on S by I. Conformal parameters $z = x + iy$ on R_1 correspond to isothermal coordinates x, y on S in terms of which I is given by

$$\lambda(x,y)(dx^2 + dy^2), \tag{2}$$

where λ is an arbitrary function. In case $\mathscr{K}, \mathscr{H} > 0$ on S, we denote by R_2 the Riemann surface determined on S by II. Conformal parameters $z = x + iy$ on R_2 correspond to bisothermal coordinates x, y on S in terms of which II is given by (2). In case $\mathscr{K} < 0$ on S, we denote by R_2' the Riemann surface determined on S by II', the form defined above in (1). Conformal parameters $z = x + iy$ on R_2' correspond to disothermal coordinates x, y on S in terms of which II' is given by (2). The existence of C^3 isothermal, and C^2 bisothermal or disothermal coordinates under the conditions given is assured. (See §4 of [2], for instance.)

In this paper we study mappings $f: S \to \hat{S}$ which yield Teichmüller mappings from R_2' to $\hat{R}_1$ or $\hat{R}_2$ or $\hat{R}_2'$. We ignore the cases in which f yields a Teichmüller mapping from R_1 or R_2 to $\hat{R}_2'$, because all results below may be easily reworded so as to cover these cases. This follows from the fact that the inverse of a Teichmüller mapping is itself a Teichmüller mapping with the same maximal dilatation as the original.

Wherever R_2 (or $\hat{R}_2$) structure is referred to, it is automatically assumed that $\mathscr{K}, \mathscr{H} > 0$ (or $\hat{\mathscr{K}}, \hat{\mathscr{H}} > 0$). Wherever R_2' (or $\hat{R}_2'$) structure is referred to it is automatically assumed that $\mathscr{K} < 0$ (or $\hat{\mathscr{K}} < 0$). By K we always denote the maximal dilatation of the Teichmüller mapping then under discussion. Lemma 1 is a direct consequence of remarks made toward the end of §2 in [6].

Lemma 1. *Suppose $f: R_2' \to \hat{R}_1$ or $f: R_2' \to \hat{R}_2$ or $f: R_2' \to \hat{R}_2'$ is a Teichmüller mapping. Then there are disothermal coordinates near all but (isolated) exceptional points on S in terms of which $\hat{I}$ or $\hat{I}\hat{I}$ or $\hat{I}\hat{I}'$ (respectively) is given by*

$$\gamma(x,y)(K^2 dx^2 + dy^2), \tag{3}$$

where γ is a positive function.

We will refer below to disothermal surfaces. Such surfaces are characterized by the existence of disothermal lines-of-curvature coordinates in the neighborhood of every point. The terms umbilic, removable umbilic, irremovable umbilic, line-of-curvature, net of lines of curvature, isothermal surface and bisothermal surface will be used as defined within §3 of [6]. Note, however, that a net of lines of curvature is assumed to be an orthogonal net except at irremovable umbilics, that is, except at actual singularities in the net. (Thus any orthogonal net of curves on a piece of sphere is a net of lines of curvature.)

LEMMA 2. *Let $f: S \to \hat{S}$ preserve a net of lines of curvature. If $f: R_2' \to \hat{R}_1$ or $f: R_2' \to \hat{R}_2$ or $f: R_2' \to \hat{R}_2'$ is a nonconformal Teichmüller mapping, then S is disothermal while $\hat{S}$ is isothermal, bisothermal or disothermal (respectively).*

Proof of Lemma 2. The directions of principal curvature on S correspond to $dy : dx$ values α_1 and α_2 which satisfy

$$-Mdx^2 + (L - N)dxdy + Mdy^2 = 0 \tag{4}$$

when x, y are isothermal, or

$$-Fdx^2 + (E - G)dxdy + Fdy^2 = 0 \tag{5}$$

when x, y are either bisothermal or disothermal. (See [5; 7].)

Suppose $f: R_2' \to \hat{R}_1$ is a nonconformal Teichmüller mapping. Then introducing the coordinates given by Lemma 1 near any nonexceptional point, we have

$$\begin{aligned} II' &= \lambda(dx^2 + dy^2), \\ \hat{I} &= \gamma(K^2dx^2 + dy^2), \end{aligned} \tag{6}$$

with $K > 1$. Lines of curvature on $\hat{S}$ correspond to $dy : dx$ values α_1 and α_2 which satisfy

$$-K^2\hat{M}dx^2 + (\hat{L} - K^2\hat{N})dxdy + \hat{M}dy^2 = 0, \tag{7}$$

since Kx, y are isothermal on $\hat{S}$. If either $dx \equiv 0$ or $dy \equiv 0$ corresponds to a direction of principal curvature, (5) and (7) yield $F = \hat{M} = 0$. But (1) and $F = M' = 0$ mean $M = 0$, making x, y disothermal lines-of-curvature coordinates on S. On the other hand $\hat{F} = \hat{M} = 0$, so that Kx, y are isothermal lines-of-curvature coordinates on $\hat{S}$. Finally, either $dx \equiv 0$ or $dy \equiv 0$ must correspond to a direction of principal curvature. Otherwise $F \neq 0$, and the usual formulas for angle measurement on S and $\hat{S}$ may be used to express the fact that the orthogonal principal directions on S correspond to orthogonal principal directions on $\hat{S}$. This yields

$$0 = E + F(\alpha_1 + \alpha_2) + G\alpha_1\alpha_2$$

and

$$0 = K^2 + \alpha_1\alpha_2.$$

Since α_1 and α_2 solve (5) divided by dx^2,

$$\alpha_1 + \alpha_2 = \frac{G - E}{F}.$$

Thus,

$$0 = 1 + \alpha_1 \alpha_2,$$

which is a contradiction, since $K > 1$.

Suppose now that $f: R_2' \to \hat{R}_2$ or $f: R_2' \to \hat{R}_2'$ is a nonconformal Teichmüller mapping. Then the coordinates given by Lemma 1 near any nonexceptional point yield

$$\text{(8)} \qquad \begin{aligned} II' &= \lambda(dx^2 + dy^2), \\ \hat{II} &= \gamma(K^2 dx^2 + dy^2), \end{aligned}$$

or

$$\text{(9)} \qquad \begin{aligned} II' &= \lambda(dx^2 + dy^2), \\ \hat{II}' &= \gamma(K^2 dx^2 + dy^2), \end{aligned}$$

respectively. In either case, directions of principal curvature on $\hat{S}$ correspond to $dy : dx$ values α_1 and α_2 which satisfy

$$\text{(10)} \qquad -K^2 \hat{F} dx^2 + (\hat{E} - K^2 \hat{G}) dx dy + \hat{F} dy^2 = 0,$$

since Kx, y are bisothermal or disothermal on $\hat{S}$. Once again, since α_1 and α_2 yield orthogonal directions on both S and $\hat{S}$, there is a contradiction unless $dx \equiv 0$ and $dy \equiv 0$ correspond to directions of principal curvature. But then, using (1), (5), (8), (9) and (10), we see that x, y are disothermal on S, while Kx, y are either bisothermal or disothermal on $\hat{S}$, as the case requires.

Note, finally, that in all three situations covered by Lemma 2, f can have no exceptional points. For $\mathscr{K} < 0$ makes S umbilic free. Thus an exceptional point p would be a regular point in the net of lines of curvature on S, giving p the index $j = 0$ in this net. However,

$$j = \frac{-m}{2},$$

where m is the order of the zero or minus the order of the pole at p of Ω, the defining quadratic differential of the nonconformal Teichmüller mapping f. This fact follows from previous arguments, which show the coincidence of the net of trajectories and orthogonal trajectories of Ω with the net of lines of curvature on S. It is helpful in this connection to rewrite (5) as we do in the course of proving Lemma 4 below. (See §3 of [5].)

3. Our theorems will deal with mappings between surfaces which preserve a net of lines of curvature. To maintain the order of presentation established in

[4; 6], we consider first the standard mapping f between parallel surfaces S and $\hat{S}$ which associates with each point p on S a point on $\hat{S}$ the fixed distance $t \neq 0$ from S along the unit normal to S at p. It is well known that f preserves lines of curvature. Moreover, if the orientation on $\hat{S}$ is the one induced upon it from S by f, then f preserves normals as well. Thus results in this section are corollaries of Lemma 5 and Theorem 2 which are proved below in §4. (The same comment applies to the corresponding results in [4; 6].) We consider first the conformal cases.

LEMMA 3. *Let $f: S \to \hat{S}$ be the standard mapping between parallel surfaces. If $f: R_2' \to \hat{R}_1$ is conformal, then S and $\hat{S}$ are Weingarten surfaces satisfying*

$$\mathscr{H} - 2t\mathscr{K} + t^2\mathscr{K}\mathscr{H} = 0 \tag{11}$$

and

$$\hat{\mathscr{H}} + t(2\hat{\mathscr{H}}^2 - \hat{\mathscr{K}}) = 0 \tag{12}$$

respectively. If $f: R_2' \to \hat{R}_2$ is conformal, then S and $\hat{S}$ are Weingarten surfaces satisfying

$$\mathscr{H} = \frac{1}{t} \tag{13}$$

and

$$\hat{\mathscr{H}} = \frac{-1}{t} \tag{14}$$

respectively. The mapping $f: R_2' \to \hat{R}_2'$ cannot be conformal.

Proof of Lemma 3. The two fundamental forms on $\hat{S}$ are given in terms of those on S by

$$\begin{aligned} \hat{I} &= I - 2tII + t^2 III, \\ \hat{II} &= II - tIII \end{aligned} \tag{15}$$

where

$$III = 2\mathscr{H}II - \mathscr{K}I. \tag{16}$$

(See p. 272 of [3].) It follows easily that

$$k_1 = \frac{\hat{k}_1}{1 + \hat{k}_1 t}, \quad k_2 = \frac{\hat{k}_2}{1 + \hat{k}_2 t}, \tag{17}$$

while

$$\hat{k}_1 = \frac{k_1}{1 - k_1 t}, \quad \hat{k}_2 = \frac{k_2}{1 - k_2 t}. \tag{18}$$

If $f: R_2' \to \hat{R}_1$ is conformal, (17), (18) and (34) of Lemma 5 yield (11) on S and (12) on $\hat{S}$. If $f: R_2' \to \hat{R}_2$ is conformal, (17), (18) and (35) of Lemma 5 yield (13) on S and (14) on $\hat{S}$. If $f: R_2' \to \hat{R}_2'$ were conformal, (18) and (36) of Lemma 5 would yield $k_1 = k_2$ which contradicts $\mathscr{K} < 0$. We need only check the theorem which follows in the nonconformal cases.

THEOREM 1. *Let $f: S \to \hat{S}$ be the standard mapping between parallel surfaces. If $f: R_2' \to \hat{R}_1$ is a Teichmüller mapping, then S and $\hat{S}$ are Weingarten surfaces satisfying*

$$(k_2 + K^2 k_1) - 2t(1 + K^2)\mathscr{K} + t^2\mathscr{K}(k_1 + K^2 k_2) = 0 \tag{19}$$

and

$$(\hat{k}_2 + K^2\hat{k}_1) - t(1 + K^2)\hat{\mathscr{K}} + 2t\hat{\mathscr{H}}(\hat{k}_2 + K^2\hat{k}_1) = 0 \tag{20}$$

respectively. If $f: R_2' \to \hat{R}_2$ is a Teichmüller mapping, then S and $\hat{S}$ are Weingarten surfaces satisfying

$$(1 + K^2) = t(k_1 + K^2 k_2) \tag{21}$$

and

$$(1 + K^2) = -t(\hat{k}_2 + K^2\hat{k}_1) \tag{22}$$

respectively. If $f: R_2' \to \hat{R}_2'$ is a Teichmüller mapping, then $K > 1$ while S and $\hat{S}$ are Weingarten surfaces satisfying

$$(1 - tk_1) = K^2(1 - tk_2) \tag{23}$$

and

$$(1 + t\hat{k}_2) = K^2(1 + t\hat{k}_1) \tag{24}$$

respectively.

Proof of Theorem 1. If $f: R_2' \to \hat{R}_1$ is a Teichmüller mapping with $K > 1$, then (17), (18) and (44) of Theorem 2 yield (19) on S and (20) on $\hat{S}$. If $f: R_2' \to \hat{R}_2$ is a Teichmüller mapping with $K > 1$, then (17), (18) and (45) of Theorem 2 yield (21) on S and (22) on $\hat{S}$. If $f: R_2' \to \hat{R}_2'$ is a Teichmüller mapping with $K > 1$, then (17), (18) and (46) of Theorem 2 yield (23) on S and (24) on $\hat{S}$. Of course, (19), (20), (21) and (22) reduce to (11), (12), (13) and (14) respectively when $K = 1$.

4. In discussing mappings which preserve both normals and a net of lines of curvature, it will be helpful to have the following definitions. Let f be a homeomorphism from S onto $\hat{S}$. Let $\hat{\Sigma} \subset S$ be the set of all preimages under f of zeros of $\hat{\mathscr{H}}$. Let $\Sigma \subset S$ be the set of zeros of $\mathscr{H}$. Let $\Sigma' = \Sigma \cap \hat{\Sigma}$. Finally, let S' be the complement of Σ' on S.

LEMMA 4. *Let $f: S \to \hat{S}$ preserve normals. If $f: R_2' \to \hat{R}_1$ is conformal, then f preserves lines of curvature on S'. If $f: R_2' \to \hat{R}_2$ or $f: R_2' \to \hat{R}_2'$ is conformal, then f preserves lines of curvature on S.*

The elementary notation used to prove Lemma 4 in [6] could be employed here too. We prefer, however, the abbreviations afforded by §4 in [7]. For this purpose, we recall the following definitions. Let R be an arbitrary Riemann surface defined on S. Let $z = x + iy$ be a conformal parameter on R. For an arbitrary quadratic form $\Omega = Adx^2 + 2Bdxdy + Cdy^2$ on S, consider the associated quadratic differential

$$\left\{\frac{A - C}{2} - iB\right\} dz^2$$

on R. The quadratic differentials associated with I, II and III are called Ω_1, Ω_2 and Ω_3 respectively. Thus $\Omega_1 = 0$ means $R = R_1$, while $\Omega_2 = 0$ means that $R = R_2$ or that $\mathscr{K} = \mathscr{H} = 0$. By (4) and (5), lines of curvature on R_1 correspond to solutions of

$$\operatorname{Im} \Omega_2 > 0, \tag{25}$$

while those on R_2 or R_2' correspond to solutions of

$$\operatorname{Im} \Omega_1 > 0. \tag{26}$$

The following statement is a conveniently weakened version of Lemma 6 in [7].

REMARK. Wherever $R = R_1$,

$$\Omega_3 = 2\mathscr{H}\Omega_2. \tag{27}$$

Wherever $R = R_2$,

$$\Omega_3 = -\mathscr{K}\Omega_1. \tag{28}$$

Wherever $R = R_2'$,

$$\Omega_3 = \mathscr{K}\Omega_1 = \mathscr{H}\Omega_2. \tag{29}$$

Proof of Lemma 4. Suppose $f: R_2' \to \hat{R}_1$ is conformal and preserves normals. Then we use (29) and (27) to express the fact that under f, $R = R_2' = \hat{R}_1$ while $\Omega_3 = \hat{\Omega}_3$. This yields

$$\mathscr{K}\Omega_1 = 2\hat{\mathscr{H}}\hat{\Omega}_2. \tag{30}$$

Lines of curvature on S correspond to solutions of (26), those on $\hat{S}$ to solutions of (25), suitably hatted. Since $\hat{\mathscr{H}}$ and $\mathscr{K} < 0$ are real valued, f preserves directions of principal curvature except perhaps on $\hat{\Sigma}$. But on $\hat{\Sigma}$, (30) yields $\Omega_1 = 0$, so that by (29), $\mathscr{H} = 0$. (The alternative $\Omega_2 = 0$ is impossible, since $\mathscr{K} < 0$ while R_2 and R_2' never coincide.) Thus $\Sigma' = \hat{\Sigma} \subset \Sigma$, and f preserves lines of curvature on S'.

Similarly, if $f: R_2' \to \hat{R}_2$ is conformal and preserves normals, then (29) and (28) yield

$$\mathscr{K}\Omega_1 = -\hat{\mathscr{K}}\hat{\Omega}_1. \tag{31}$$

Lines of curvature on S correspond again to solutions of (26), those on $\hat{S}$ to solutions of (26), suitably hatted. Here, since $\mathscr{K} < 0$ and $\hat{\mathscr{K}} > 0$ are real, f preserves lines of curvature on all of S.

Suppose finally that $f: R_2' \to \hat{R}_2'$ is conformal and preserves normals. By (29),

$$\mathscr{K}\Omega_1 = \hat{\mathscr{K}}\hat{\Omega}_1 \tag{32}$$

with both $\mathscr{K} < 0$ and $\hat{\mathscr{K}} < 0$ real. Lines of curvature on S and $\hat{S}$ are determined as in the previous case. Thus, once again, f preserves lines of curvature everywhere.

The method just employed yields the following statement which has its proper place among the results in [4].

LEMMA 4′. *Let $f: S \to \hat{S}$ preserve normals. If $f: R_1 \to \hat{R}_1$ is conformal, f preserves a net of lines of curvature on S'.*

Proof of Lemma 4′. In this case, (27) yields

$$\mathscr{H}\Omega_2 = \hat{\mathscr{H}}\hat{\Omega}_2. \tag{33}$$

Lines of curvature on S and $\hat{S}$ correspond to solutions of (25), suitably hatted for $\hat{S}$. Since $\mathscr{H}$ and $\hat{\mathscr{H}}$ are real, f preserves lines of curvature except, perhaps, on $\hat{\Sigma} \cup \Sigma$. But now consider Σ_1, the complement of $\hat{\Sigma}$ in $\Sigma \cup \hat{\Sigma}$. For $p \in \Sigma_1, \mathscr{H} = 0$ at p while $\hat{\mathscr{H}} \neq 0$ at $f(p)$. By (33), $\hat{\Omega}_2 = 0$ at $f(p)$, so that $\hat{R}_1$ and $\hat{R}_2$ must coincide there. Thus $f(p)$ is an umiblic on $\hat{S}$. It follows that the conformal mapping $f: R_1 \to \hat{R}_1$ will carry any net of lines of curvature on Σ_1 to a net of lines of curvature on $f(\Sigma_1)$. On the other hand, consider Σ_2, the complement of Σ in $\Sigma \cup \hat{\Sigma}$. For $p \in \Sigma_2, \mathscr{H} \neq 0$ at p, while $\hat{\mathscr{H}} = 0$ at $f(p)$. Here (33) forces R_1 and R_2 to coincide at p, making p an umbilic on S. On Σ_2, f will preserve only those lines of curvature which are the preimages of lines of curvature on $f(\Sigma_2)$. (Picture here the case in which Σ_2 is a piece of sphere, and $f(\Sigma_2)$ a piece of nonplanar minimal surface.) In any case, f will preserve on S' the preimage of any net of any net of lines of curvature which exists on $f(S')$. This restriction to S' here and in the first case covered by Lemma 4, may be explained as follows.

REMARK. If we use ordinary conformal structure on the unit sphere oriented by its inner normal, then the spherical image mapping of an umbilic free minimal surface is conformal either from R_1 or from R_2'. (See Theorem 3 below.) Thus an f obtained by composing the spherical image mapping of an umbilic free minimal surface S with the inverse of the spherical image mapping of an umbilic free minimal surface $\hat{S}$ will be conformal from R_1 or R_2' to $\hat{R}_1$ or $\hat{R}_2'$, and will preserve nornals. But only in rare instances will such an f preserve lines of curvature.

LEMMA 5. *Let $f: S \to \hat{S}$ preserve normals. If $f: R_2' \to \hat{R}_1$ is conformal, then*

$$k_1\hat{k}_2^2 = -k_2\hat{k}_1^2 \tag{34}$$

holds at points in correspondence under f. If $f: R_2' \to \hat{R}_2$ is conformal, then

$$k_1\hat{k}_2 = -k_2\hat{k}_1 \tag{35}$$

holds at points in correspondence under f. If $f: R_2' \to \hat{R}_2'$ is conformal, then

$$k_1\hat{k}_2 = k_2\hat{k}_1 \tag{36}$$

holds at points in correspondence under f.

Proof of Lemma 5. Suppose $f: R_2' \to \hat{R}_1$ is conformal. By Lemma 1 of [7], any choice of lines-of-curvature coordinates near a point of S yields

$$\begin{aligned} I &= Edx^2 + Gdy^2, \\ II &= k_1Edx^2 + k_2Gdy^2, \\ II' &= k_1Edx^2 - k_2Gdy^2. \end{aligned} \tag{37}$$

Here $II' \propto I$, so that, by Lemma 4, we have

$$\begin{aligned} \hat{I} &= \gamma(k_1Edx^2 - k_2Gdy^2), \\ \hat{II} &= \gamma(\hat{k}_1k_1Edx^2 - \hat{k}_2k_2Gdy^2) \end{aligned} \tag{38}$$

on S'. Using (16) to express the preservation of normals, we have

$$2\mathscr{H}II - \mathscr{K}I = 2\hat{\mathscr{H}}\hat{II} - \hat{\mathscr{K}}\hat{I}. \tag{39}$$

Equating coefficients, using (37) and (39), yields

$$k_1 = \gamma\hat{k}_1^2,$$

$$k_2 = -\gamma\hat{k}_2^2$$

on S', or (34). But on Σ', $k_1 = -k_2$, while $\hat{k}_1 = -\hat{k}_2$ on $f(\Sigma')$. Thus (34) holds everywhere on S.

Suppose now that $f: R_2' \to \hat{R}_2$ is conformal. Near any point on S, choose coordinates in terms of which (37) holds. Here $II' \propto \hat{II}$, so that Lemma 4 yields

$$\begin{aligned} \hat{I} &= \gamma\left(\frac{k_1}{\hat{k}_1}Edx^2 - \frac{k_2}{\hat{k}_2}Gdy^2\right), \\ \hat{II} &= \gamma(k_1Edx^2 - k_2Gdy^2). \end{aligned} \tag{40}$$

Using this in (39), and equating coefficients, we have

$$k_1 = \gamma \hat{k}_1, \qquad k_2 = -\gamma \hat{k}_2, \tag{41}$$

or (35).

Suppose, finally, that $f: R_2' \to \hat{R}_2'$ is conformal. As before, choose coordinates near any point on S in terms of which (37) holds. Here $II' \propto \hat{II}'$, so that by Lemma 1 of [7], and Lemma 4 above,

$$\hat{I} = \gamma\left(\frac{k_1}{\hat{k}_1}Edx^2 + \frac{k_2}{\hat{k}_2}Gdy^2\right), \qquad \hat{II} = \gamma(k_1Edx^2 + k_2Gdy^2). \tag{42}$$

Using this in (39), and equating coefficients, we have

$$k_1 = \gamma \hat{k}_1, \qquad k_2 = \gamma \hat{k}_2, \tag{43}$$

or (36).

Note now that if we place the values (41) in $\hat{I}$ of (40), we have $\hat{I} = \gamma^2 I$, making $f: R_1 \to \hat{R}_1$ conformal. Similarly, placing the values (43) in $\hat{I}$ of (42), we have $\hat{I} = \gamma^2 I$, making $f: R_1 \to \hat{R}_1$ conformal. The following statement is easily checked. (See Theorem 2 of [4].)

Remark. Let $f: S \to \hat{S}$ preserve normals, and $\mathscr{K} < 0$ on S. If $\hat{\mathscr{K}} > 0$ on $\hat{S}$, $f: R_1 \to \hat{R}_1$ is conformal iff $f: R_2' \to \hat{R}_2$ is conformal. If $\hat{\mathscr{K}} < 0$ on $\hat{S}$, $f: R_1 \to \hat{R}_1$ is conformal iff $f: R_2' \to R_2'$ is conformal.

We turn now to Teichmüller mappings which preserve normals and lines of curvature. We obtain, just as in Lemma 5, joint Weingarten conditions $W(k_1, k_2; \hat{k}_1, \hat{k}_2) = 0$ relating the principal curvatures at points of S and $\hat{S}$ in correspondence under f. Only the nonconformal cases need to be checked in the theorem which follows.

Theorem 2. *Let f preserve normals and lines of curvature. If $f: R_2' \to \hat{R}_1$ is a Teichmüller mapping, then*

$$k_1\hat{k}_2^2 = -K^2k_2\hat{k}_1^2 \tag{44}$$

at points in correspondence under f. If $f: R_2' \to \hat{R}_2$ is a Teichmüller mapping, then

$$k_1\hat{k}_2 = -K^2k_2\hat{k}_1 \tag{45}$$

holds at points in correspondence under f. If $f: R_2' \to \hat{R}_2'$ is a Teichmüller mapping, then

(46 $$k_1\hat{k}_2 = K^2k_2\hat{k}_1$$

holds at points in correspondence under f.

Proof of Theorem 2. Assuming $K > 1$, we may choose near any point on S the coordinates described by Lemma 1 and Lemma 2 which are relevant to the case under consideration. In all three situations covered by Theorem 2, we have

$$\begin{aligned} I &= \lambda\left(\frac{dx^2}{k_1} - \frac{dy^2}{k_2}\right), \\ II &= \lambda(dx^2 - dy^2), \\ II' &= \lambda(dx^2 + dy^2), \end{aligned} \tag{47}$$

since x, y are lines-of-curvature coordinates on S. If $f: R_2' \to \hat{R}_1$ is a Teichmüller mapping with $K > 1$, then

$$\begin{aligned} \hat{I} &= \gamma(K^2 dx^2 + dy^2), \\ \hat{II} &= \gamma(K^2 \hat{k}_1 dx^2 + \hat{k}_2 dy^2). \end{aligned}$$

Using these values in (39) and equating coefficients, we have

$$\begin{aligned} \lambda k_1 &= \gamma K^2 \hat{k}_1^2, \\ \lambda k_2 &= -\gamma \hat{k}_2^2, \end{aligned}$$

or (44). If $f: R_2' \to \hat{R}_1$ is a Teichmüller mapping with $K > 1$, then

$$\begin{aligned} \hat{I} &= \gamma\left(\frac{K^2}{\hat{k}_1} dx^2 + \frac{1}{\hat{k}_2} dy^2\right), \\ \hat{II} &= \gamma(K^2 dx^2 + dy^2). \end{aligned}$$

Equating coefficients in (39) yields

$$\begin{aligned} \lambda k_1 &= \gamma K^2 \hat{k}_1, \\ \lambda k_2 &= -\gamma \hat{k}_2, \end{aligned}$$

or (45). If $f: R_2' \to \hat{R}_2'$ is a Teichmüller mapping with $K > 1$, then

$$\begin{aligned} \hat{I} &= \gamma\left(\frac{K^2}{\hat{k}_1} dx^2 - \frac{1}{\hat{k}_2} dy^2\right), \\ \hat{II} &= \gamma(K^2 dx^2 - dy^2), \\ \hat{II}' &= \gamma(K^2 dx^2 + dy^2). \end{aligned}$$

Here, (39) yields

$$\begin{aligned} \lambda k_1 &= \gamma K^2 \hat{k}_1, \\ \lambda k_2 &= \gamma \hat{k}_2, \end{aligned}$$

or (46).

5. The spherical image mapping of a surface into the unit sphere and the identity mapping of a surface onto itself are particular examples of the kind of mapping discussed in §4. All comments below are therefore corollaries of Lemma 5 and Theorem 2.

THEOREM 3. *Let f be the spherical image mapping of S onto $\hat{S}$, part of the unit sphere. If $f: R_2' \to \hat{R}_1$ is a Teichmüller mapping, then S is both isothermal and disothermal, while*

$$k_1 = -K^2 k_2. \tag{48}$$

Proof of Theorem 3. When $K = 1$, use Lemma 5, $\hat{k}_1 = \hat{k}_2$, and the fact that a minimal surface is both isothermal and disothermal where $\mathscr{K} < 0$. (Indeed, $R_1 = R_2'$ wherever $\mathscr{H} = 0$ and $\mathscr{K} < 0$.) When $K \neq 1$, use Lemma 2 and Theorem 2. To show that S is isothermal, substitute (48) in (47), and use coordinates x, Ky on S.

THEOREM 4. *If the identity mapping $f: S \to S$ is a Teichmüller mapping $f: R_2' \to R_1$, then S is both isothermal and disothermal, while*

$$k_2 = -K^2 k_1.$$

Proof of Theorem 4. Use Lemma 2, Lemma 5, Theorem 2, and the comment just made about minimal surfaces.

REMARK. Let $\mathscr{K} < 0$ on S. Recalling Theorem 3 of [4], we see that the following statements are equivalent.

1. The spherical image mapping of S is a Teichmüller mapping from R_1.
2. The spherical image mapping of S is a Teichmüller mapping R_2'.
3. The identity mapping of S is a Teichmüller mapping from R_1 to R_2'.

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