SPACES OF VECTOR VALUED REAL ANALYTIC FUNCTIONS

BY
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In this paper we study the spaces of weakly and strongly analytic functions with values in a locally convex topological vector space F and we look for conditions on F such that these two spaces (which are different in general) should coincide.

In the case of vector valued C^∞ functions and of vector valued holomorphic functions, Grothendieck proved (cf. [4; 7]) that it suffices to assume F complete (even less, quasi-complete, i.e., closed bounded sets are complete) to conclude that the two notions of weakly C^∞ (resp. weakly holomorphic) functions and strongly C^∞ (resp. holomorphic) functions coincide.

As we show with an example (cf. §2) the sole condition of completeness of F does not imply strong analyticity from weak analyticity. On the other hand, it is known that if F is a Banach space then the two notions of real analyticity are the same [3]. For these reasons it is natural to raise the question of finding less restrictive conditions on F such that this occurs.

The problem presents two aspects, one concerning the algebraic identification of the two spaces of analytic functions and the other the identification both in the algebraic and topological senses when these spaces are equipped with natural topologies. In order to deal with the algebraic case, we introduce the definition of quasi-$(\mathscr{D}\mathscr{F})$ spaces (cf. §2, Definition 2) which generalizes the notion of $(\mathscr{D}\mathscr{F})$ spaces introduced by Grothendieck in [5]. A quasi-$(\mathscr{D}\mathscr{F})$ space still has one of the important properties of $(\mathscr{D}\mathscr{F})$ spaces, namely, its strong dual is a $(\mathscr{F})$ space, a a property we use in an essential way to prove Theorem 1.

The identification in the algebraic and topological senses can be proven when F is a $(\mathscr{D}\mathscr{F})$ space. The proof of this result was suggested by a similar result of Grothendieck (cf. [6, Chapter II, p. 82, §4]) concerning spaces of vector valued distributions. However, new difficulties arise from the fact that the topology of the space of strongly analytic functions is a generalized rather than a strict inductive limit topology (cf. [1, §II]), i.e., defined by an increasing sequence of spaces each of which induces in the previous one a topology weaker than the given one.

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§1 is devoted to definitions, notations and terminology. We begin §2 by defining weakly and strongly real analytic functions with values in a complete space and we give an example of a complete space in which the two notions are different. We present two lemmas stating, respectively, criteria for strong analyticity and for weak analyticity which we use throughout this paper. Next we define quasi-$(\mathscr{D}\mathscr{F})$ spaces and prove Theorem 1 which states that the two spaces of analytic functions coincide when F is a quasi-$(\mathscr{D}\mathscr{F})$ space.

In §3 we discuss the problem of identification in the algebraic and topological senses of the two spaces of analytic functions, using results of Grothendieck's theory of topological tensor products.

In §4 we apply the results of the previous sections to study weak and strong analyticity of families of operators depending upon a real parameter. We prove that if E is a barrelled space, F a complete quasi-$(\mathscr{D}\mathscr{F})$ space and (T_t) a family of continuous linear maps from E into F depending upon t, then the various notions of analyticity we can define for the family (T_t) (cf. §4, Definition 4) are the same. This question, as well as this paper, was motivated by the reading of a recent paper of Browder [3].

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1. **Notation and terminology.** In what follows the topological vector spaces that we consider are always locally convex and separated over the field of complex numbers. If E is a such space, we denote by E' its dual and by $s(E,E')$ (resp. $s(E',E)$) the weak topology on E (resp. E'), given by the natural pairing between E and E'.

If E and F are two topological vector spaces, we denote by $L(E,F)$ the space of all continuous linear maps from E into F. If $\mathfrak{S}$ is a set of bounded subsets of E, we denote by $L_{\mathfrak{S}}(E,F)$ the space $L(E,F)$ endowed with the topology of the uniform convergence on the sets of $\mathfrak{S}$ (also called the topology of $\mathfrak{S}$-convergence on $L(E,F)$). When $\mathfrak{S}$ is the set of all finite subsets of E we obtain, as it is well known, the topology of simple convergence and the space $L(E,F)$ endowed with this topology is denoted by $L_s(E,F)$. When $\mathscr{G}$ is the set of all bounded sets of E the topology obtained is that of the uniform convergence on bounded sets of E and is denoted by $L_b(E,F)$. In particular, E'_s (resp. E'_b) will be the dual of E with the weak topology (resp. strong) topology.

A topological vector space is called barrelled if every weakly bounded set of its dual is equicontinuous. An equivalent definition is the following: E is barrelled if for any topological vector space F, any subset M of $L(E,F)$ bounded in the topology of simple convergence is equicontinuous.

In [1], we discussed the space $\mathscr{A}(K)$ whose definition and most important properties we recall here. Let K be a compact subset of R (for convenience of notation we shall consider throughout this paper functions of one variable, the passage to the case of n variables being straightforward) and denote by $\mathscr{A}(K)$ the space of classes of holomorphic functions defined in open sets of C containing K, two functions being identified if they coincide in an open neighborhood of K in C. If Ω is an open set of C, denote by $\mathscr{H}(\Omega)$ the space of complex analytic functions on Ω endowed with the topology of uniform convergence on compact sets of Ω. On $\mathscr{A}(K)$ we define the inductive limit topology of the spaces $\mathscr{H}(\Omega)$. With this topology $\mathscr{A}(K)$ is a generalized $(\mathscr{L}\mathscr{F})$ space, which is a Montel, barrelled, complete, bornological space. If M is a bounded set of $\mathscr{A}(K)$ then the functions of M can be extended as complex analytic functions to a suitable open neighborhood Ω of K in C and the set of the extended functions is bounded in $\mathscr{H}(\Omega)$ (cf. [1, §II]). Furthermore, $\mathscr{A}(K)$ is a $(\mathscr{D}\mathscr{F})$ space (cf. [5, p. 80, example (b)]).

2. **Analyticity and weak analyticity.**

DEFINITION 1. *Let U be an open set of R, F a complete topological vector space and ϕ a mapfrom U into F. We shall say that:*

(i) *ϕ is weakly analytic on U if, for each $t_0 \in U$ and $f' \in F'$, there exists an $\varepsilon = \varepsilon(t_0, f') > 0$, such that*

$$\langle \phi(t), f' \rangle = \sum a_p(f')(t - t_0)^p, \tag{1}$$

the series being convergent for $|t - t_0| < \varepsilon$;

(ii) *ϕ is strongly analytic or analytic on U if, for each $t_0 \in U$, there exists an $\varepsilon = \varepsilon(t_0) > 0$, such that*

$$\phi(t) = \sum a_p(t - t_0)^p \tag{2}$$

where a_p are elements of F, the series converging in F for $|t - t_0| < \varepsilon$.

Obviously (ii) implies (i). Also if ϕ is analytic in one of the above senses then ϕ is C^∞ in the corresponding sense; furthermore one can easily check that

$$a_p(f') = (1/p!)\langle \phi^{(p)}(t_0), f' \rangle, \tag{3}$$

$$a_p = (1/p!)\phi^{(p)}(t_0). \tag{4}$$

Grothendieck has shown (cf. [4; 8]) that for vector valued functions defined in an open U with values in a complete space F the notions of weak C^∞ differentiability (i.e., for each $f' \in F'$ the numerical function $\phi_{f'}(t) = \langle \phi(t), f' \rangle$ is C^∞) and of C^∞ differentiability coincide. We are going to investigate under what conditions on F it is possible to conclude that the two above definitions coincide. The following example shows that, as opposed to what happens in the differentiable case, the assumption that F is complete is not enough to derive analyticity from the weak analyticity. Consider the space $\mathscr{A}(K)$. Its strong dual $(\mathscr{A}(K))'$ is Frechet space, since it is the dual of a $(\mathscr{D}\mathscr{F})$ space (cf. the remark following Definition 2). Consider now the following map $\delta : t \in K \to \delta_t \in (\mathscr{A}(K))'$, where δ_t is defined by $\delta_t(g) = g(t)$, for each $g \in \mathscr{A}(K)$. It is clear that δ is weakly analytic but not analytic.

The following two lemmas give us a criterion of analyticity and of weak analyticity, respectively. Lemma 2 is the analogue in the analytic case of a similar one for the differentiable case due to Grothendieck (cf. [4, p. 233, Proposition 14]). We present the proofs here for the convenience of the reader.

LEMMA 1. *Let F be a complete space and $\phi : U \to F$. Then ϕ is analytic at $t_0 \in U$ if and only if ϕ is C^∞ and there exists a real number $r > 0$ such that the set*

$$A = \{(1/p!)\,\phi^{(p)}(t_0)\,r^p,\ p = 0,1,2,\cdots\}$$

is bounded in F.

Proof. The necessity is an immediate consequence of the definition of analyticity. Conversely, denote by B the closed, circled convex hull of the set A and let F_B be the subspace of F spanned by B. Define the norm

$$\| u \| = \operatorname{Inf}\{\lambda > 0 : u \in \lambda B\}.$$

It is well known (cf. [4, p. 190, Lemma 1]) that F_B is a Banach space and that the induced topology of F is coarser than the topology given by the norm.

We have now for all $t \in U$ such that $|t - t_0| < r$:

$$(1/p!)\,\phi^{(p)}(t_0)(t - t_0)^p = (1/p!)\,\phi^{(p)}(t_0)\,r^p((t - t_0)/r)^p \in ((t - t_0)/r)^p B.$$

It follows from the definition of the norm in F_B that

$$\|(1/p!)\,\phi^{(p)}(t_0)(t - t_0)^p\| \leqq (|t - t_0|/r)^p$$

and this implies that the series $\sum(1/p!)\,\phi^{(p)}(t_0)(t - t_0)^p$ converges in F_B, hence n F, for $|t - t_0| < r$, q.e.d.

LEMMA 2. *Let U be an open set of R and F a complete space. The following conditions are equivalent*:

(i) *$\phi : U \to F$ is weakly analytic*;

(ii) *for each compact subset K of U, the map $\Phi : F' \to \mathscr{A}(K)$ defined by*

$$\Phi(f')(t) = \langle \phi(t), f' \rangle \tag{5}$$

transforms equicontinuous sets of F' into bounded sets of $\mathscr{A}(K)$.

Proof. Condition (ii) implies condition (i), trivially, so all we have to prove is that (i) implies (ii). Let A be a weakly closed equicontinuous set of F' and define F'_A as the subspace of F' spanned by A with norm

$$\|f'\|_A = \operatorname{Inf}\{\lambda > 0 : f' \in \lambda A\}.$$

With this norm F'_A is a Banach space, for our hypothesis on A implies that A is weakly compact, hence complete (cf. [4, p. 190, Lemma 1]). The map Φ is obviously continuous from F'_A into $\mathscr{A}(K)$, where $\mathscr{A}(K)$ is endowed with the topology of pointwise convergence. By applying the closed graph theorem as proven by Grothendieck (cf. [4, p. 271, Theorem 2]), which states that a linear map from a Frechet space into a $(\mathscr{L}\mathscr{F})$ space is continuous if and only if its graph is closed, we conclude that Φ is continuous from F'_A into $\mathscr{A}(K)$ with its inductive limit topology. Consequently, $\Phi(A)$ is a bounded set in the natural topology of $\mathscr{A}(K)$, q.e.d.

COROLLARY. *Suppose F is a complete space and let $\phi: U \to F$ be a weakly analytic function. Then Φ is continuous with respect to the weak topology $s(F',F)$ of F' and the weak topology of $\mathscr{A}(K)$.*

Proof. By Lemma 2, Φ maps equicontinuous sets of F' into relatively compact sets of $\mathscr{A}(K)$, since $\mathscr{A}(K)$ is a Montel space. As one can see, restricted to each equicontinuous set A of F', Φ is continuous from A in the topology induced by $s(F',F)$ into $\mathscr{A}(K)$ endowed with the pointwise convergence topology. It follows that the restriction of Φ to A is continuous with respect to the topology $s(F',F)$ and the weak topology $s(\mathscr{A}(K), (\mathscr{A}(K))')$, because, $\Phi(A)$ being weakly relatively compact, on $\Phi(A)$ the pointwise convergence topology and the weak topology $s(\mathscr{A}(K), (\mathscr{A}(K))')$ coincide. It follows that for each continuous linear functional T on $\mathscr{A}(K)$, the linear functional $T \circ \Phi$ on F' restricted to equicontinuous sets A of F' is continuous with respect to $s(F',F)$. Now, F being complete, we apply Grothendieck's criterion for completeness of locally convex topological vector spaces, which states that F is complete if and only if each linear functional on F' whose restrictions to equicontinuous sets of F' are continuous in $s(F',F)$ is continuous in F' endowed with $s(F',F)$ (cf. [4, p. 129, Corollary 2]). Hence, $T \circ \Phi$ is $s(F',F)$-continuous in F' for each T in the dual of $\mathscr{A}(K)$, or, equivalently, that Φ is continuous from F' endowed with the weak topology $s(F',F)$ into $\mathscr{A}(K)$ with the weak topology $s(\mathscr{A}(K), (\mathscr{A}(K))')$, q.e.d.

Lemma 2 and its corollary state that the space of weakly analytic functions defined on a compact subset K of R with values in a complete space F has a natural identification with the space of all linear maps from F' into $\mathscr{A}(K)$ continuous in the topology $s(F',F)$ of F' and the weak topology of $\mathscr{A}(K)$ and transforming equicontinuous sets of F' into relatively compact sets of $\mathscr{A}(K)$. It has been shown (cf. [9, Exposé 8, p. 4]) that this last space can be identified with the space $L(F'_c, \mathscr{A}(K))$ of continuous linear maps from F'_c into $\mathscr{A}(K)$, where F'_c denotes F' with the topology of uniform convergence on compact subsets of F.

Now, denote by $\mathscr{A}(K,F)$ the space of vector valued real analytic functions defined on K with values in F. The problem we want to consider is to find conditions on F such that the two spaces $\mathscr{A}(K,F)$ and $L(F'_c, \mathscr{A}(K))$ coincide.

We can solve it in the case when the strong dual F'_b of F is a Frechet space (i.e., metrisable and complete). In [5], Grothendieck introduced and studied extensively the so-called $(\mathscr{D}\mathscr{F})$ spaces (see Definition 3, below). Among their properties, let us mention that strong duals of $(\mathscr{D}\mathscr{F})$ spaces are $(\mathscr{F})$ spaces. We are going to define here another class of spaces that we call quasi-$(\mathscr{D}\mathscr{F})$ spaces, for which the strong dual is still a $(\mathscr{F})$ space.

DEFINITION 2. *A topological vector space F is called quasi-$(\mathscr{D}\mathscr{F})$ if it verifies the two conditions*:

(1) *there exists a fundamental sequence of bounded sets in F (i.e., any given bounded set is contained in some bounded set of the sequence)*;

(2) *if (f_i') is a bounded sequence in F_b' then (f_i') is an equicontinuous set of F'.*

A $(\mathscr{D}\mathscr{F})$ space (cf. Definition 3) is quasi-$(\mathscr{D}\mathscr{F})$ and it is reasonable to expect that the class of quasi-$(\mathscr{D}\mathscr{F})$ contains as a proper subset the class of $(\mathscr{D}\mathscr{F})$ spaces. However, we do not know an example of a quasi-$(\mathscr{D}\mathscr{F})$ space which is not $(\mathscr{D}\mathscr{F})$.

It follows from Definition 2, that the strong dual of F is a Frechet space. In fact, using condition (1), we can deduce that F_b' is metrisable just by taking polars of the sets belonging to the fundamental sequence. Consider, next, a Cauchy sequence (f_i') in F_b'. Since (f_i') is bounded in F_b' it follows by (2) that this sequence is an equicontinuous set of F'. On the other hand, (f_i') converges weakly to a linear map f' on F' which has to be continuous since it is an element of the weak closure of an equicontinuous set. Finally, (f_i') being a Cauchy sequence in F_b' and converging weakly to f', converges strongly to f' and this proves that F_b' is complete.

THEOREM 1. *Let U be an open set of R and F be a complete quasi-$(\mathscr{D}\mathscr{F})$ space. If ϕ defined on U with values in F is weakly analytic, then ϕ is analytic.*

Proof. 1. By the corollary of Lemma 2, we know that for each compact subset K of U, the map

$$\Phi : f' \in F' \to \phi_{f'}(t) = \langle \phi(t), f' \rangle \in \mathscr{A}(K)$$

is continuous with respect to the topology $s(F', F)$ of F' and the weak topology of $\mathscr{A}(K)$. Since F_b' is a Frechet space we can apply the closed graph theorem (cf. [4, p. 271, Theorem 2]) and conclude that Φ is continuous from F_b' into $\mathscr{A}(K)$.

2. As we remarked before, $\mathscr{A}(K)$ is a $(\mathscr{D}\mathscr{F})$ space, hence its strong dual is a $(\mathscr{F})$ space. If we consider the bilinear functional $B(f', T)$ defined on $F_b' \times (\mathscr{A}(K))'$ by

$$B(f', T) = \langle \Phi(f'), T \rangle$$

where $\langle , \rangle$ denotes the pairing between $\mathscr{A}(K)$ and its strong dual $(\mathscr{A}(K))'$, it is clear that B is continuous in each variable separately. Since F_b' and $(\mathscr{A}(K))'$ are both Frechet spaces, it follows by the Banach-Steinhaus theorem (cf. [2, Chapter III, §4, Proposition 2]) that B is continuous in both variables and hence that Φ maps a suitable neighborhood of zero W in F_b' into a bounded set of $\mathscr{A}(K)$.

3. Consider the bounded set

$$\{\Phi(f') = \phi_{f'} : f' \in W\}$$

in $\mathscr{A}(K)$. By the property of bounded sets of $\mathscr{A}(K)$ recalled in §1, there exists an open neighborhood Ω of K in the complex plane such that all the functions $\phi_{f'}, f' \in W$ can be extended to complex analytic functions on Ω and the set of such extended functions is bounded in $\mathscr{H}(\Omega)$. We keep the same notation $\phi_{f'}$ for the extended function.

It follows by Cauchy's integral formula that there exist two positive constants C_W and k such that

$$|\phi_{f'}^{(p)}(t_0)| \leqq C_W p! k^p \text{ for all } f' \in W \tag{6}$$

where t_0 is an arbitrary but fixed element of K and $\phi_{f'}^{(p)}$ is the pth derivative of $\phi_{f'}$.

4. To achieve the proof of the theorem, all we have to prove, by Lemma 1, is that the set

$$A = \{(1/p!)\phi^{(p)}(t_0) r^p,\ p = 0,1,2,\cdots\},$$

where $r = 1/k$, is bounded in F. For this, given a zero neighborhood V in F we have to find a $\lambda > 0$ such that $\lambda A \subset V$. Since zero neighborhoods in F correspond, by polarity, to equicontinuous sets in F' and vice versa, this amounts to showing that, given an equicontinuous set H in F' there exists an $M > 0$ such that:

$$|\langle (1/p!)\phi^{(p)}(t_0) r^p, f' \rangle| \leqq M \tag{7}$$

for all $f' \in H$ and $p = 0,1,2,\cdots$. Let H be an equicontinuous set in F'. Being equicontinuous, H is bounded in F'_b and thus there exists a $\mu > 0$ such that $\mu H \subset W$, where W is the zero neighborhood in F'_b defined above. It follows from (6) that

$$|\phi_{\mu f'}^{(p)}(t_0)| \leqq C_W \cdot p! k^p, \text{ for all } f' \in H,\ p = 0,1,\cdots.$$

Hence

$$|\langle (1/p!)\phi^{(p)}(t_0) r^p, \mu f' \rangle| \leqq C_W, \text{ for all } f' \in H,\ p = 0,1,\cdots$$

and from here we get (7) by taking $M = (1/\mu) \cdot C_W$, q.e.d.

Using the remarks just folowing Lemma 2 and its corollary we can say that under conditions of Theorem 1, the space $\mathscr{A}(K,F)$ can be identified in the algebraic sense with the space $L(F'_c, \mathscr{A}(K))$ of weakly analytic functions defined on K with values in F. In the next section we are going to define natural topologies on these spaces and to find conditions on F in order that they coincide.

3. **The space** $\mathscr{A}(K) \hat{\otimes}_\pi F$. The natural topology on $L(F'_c, \mathscr{A}(K))$ is that of the uniform convergence on equicontinuous sets of F' (cf. [9] for a detailed discussion

of this space). Denote by $L_\varepsilon(F'_c, \mathscr{A}(K))$ the space $L(F'_c, \mathscr{A}(K))$ equipped with this topology. It is a complete space (cf. [9, Exposé 8, Proposition 5]).

Consider now the tensor product $\mathscr{A}(K) \otimes F$. It can be identified in an obvious way with a subspace of $L(F'_c, \mathscr{A}(K))$ and it is known that this tensor product is dense in $L_\varepsilon(F'_c, \mathscr{A}(K))$ (cf. [9, Exposés 14, 15]). Denote by $\mathscr{A}(K)\hat{\otimes}_\varepsilon F$ the completion of $\mathscr{A}(K) \otimes F$ with respect to the induced topology of $L_\varepsilon(F'_c, \mathscr{A}(K))$.

On the other hand, we can define on $\mathscr{A}(K) \otimes F$ the projective tensor product topology (or π-topology), namely, the unique locally convex topology on $\mathscr{A}(K)\otimes F$ whose dual is the space of all continuous bilinear functionals on $\mathscr{A}(K) \times F$ (cf. [6, Chapter I, §1]). Let $\mathscr{A}(K)\hat{\otimes}_\pi F$ be the complexion of $\mathscr{A}(K) \otimes F$ with respect to this topology. It is known that $\mathscr{A}(K)$ is a nuclear space (cf. [6, Chapter II, p. 48, Corollary 1]) which amounts to saying that $\mathscr{A}(K)\hat{\otimes}_\pi F = \mathscr{A}(K)\hat{\otimes}_\varepsilon F$, this identification being in both the algebraic and topological senses. We can summarize the above results in the following

THEOREM 2. *Let U be an open set of R, K any compact subset of U and F a complete space. Then the space of weakly analytic functions defined on K with values in F can be identified with any of the spaces*

$$L_\varepsilon(F', \mathscr{A}(K)) = \mathscr{A}(K)\hat{\otimes}_\varepsilon F = \mathscr{A}(K)\hat{\otimes}_\pi F.$$

On $\mathscr{A}(K, F)$ let us introduce the limit inductive topology of $\mathscr{H}(\Omega, F)$ where Ω is an open set in C containing K and $\mathscr{H}(\Omega, F)$ the space of holomorphic functions with values in F endowed with the topology of uniform convergence on compact sets of Ω. By remarking that $\mathscr{H}(\Omega, F) = \mathscr{H}(\Omega)\hat{\otimes}_\pi F$ (cf. [6, Chapter II, p. 81, Example 2]) one can prove that the subspace $\mathscr{A}(K) \otimes F$ is dense in $\mathscr{A}(K, F)$. From this fact and Theorem 2, in order to prove that the space $\mathscr{A}(K, F)$ can be identified algebraically and topologically with the space of weakly analytic functions defined on K it is enough to prove that on $\mathscr{A}(K) \otimes F$ the π-topology and the induced topology of $\mathscr{A}(K, F)$ coincide. This we can prove when F is a quasi-$(\mathscr{DF})$ space, verifying the additional property (P) (cf. Lemma 3 below). In order to motivate the introduction of property (P) let us recall the definition of $(\mathscr{DF})$ spaces and prove one of their special properties.

DEFINITION 3 [5, DEFINITION 1]. *A topological vector space F is called $(\mathscr{DF})$ if the following two conditions are verified*:

(1) *there exists a fundamental sequence of bounded sets in F*;

(2) *if (M_i) is a sequence of equicontinuous sets of F' such that $M = \bigcup_i M_i$ is bounded in F'_b then M is equicontinuous.*

We need the following known property of $(\mathscr{DF})$ spaces.

LEMMA 3 [5, LEMMA 2]. *Suppose F is a $(\mathscr{DF})$ space. Then F verifies* (P) *given a sequence (V_i) of zero neighborhoods in F we can find a zero neighborhood V and a sequence (λ_i) of positive numbers such that $\lambda_i V \subset V_i$, for all i.*

Proof. Let M_i be the polar of V_i, for each i. Since V_i is a zero neighborhood in F, M_i is an equicontinuous set, hence bounded in F'_b. We already remarked in §2 that F'_b is a Frechet space. It follows then that there exists a sequence (μ_i) of positive numbers such that $M = \bigcup_i \mu_i M_i$ is bounded in F'_b (cf. [4, p. 286, Theorem 1]), hence by condition (2) is equicontinuous. The proof follows by taking again the polars of M and M_i, q.e.d.

THEOREM 3. *Let F be a quasi-($\mathscr{DF}$) space verifying property* (P). *Then on $\mathscr{A}(K) \otimes F$ the π-topology and the induced topology of $\mathscr{A}(K,F)$ coincide.*

Proof. Let (Ω_i) be a decreasing sequence of open neighborhoods of K in C, $\mathscr{H}(\Omega_i, F)$ the space of holomorphic functions on Ω_i with values in F and u_i the restriction map of $\mathscr{H}(\Omega_i, F)$ into $\mathscr{A}(K,F)$. To prove that the above described topologies coincide on $\mathscr{A}(K) \otimes F$ we just have to show that they have the same dual with the same equicontinuous sets. It is easy to see that any equicontinuous set in $B(\mathscr{A}(K), F)$ dual of $\mathscr{A}(K) \otimes F$ with the π-topology is equicontinuous with respect to the inductive limit topology.

Suppose, now, that M is an equicontinuous set in the dual of $\mathscr{A}(K) \otimes F$ equipped with the inductive limit topology. It follows from the definition of this topology that the set

$$M_i = \{u \circ (u_i \otimes 1) : u \in M\}$$

is, for each i, an equicontinuous set of the dual of $\mathscr{H}(\Omega_i, F)$. (Here 1 denotes the identity map of F.) Since $\mathscr{H}(\Omega_i, F) = \mathscr{H}(\Omega_i) \hat{\otimes}_\pi F$, there exists then a zero neighborhood U_i in $\mathscr{H}(\Omega_i)$ and a zero neighborhood V_i in F such that:

$$|u(u_i(g), f)| \leqq 1$$

for all $g \in U_i$ and $f \in V_i$. Now, F verifying property (P), there exists a zero neighborhood V and a sequence (λ_i) of positive numbers such that $\lambda_i V \subset V_i$, for all i. Next, let U be the zero neighborhood in $\mathscr{A}(K)$ obtained by taking the convex hull of $\bigcup_i u_i(\lambda_i U_i)$. It follows that

$$|u(g,f)| \leqq 1$$

for all $g \in U, f \in V$, hence M is equicontinuous in $\mathscr{A}(K) \otimes_\pi F$, q.e.d.

For ($\mathscr{DF}$) spaces, Theorem 3 was known [6, Chapter I, Proposition 6]. We remark that we do not know an example of a quasi-($\mathscr{DF}$) space verifying property (P) which is not a ($\mathscr{DF}$) space, a question interesting to be investigated.

4. **Analytic families of operators.** Let U be an open set of R, E a barrelled space, F a complete space, T a map from U into $L(E,F)$ and denote by T_t the element of $L(E,F)$ image of $t \in U$ by T.

DEFINITION 4. *We shall say that*:

(i) *T is scalarly analytic on U if for each $e \in E$ and each $f' \in F'$ the numerical* *function* $t \in U \to \langle T_t e, f' \rangle$ *is analytic on* U, *i.e., for each* $t_0 \in U$, *there exists an* $\varepsilon = \varepsilon(t_0, e, f') > 0$ *such that*

$$\langle T_t e, f' \rangle = \sum a_p(e, f')\,(t - t_0)^p, \tag{8}$$

the series (8) *being convergent for* $|t - t_0| < \varepsilon$;

(ii) T *is weakly analytic on* U *if, for each* $t_0 \in U$ *and* $e \in E$, *there exists an* $\varepsilon = \varepsilon(t_0, e) > 0$, *such that*

$$T_t e = \sum a_p(e)(t - t_0)^p, \tag{9}$$

the series converging in F *for* $|t - t_0| < \varepsilon$;

(iii) T *is strongly analytic if* T *is an analytic function on* U *with values in* $L_b(E,F)$, *i.e., if, for each* $t_0 \in U$, *there exists an* $\varepsilon = \varepsilon(t_0) > 0$, *such that*

$$T_t = \sum a_p(t - t_0)^p, \tag{10}$$

the series converging in $L_b(E,F)$ *for* $|t - t_0| < \varepsilon$.

It is obvious that (iii) implies (ii) and (ii) implies (i). Suppose T is scalarly analytic; then the numerical function $t \in U \to \langle T_t e, f' \rangle$ is C^∞ for each $e \in E$ and $f' \in F'$. We say that in this case T is scalarly C^∞. Also if T is weakly (strongly) analytic then T is weakly (strongly) C^∞. It can be shown by applying results of Grothendieck [4, pp. 238–245], and Schwartz [8] about vector valued differentiable functions that under the hypothesis E barrelled and F complete these three notions of differentiability coincide.

THEOREM 4. *Suppose* E *is barrelled,* F *is a complete quasi-*$(\mathscr{D}\mathscr{F})$ *space and* T *a map from* U *into* $L(E,F)$. *If* T *is scalarly analytic then* T *is strongly analytic.*

Proof. To prove that T is strongly analytic it suffices, by Lemma 1, to show that there exists an $r > 0$, such that the set

$$\{(1/p!) \cdot T_{t_0}^{(p)} \cdot r^p,\ p = 0, 1, 2, \cdots\}$$

is bounded in $L_b(E,F)$, where t_0 is a fixed element of U and $T_{t_0}^{(p)}$ denotes the derivative of T at t_0. Since E is barrelled, it suffices to show that this set is bounded in $L_s(E,F)$ (cf. [2, Chapter III, §3, Theorem 2]), or, in other words, that for each fixed $e \in E$, the set

$$\{(1/p!) \cdot T_{t_0}^{(p)}(e) \cdot r^p,\ p = 0, 1, 2, \cdots\}$$

is bounded in F. But this amounts to show that, for each fixed $e \in E$, the vector valued function

$$\Phi_e \colon t \in U \to T_t(e) \in F$$

is analytic. We apply Theorem 1, which completes the proof.

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