

ORTHOGONAL CONJUGACIES IN ASSOCIATIVE AND LIE ALGEBRAS⁽¹⁾

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1. Introduction. If A is a (finite-dimensional) associative algebra over a field F with radical R such that A/R is separable (semisimple, and remains semisimple under any extension of F), then the well-known Wedderburn principal theorem states that A is a semi-direct sum $T + R$, where T is a subalgebra of A isomorphic to A/R . T is a maximal separable subalgebra of A and will be also called a Wedderburn factor of A . The Malcev theorem (see [6]) states that any two maximal separable subalgebras of A are conjugate in A via an inner automorphism given by conjugation by an element $1 - z$, where $z \in R$ (A need not contain an identity). Let G be a semisimple group of automorphisms and anti-automorphisms of A (see §2 for definitions and terminology). In certain circumstances, G will leave invariant a Wedderburn factor of A (see [7; 8] and Theorem 1 of §3 for an exact description). In [9], we proved a uniqueness theorem for G -invariant Wedderburn factors for F of characteristic 0 and G finite. In [11], this was generalized to characteristic F not two and G finite of order not divisible by the characteristic of F . In §3 we generalize this result to arbitrary semisimple G and characteristic F not two (Theorem 2 and Corollary 1). It is shown that any two G -invariant Wedderburn factors of A are G -orthogonally conjugate in A .

Let L be a Lie algebra over a field F of characteristic 0. Then, if R denotes the radical (maximal solvable ideal) of L , then the well-known Levi theorem (see [5]) says that L is a semi-direct sum $T + R$ where T is a subalgebra of L isomorphic to L/R . T is a maximal semisimple subalgebra of L and will also be called a Levi factor of L . The Malcev-Harish-Chandra theorem (see [4; 6]) states that any two Levi factors of L are conjugate by an automorphism $\exp(\text{Ad } z)$ of L , where z is in the nil radical (maximal nilpotent ideal) of L . If G is a semisimple group of automorphisms of L , then L will contain G -invariant Levi factors (see [7]). In [10], we proved a uniqueness theorem for G -invariant Levi factors of L assuming G was finite. In §4, we generalize this result to an arbitrary semisimple G (Theorem 4 and Corollary 3). It is shown that for any two G -in-

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variant Levi factors of L , there exists a fixed point z of G in the nil radical of L such that $\exp(\text{Ad } z)$ carries one onto the other.

Let L be a solvable Lie algebra over a field of characteristic 0 and G a semi-simple group of automorphisms of L . Then G will leave invariant a Cartan subalgebra of L (see [7]). In general, any two Cartan subalgebras of L will be conjugate via an automorphism $\exp(\text{Ad } z)$, where z is in the intersection of all the terms of the chain of powers of L (see [1]). In §5, we show that if the two Cartan subalgebras are G -invariant, then z may be taken as a fixed point of G (see Theorem 6).

Finally, we conclude with some examples of invariant substructures in §6.

All algebras considered here will be finite-dimensional over the base field.

2. Preliminaries. Let A be an associative algebra over the field F . If A does not have an identity, let A_1 be the algebra obtained by adjoining F to A . Let G be a group of automorphisms and antiautomorphisms of A . If A is commutative, consider the elements of G as automorphisms. If A is not commutative, then each element of G is either an automorphism of A or an antiautomorphism of A , but not both. G acts on A_1 by letting $t(\alpha) = \alpha$ for $\alpha \in F$, $t \in G$.

DEFINITION 1. An element z of A_1 is said to be G -symmetric if z is a fixed point of the automorphisms in G and is sent into $-z$ by the antiautomorphisms in G .

The G -symmetric elements of A_1 form a Lie algebra over F , i.e., a subspace closed under $[x, y] = xy - yx$.

DEFINITION 2. A regular element w of A_1 is G -orthogonal if w is a fixed point of the automorphisms in G and is sent into w^{-1} by the antiautomorphisms in G .

The G -orthogonal elements of A_1 form a multiplicative group. This group contains as a subgroup the collection of G -orthogonal elements of the form $1 - z$, where $z \in R$, the radical of A .

DEFINITION 3. An element $q \in A_1$ is G -quasi-orthogonal if q is quasi-regular (i.e., q has a quasi-inverse q' such that $q + q' - qq' = 0 = q + q' - q'q$), q is a fixed point of the automorphisms in G and is sent into its quasi-inverse q' by the antiautomorphisms in G .

The collection of G -quasi-orthogonal elements of A_1 is a group under the composition $x \circ y = x + y - xy$. An element $q \in A_1$ is G -quasi-orthogonal if and only if $1 - q$ is G -orthogonal.

The proofs of the assertions made here and other relations between the properties of G -symmetry, G -orthogonality, and G -quasi-orthogonality can be found in [11].

DEFINITION 4. A G -orthogonal conjugacy of A is an inner automorphism of A given by conjugation by a G -orthogonal element of A_1 .

The G -orthogonal conjugacies of A form a group. This group contains as a

subgroup the conjugations by G -orthogonal elements $1 - z$ for $z \in R$. A G -orthogonal conjugacy commutes with each element of G .

DEFINITION 5. Two subalgebras of A are said to be G -orthogonally conjugate if there exists a G -orthogonal conjugacy of A carrying one onto the other.

The relation of G -orthogonal conjugacy is an equivalence relation among the subalgebras of A . It is also an equivalence relation among the G -invariant subalgebras of A . Any G -orthogonal conjugacy will carry a G -invariant subalgebra onto another G -invariant subalgebra.

If $x \in A$, then $\text{Ad } x$ will denote the derivation $a \rightarrow [x, a] = xa - ax$ of A . If $x \in R$, then $\text{Ad } x$ is a nilpotent derivation of A and $\exp(\text{Ad } x) = I + \text{Ad } x + (\text{Ad } x)^2/2! + \dots$ is an automorphism of A .

Now let L be a Lie algebra over a field of characteristic zero. If $x \in L$ then, $\text{Ad } x$ will denote the derivation $1 \rightarrow [x1]$ of L . If x is in the nil radical of L , then $\text{Ad } x$ is a nilpotent derivation of L and $\exp(\text{Ad } x) = I + \text{Ad } x + (\text{Ad } x)^2/2! + \dots$ is an automorphism of L . If G is a group of automorphisms of L , and x is a fixed point of G in the nil radical of L , then $\text{Ad } x$ will commute with the elements of G , and so will $\exp(\text{Ad } x)$ which is a polynomial in $\text{Ad } x$. Any such $\exp(\text{Ad } x)$ will hence carry a G -invariant subalgebra onto another G -invariant subalgebra, and also a G -invariant Cartan subalgebra onto another G -invariant Cartan subalgebra. In reference to other terminology and properties of Lie algebras, see [3] or [5].

A group G of nonsingular linear transformations is *semisimple* (fully reducible) if every G -invariant subspace has a G -invariant complementary subspace.

3. **The associative algebra case.** We first summarize what is known concerning the existence of invariant Wedderburn factors (maximal separable subalgebras) in the following theorem.

THEOREM 1. *Let A be an associative algebra over a field F . Let R denote the radical of A , and assume A/R is separable. Let G be a group of automorphisms and antiautomorphisms of A . If the characteristic of F is zero, assume G is semisimple. If the characteristic of F is $p \neq 0$, assume G is finite of order not a multiple of p . (G is then semisimple.) Then G leaves invariant some maximal separable subalgebra of A .*

Proof. If A is abelian, then by the Malcev theorem, A contains a unique maximal separable subalgebra, which will be left invariant by G . Hence we assume A is not abelian, and that each element of G is either an automorphism or an antiautomorphism of A , but not both. The case of characteristic p is proved in [8]. Now assume that F has characteristic zero and G is semisimple. The case $R^2 = \{0\}$ is essentially proved in Lemma 5.1 of [7] (the statement of the lemma omits the hypothesis that the group is fully reducible). Change the notation of the lemma so that $\mathfrak{G} = A$, $\mathfrak{R} = R$, $\Gamma = G$, and $\mathfrak{M} = M$, any Wedderburn factor

of A . M exists and Hypothesis (1) of the lemma is satisfied by the Wedderburn principal theorem. Hypothesis (2) is satisfied by the Malcev theorem, and since automorphisms and antiautomorphisms permute Wedderburn factors. Hypothesis (3) is clear. In the lemma, G is assumed to consist only of automorphisms. The proof of the lemma can be easily extended if G is also allowed to possess antiautomorphisms. Another way of obtaining the result is to replace G by the group H of automorphisms of A (considered as a Lie algebra) obtained by replacing the antiautomorphisms of A by their negatives. Then, by the lemma, H (and hence G) will leave fixed a Wedderburn factor of A . The passage from the case $R^2 = \{0\}$ to the general case may then be effected by the induction argument described in the proof of Theorem 1 in [8], together with the known facts that the homomorphic images of a semisimple group obtained by restricting it to an invariant subspace or letting it act on a factor space modulo an invariant subspace are also semisimple.

We remark that it does not seem to be known whether the conclusion of Theorem 1 holds when F is of characteristic p and G is an infinite semisimple group. Examples are given in §6 to show that it can happen that such a G leaves invariant maximal separable subalgebras, so that the uniqueness result to be presented now (Corollary 1) can apply in this situation. No general result or counterexample seems to be available. The methods of [7] for characteristic zero depend heavily on the technique of algebraic groups (e.g., rational representations preserve semisimple groups) and do not carry over to characteristic p . On the other hand, if G^* denotes the algebraic hull of G (i.e., the closure of G in the Zariski topology; see [2]), then G^* is an algebraic group of automorphisms and antiautomorphisms of A (since the group of all automorphisms and anti-automorphisms of A is algebraic) which is semisimple (since G and G^* have the same invariant subspaces). Hence, in all of this discussion, we could assume that G is an algebraic group, if necessary.

We now turn to the uniqueness question for maximal separable subalgebras of A left invariant by G . The assumptions will be that A/R is separable, and that G is a semisimple group of automorphisms and antiautomorphisms of A . If $z \in R$, we denote by C_{1-z} the inner automorphism of A defined by $C_{1-z}(a) = (1-z)a(1-z)^{-1}$. Let T be a maximal separable subalgebra of A left invariant by G , and let S be any separable subalgebra of A left invariant by G .

LEMMA 1. *Let $Z = \{z \in R; C_{1-z}(S) \subseteq T\}$. If $R^2 = \{0\}$, then Z is a non-empty flat subset of A , i.e., if $z_1, z_2, \dots, z_n \in Z$, $\alpha_1, \alpha_2, \dots, \alpha_n \in F$, and $\sum_{i=1}^n \alpha_i = 1$, then $\sum_{i=1}^n \alpha_i z_i \in Z$.*

Proof. Z is nonempty by the Malcev theorem. Let $y = \sum_{i=1}^n \alpha_i z_i$, and let $s \in S$. Then

$$\begin{aligned}
C_{1-y}(s) &= (1-y)s(1+y) = s + sy - ys \\
&= \sum_{i=1}^n (\alpha_i s) + \sum_{i=1}^n \alpha_i (sz_i) + \sum_{i=1}^n \alpha_i (-z_i s) \\
&= \sum_{i=1}^n \alpha_i (s + sz_i - z_i s) = \sum_{i=1}^n \alpha_i C_{1-z_i}(s) \in T.
\end{aligned}$$

In the following sequence of lemmas, we continue to assume that $R^2 = \{0\}$. Since Z is nonempty, we choose an $x \in Z$ and fix x as a reference point.

LEMMA 2. *Let $W = \{z - x; z \in Z\}$. Then W is a subspace of A .*

Proof. Let $z_1, z_2 \in Z$, $\alpha_1, \alpha_2 \in F$. Then

$$\alpha_1(z_1 - x) + \alpha_2(z_2 - x) = \alpha_1 z_1 + \alpha_2 z_2 + (1 - \alpha_1 - \alpha_2)x - x \in W$$

since $\alpha_1 z_1 + \alpha_2 z_2 + (1 - \alpha_1 - \alpha_2)x \in Z$ by Lemma 1.

LEMMA 3. *Z is an affine subspace of A .*

Proof. Z is the translate of W under the affine transformation $a \rightarrow a + x$ of A .

LEMMA 4. *If t is an automorphism in G , then t leaves Z invariant.*

Proof. Let $s \in S$, $z \in Z$. Then $s = t(u)$ for some $u \in S$. $C_{1-t(z)}(s) = s + st(z) - t(z)s = t(u) + t(u)t(z) - t(z)t(u) = t(u + uz - zu) = tC_{1-z}(u) \in T$. Hence $t(z) \in Z$.

LEMMA 5. *If t is an antiautomorphism in G , then t reflects Z through the origin, i.e., if $z \in Z$, then $t(z) \in -Z = \{-z; z \in Z\}$.*

Proof. Let $z \in Z$, $s \in S$, $s = t(u)$, $u \in S$. $C_{1+t(z)}(s) = s + t(z)s - st(z) = t(u) + t(z)t(u) - t(u)t(z) = t(u + uz - zu) = tC_{1-z}(u) \in T$. Hence $-t(z) \in Z$, $t(z) \in -Z$.

LEMMA 6. *G leaves the subspace W of A invariant.*

Proof. Let $z \in Z$. Let t be an automorphism in G . Then

$$t(z - x) = (t(z) - t(x) + x) - x \in W$$

by Lemmas 1 and 4. Now let t be an antiautomorphism in G . Then $t(z - x) = (-(-t(z)) + (-t(x)) + x) - x \in W$ by Lemmas 1 and 5.

LEMMA 7. *There exists a G -symmetric element z of R such that $C_{1-z}(S) \subseteq T$.*

Proof. Since G is semisimple, we may write $A = W \oplus Y$, where Y is a G -invariant subspace of A . Now Y will intersect Z in exactly one point, call it z . Then clearly $Y \cap (-Z) = \{-z\}$. It follows now from Lemmas 4 and 5 that z is a fixed point of the automorphisms in G and is sent into its negative by the antiautomorphisms in G .

Lemma 7 will be used in the case $R^2 = \{0\}$ of the following theorem.

THEOREM 2. *Let A be an associative algebra over a field F of characteristic not two. Let R be the radical of A and assume A/R is separable. Let G be a semi-simple group of automorphisms and antiautomorphisms of A . Let T be a maximal separable subalgebra of A left invariant by G and let S be any separable subalgebra of A left invariant by G . Then there exists a G -quasi-orthogonal element z of R (i.e., $1 - z$ is G -orthogonal in the algebra A_1 obtained from A by adjunction of an identity, if necessary) such that the inner automorphism C_{1-z} of A carries S into T .*

Proof. If A is commutative, the same remark in the proof of Theorem 1 holds, and we may take $z = 0$. Hence we now assume A is not commutative. If $R^2 = \{0\}$, then by Lemma 7, there exists a G -symmetric element z of R such that $C_{1-z}(S) \subseteq T$. Since $(1 - z)^{-1} = 1 + z$, $1 - z$ is G -orthogonal. We note that the restriction that characteristic $F \neq 2$ is not used when $R^2 = \{0\}$. Now let $R^2 \neq \{0\}$. We proceed by induction on the dimension of A . We first note that G induces semisimple groups (also denoted by G) of automorphisms and antiautomorphisms of the algebras R , $T + R^2$, and A/R^2 , all of which have dimension less than that of A . Let $a \rightarrow \bar{a} = a + R^2$ denote the natural homomorphism of A onto A/R^2 . The radical of A/R^2 is R/R^2 . \bar{T} is a maximal separable subalgebra of \bar{A} left invariant by G , and \bar{S} is a separable subalgebra of \bar{A} left invariant by G . Hence by induction (or by the case $R^2 = \{0\}$), there exists an element $v \in R$ such that $C_{\bar{1}-\bar{v}}(\bar{S}) = (\bar{1} - \bar{v})\bar{S}(\bar{1} - \bar{v})^{-1} \subseteq \bar{T}$, and \bar{v} is G -symmetric, i.e., $t(v) - v \in R^2$ for t an automorphism in G , and $t(v) + v \in R^2$ for t an antiautomorphism in G . Now we write $R = R^2 \oplus U$, where U is a G -invariant subspace. We write $-v/2 = x + u$ for $x \in R^2$, $u \in U$. Let t be an automorphism in G . Then $t(u) - u = -t(v)/2 - t(x) + v/2 + x = -\frac{1}{2}(t(v) - v) + x - t(x) \in R^2 \cap U = \{0\}$. Let t be an antiautomorphism in G . Then $t(u) + u = -\frac{1}{2}(t(v) + v) - x - t(x) \in R^2 \cap U = \{0\}$. Hence u is a G -symmetric element of R . We now set $y = -2u(1 - u)^{-1}$. Clearly $y \in R$ and $\bar{y} = -2\bar{u}(\bar{1} - \bar{u})^{-1} = -2(-\bar{v}/2)(\bar{1} + \bar{v}/2) = \bar{v}$. Hence $(1 - y)S(1 - y)^{-1} + R^2 \subseteq T + R^2$. Furthermore, $1 - y = 1 + 2u(1 - u)^{-1} = (1 + u)(1 - u)^{-1}$ which is G -orthogonal since u is G -symmetric. We now apply induction to the algebra $T + R^2$, whose radical is R^2 . T is a G -invariant maximal separable subalgebra and $(1 - y)S(1 - y)^{-1}$ is a G -invariant separable subalgebra, since C_{1-y} commutes with the elements of G . By induction, there exists an element $r \in R^2$ such that $1 - r$ is G -orthogonal and $(1 - r)(1 - y)S(1 - y)^{-1}(1 - r)^{-1} \subseteq T$. Then $z = r + y - ry$ is the desired element of R satisfying the conclusion of Theorem 2.

COROLLARY 1. *Let A be an associative algebra over a field F of characteristic not two. Let R be the radical of A , and assume that A/R is separable. Let G be a semisimple group of automorphisms and antiautomorphisms of A . Then*

any two G -invariant maximal separable subalgebras of A are G -orthogonally conjugate. If the characteristic of F is zero, then the G -orthogonal conjugacy may be written in the form $\exp(\text{Ad } x)$, where x is a G -symmetric element of R .

Proof. The first statement follows immediately from Theorem 2. If the characteristic of F is zero, let z be as described in Theorem 2, and let $x = \log(1 - z) = -z - z^2/2 - z^3/3 - \dots$. If t is an automorphism in G , then $t(z) = z$ implies $t(x) = x$. If t is an antiautomorphism in G , then $t(x) = -t(z) - t(z)^2/2 - t(z)^3/3 - \dots = \log(1 - t(z)) = \log(1 - z)^{-1} = -\log(1 - z) = -x$. Hence x is a G -symmetric element of R . Finally $C_{1-z} = C_{\exp(\log(1-z))} = \exp(\text{Ad}(\log(1 - z))) = \exp(\text{Ad } x)$.

COROLLARY 2. *Let A and G be as in Corollary 1. Assume that A possesses a G -invariant maximal separable subalgebra. Then any G -invariant separable subalgebra of A is contained in a G -invariant maximal separable subalgebra of A .*

Proof. This follows directly from Theorem 2 and the fact that the inverse of a G -orthogonal conjugacy is a G -orthogonal conjugacy.

4. The Lie algebra case. In [7], the following theorem is proved.

THEOREM 3. *Let L be a Lie algebra over a field of characteristic zero. Let G be a semisimple group of automorphisms of L . Then G leaves invariant a maximal semisimple subalgebra of L .*

We recall that a maximal semisimple subalgebra of L is also called a Levi factor of L . Let N be the nil radical (maximal nilpotent ideal) of L . Then we may prove a result analogous to Theorem 2 and Corollary 1. Since the proof is parallel to the proof of Theorem 2, we outline it here.

Let G be a semisimple group of automorphisms of the Lie algebra L . Since t is an automorphism of L if and only if $-t$ is an antiautomorphism of L , it is only necessary to consider automorphisms of L . Let T be a G -invariant Levi factor of L , and S any G -invariant semisimple subalgebra of L .

LEMMA 8. *Let $Z = \{z \in N; \exp(\text{Ad } z)S \subseteq T\}$. If $N^2 \equiv [N, N] = \{0\}$, then Z is a nonempty flat subset of L .*

Proof. Using the fact that $(\text{Ad } z)^2 = 0$ for $z \in N$, the proof is analogous to the proof of Lemma 1.

We continue to assume that $N^2 = \{0\}$. We now choose a fixed $x \in Z$ as reference point.

LEMMA 9. $W = \{z - x; z \in Z\}$ is a subspace of L .

LEMMA 10. Z is an affine subspace of L .

LEMMA 11. *If $t \in G$, then t leaves Z invariant.*

Proof. Let $z \in Z$, $s \in S$, $s = t(u)$, $u \in S$. Then $t(z) \in N$ and $\exp(\text{Ad}(t(z))) = I + \text{Ad}(t(z))$, I the identity mapping of L . Then $\exp(\text{Ad}(t(z)))(s) = t(u) + [t(z), t(u)] = t(u + [z, u]) = t((\exp(\text{Ad } z))u) \in T$ since $z \in Z$. Hence $t(z) \in Z$.

LEMMA 12. *G leaves W invariant.*

Proof. Lemmas 8 and 11.

LEMMA 13. *There exists a fixed point z of G in N such that $\exp(\text{Ad } z)(S) \subseteq T$.*

Proof. Write $L = W \oplus Y$, Y G -invariant. Then Z intersects Y in exactly one point—call it z . Then, since Z and Y are G -invariant, z is a fixed point of G .

Lemma 13 will cover the case $N^2 = \{0\}$ of the following theorem.

THEOREM 4. *Let L be a Lie algebra over a field of characteristic zero. Let N be the nil radical of L. Let G be a semisimple group of automorphisms of L. Let T be a G-invariant Levi factor of L, S any G-invariant semisimple subalgebra of L. Then there exists a fixed point z of G in N such that $\exp(\text{Ad } z)$ carries S into T.*

Proof. If $N^2 = \{0\}$, Lemma 13 applies. Assume $N^2 \neq \{0\}$, and use induction on the dimension of L . Let $a \rightarrow \bar{a} = a + N^2$ denote the natural homomorphism of L onto L/N^2 . Then there exists a fixed point \bar{v} of G in \bar{N} such that $\exp(\text{Ad } \bar{v})(\bar{S}) \subseteq \bar{T}$, i.e., $v \in N$ and $t(v) - v \in N^2$ for $t \in G$. Now write $N = N^2 \oplus U$, where U is G -invariant. Write $v = x + u$, $x \in N^2$, $u \in U$. Then $t(u) - u \in N^2 \cap U = \{0\}$, so u is a fixed point of G in N , and $\bar{u} = \bar{v}$ so that $\exp(\text{Ad } u)S + N^2 \subseteq T + N^2$. Now we use the induction hypothesis on the Lie algebra $T + N^2$ to obtain a fixed point r of G in N^2 such that $\exp(\text{Ad } r) \exp(\text{Ad } u)S \subseteq T$. Using the Baker-Campbell-Hausdorff formula (see [5]), we may write $\exp(\text{Ad } r) \exp(\text{Ad } u) = \exp(\text{Ad } z)$, where z is in the Lie algebra generated by r and u . Since the fixed points of G form a Lie algebra, z is a fixed point of G and $\exp(\text{Ad } z)S \subseteq T$.

COROLLARY 3. *Let L be a Lie algebra over a field of characteristic 0. Let G be a semisimple group of automorphisms of L. Let S and T be G-invariant Levi factors of L. Then there exists a fixed point z of G in the nil radical of L such that S and T are strictly conjugate in L under the automorphism $\exp(\text{Ad } z)$.*

COROLLARY 4. *Let L and G be as in Corollary 1. Then any G-invariant semisimple subalgebra of L is contained in a G-invariant Levi factor.*

These two corollaries are immediate consequences of Theorem 4.

5. **Invariant Cartan subalgebras of solvable Lie algebras.** In [7], the following theorem is proved.

THEOREM 5. *Let L be a solvable Lie algebra over a field of characteristic 0.*

Let G be a semisimple group of automorphisms of L . Then G leaves invariant a Cartan subalgebra of L .

We now turn to the uniqueness of G -invariant Cartan subalgebras of L . Let $L^n = [L^{n-1}, L]$ denote the terms of the descending chain of powers of L , and let $L^\infty = \bigcap_{n=1}^\infty L_n$. The collection of automorphisms $\{\exp(\text{Ad } x); x \in L^\infty\}$ of L forms a group by the Baker-Campbell-Hausdorff formula. By the same formula, this group will contain as a subgroup the collection of automorphisms $\{\exp(\text{Ad } x); x \text{ a fixed point of } G \text{ in } L^\infty\}$ of L . It is this latter group which will give us the conjugacy between any two G -invariant Cartan subalgebras of L .

THEOREM 6. *Let L be a solvable Lie algebra over a field of characteristic 0. Let G be a semisimple group of automorphisms of L . Let S and T be G -invariant Cartan subalgebras of L . Then G has a fixed point z in L^∞ such that $\exp(\text{Ad } z)$ carries S onto T .*

Proof. The proof will consist of an extension of the conjugacy argument of [1]. We use induction on the dimension of L , and first assume

(1) L^∞ is not abelian.

Let Z be the center of L^∞ . Z is an ideal in L by the Jacobi identity. Z is not zero since it contains the last nonzero term in the descending chain of powers of L^∞ , which is nilpotent. Let $a \rightarrow \bar{a} = a + Z$ denote the natural homomorphism of L onto L/Z . G induces semisimple groups of automorphisms (also denoted G) of L^∞/Z and \bar{L} , and \bar{S} , \bar{T} are G -fixed Cartan subalgebras of \bar{L} (see [3]). \bar{L} has smaller dimensions than L . Hence, by induction, there is a fixed point \bar{x} of G in $\bar{L}^\infty = \bar{L}^\infty$ such that $\exp(\text{Ad } \bar{x})\bar{S} = \bar{T}$. We write $L^\infty = Z \oplus U$, where U is a G -invariant subspace. Write $x = v + u$, $v \in Z$, $u \in U$. Then $\bar{x} = \bar{u}$. If $t \in G$, then $t(u) - u = t(x) - x + v - t(v) \in Z \cap U = \{0\}$. Hence u is a fixed point of G in L^∞ and $\exp(\text{Ad } u)S + Z = T + Z$. Now since $\text{Ad } u$ commutes with the elements of G , we have that $\exp(\text{Ad } u)S$ and T are G -invariant Cartan subalgebras of $T + Z$. Also, $(T + Z)^\infty = T + Z^\infty = T^\infty = 0$ since T is nilpotent. Hence $(T + Z)^\infty \subseteq Z$. If $L = T + Z$, then $L^\infty \subseteq Z$ and this contradicts the assumption (1) that L^∞ is not abelian. Hence $T + Z$ is a proper G -fixed subalgebra of L , and by induction there is a fixed point r of G in $(T + Z)^\infty \subseteq Z$ such that $\exp(\text{Ad } r)\exp(\text{Ad } u)S = T$. Set $z = r + u$. Then, since $[r, u] = 0$, $\exp(\text{Ad } z) = \exp(\text{Ad } r)\exp(\text{Ad } u)$ maps S onto T and z is clearly a fixed point of G in L^∞ .

(2) L^∞ is abelian.

By the result in [1], there is a $z \in L^\infty$ such that $\exp(\text{Ad } z)S = T$. Let $t \in G$. Then $\exp(\text{Ad}(t(z)))S = t \exp(\text{Ad } z)t^{-1}S = t(T) = T = \exp(\text{Ad } z)S$. Now since $[z, t(z)] = 0$, $\exp(\text{Ad}(t(z) - z))S = S$. Taking logarithms, this implies that $\text{Ad}(t(z) - z)S \subseteq S$, i.e., $[S, t(z) - z] \subseteq S$. But since S is a Cartan subalgebra of L , S is its own normalizer in L , so that $t(z) - z \in S$. Now, in general, $S \cap L^\infty$ is contained in the derived algebra of L^∞ , so that, by (2), $S \cap L^\infty = \{0\}$, and z is a fixed point of G in L^∞ . This completes the proof of Theorem 6.

6. **Some examples.** Let A denote the six-dimensional triangular (associative) algebra of three-by-three matrices (a_{ij}) over F with $a_{ij} = 0$ for $j > i$. The radical of A is the three-dimensional subalgebra R consisting of all (a_{ij}) with $a_{ij} = 0$ for $j \geq i$. The subalgebra D of R consisting of all (a_{ij}) with $a_{ij} = 0$ for $i \neq j$ is a maximum separable subalgebra of A . Let $v \in F, v \neq 0$. Let G be the cyclic group generated by the inner automorphism C_V of A given by conjugation by the diagonal matrix $V = \text{diag}(1, v, 1) \in A$. The matrix of C_V with respect to the usual basis of matrix units of A is $\text{diag}(1, v, 1, 1, v^{-1}, 1)$, so that C_V and hence G are semisimple. To obtain the form of the maximal separable subalgebras of A , we apply the Malcev theorem to D . If

$$W = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix} \in R,$$

we conjugate D by $I - W, I$ the identity mapping of A . The result is the subalgebra

$$S(x, y, z) = \left\{ \begin{pmatrix} a & 0 & 0 \\ (-a + b)x & b & 0 \\ (-a + c)y + (-b + c)xz & (-b + c)z & c \end{pmatrix}; a, b, c \in F \right\}.$$

The subalgebras $S(x, y, z)$ for $x, y, z \in F$ are the maximal separable subalgebras of A . The correspondence $(x, y, z) \rightarrow S(x, y, z)$ is one-to-one on $F \times F \times F$. Now, to determine which $S(x, y, z)$ are G -invariant, we apply C_V to $S(x, y, z)$. A direct calculation will show that $C_V(S(x, y, z)) = S(vx, y, v^{-1}z)$. Hence $S(0, y, 0)$ will be a G -invariant Wedderburn factor of A for any $y \in F$. The cardinality of $\{S(0, y, 0); y \in F\}$ is the same as that of F . If v is a root of unity, then G will be a finite semisimple group. If F is infinite, and v is not a root of unity, then G will be an infinite semisimple group of automorphisms leaving invariant an infinite number of maximum separable subalgebras. Hence we can find examples of infinite semisimple groups of automorphisms of an associative algebra over fields of arbitrary characteristic which leave invariant maximal separable subalgebras. As an illustration of Theorem 2, it may be directly verified that if $y_1, y_2 \in F$, then

$$U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_2 - y_1 & 0 & 0 \end{pmatrix}$$

is a fixed point of G in R , and conjugation by $I - U$ maps $S(0, y_1, 0)$ onto $S(0, y_2, 0)$. We also remark that the maximal separable subalgebras of A in this example are also Cartan subalgebras of A considered as a solvable Lie algebra.

To illustrate Theorem 6, $-U$ is a fixed point of G in A^∞ and $\exp(\text{Ad}(-U)) = I + \text{Ad}(-U) = C_{I-U}$ maps $S(0, y_1, 0)$ onto $S(0, y_2, 0)$.

We conclude with an example to show that the assumption that G be semi-simple cannot be dropped if G is to leave invariant a maximal separable subalgebra. Let A be the three-dimensional associative algebra of two-by-two matrices (a_{ij}) over F such that $a_{12} = 0$. The radical R of A is the one-dimensional subalgebra consisting of all (a_{ij}) with $a_{11} = a_{12} = a_{22} = 0$. The subalgebra D of A consisting of all (a_{ij}) with $a_{12} = a_{21} = 0$ is a maximal separable subalgebra of A . By applying the Malcev theorem to D as above, we obtain the complete collection of maximal separable subalgebras $\{S(x); x \in F\}$ of A , where

$$S(x) = \left\{ \begin{pmatrix} a & 0 \\ (a-b)x & b \end{pmatrix}; a, b \in F \right\}.$$

The correspondence $x \rightarrow S(x)$ is one-to-one on F . Let

$$U = \begin{pmatrix} 1 & 0 \\ w & v \end{pmatrix} \in A,$$

with $v \neq 0$. Let G be the cyclic group generated by the inner automorphism C_U of A given by conjugation by U . The matrix of C_U with respect to the usual basis of matrix units of A is

$$\begin{bmatrix} 1 & w & 0 \\ 0 & v & 0 \\ 0 & -w & 1 \end{bmatrix}.$$

A direct calculation will show that $C_U S(x) = S(w + vx)$. First let $v \neq 1$. Then $S(w(1-v)^{-1})$ is a (unique) G -invariant maximal separable subalgebra and G is semisimple, since the minimum polynomial $m(\lambda)$ of C_U is easily seen to be $(\lambda - 1)(\lambda - v)$ which has distinct factors. If F is infinite, and v is not a root of unity, then

$$U^n = \begin{pmatrix} 1 & 0 \\ w(1 + v + \dots + v^{n-1}) & v^n \end{pmatrix}$$

will not be in the center of A for any n , so that G will be an infinite semisimple group. If $v^n = 1$, then $U^n = I$. Hence, if v is a primitive n th root of unity, then G will be a finite semisimple group of order n .

Now assume $v = 1$ and $w \neq 0$. Then $w + vx = x$ has no solution $x \in F$ so that G does not leave invariant any maximal separable subalgebra of A . G is not semisimple since the minimal polynomial $m(\lambda)$ of C_U may easily be seen to be $(\lambda - 1)^2$. In this case,

$$U^n = \begin{pmatrix} 1 & 0 \\ nw & 1 \end{pmatrix},$$

so that G is infinite if F has characteristic 0, and will be of order p if F has characteristic p . We also note that the subalgebras $S(x)$ are Cartan subalgebras of A considered as a solvable Lie algebra.

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