CHANDRASEKHAR'S X AND Y EQUATIONS(1)

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1. **Introduction.** In his study of radiative transfer in homogeneous plane-parallel atmospheres of finite thickness, Chandrasekhar [4] introduces nonlinear integral equations for determining certain X and Y functions. For semi-infinite atmospheres the Y function is identically zero, and the X function is usually denoted by H. In two recent papers Fox [5] and Busbridge [2] have studied the existence and uniqueness questions for the H equation.

In a recent paper we [7] have given an exact criterion for uniqueness of solutions to the X and Y equations. In case of nonuniqueness we have given a simple representation of all solutions in terms of a particular solution studied by Busbridge [1]. These results contain those of Busbridge for the H equation [2].

The purpose of this paper is two-fold. First we complete the X and Y equations by additional linear constraints so that a unique pair of functions is specified by the requirement of analyticity in a half-plane. These constraints also serve to complete certain linear singular integral equations for X and Y functions.

The second purpose of this paper is to transform the linear singular equations and linear constraints into a form suitable for numerical computation. We first use the theory of singular integral equations [8] to obtain Fredholm equations for the values on the interval $[0, 1]$ of X and Y functions which are analytic in a half-plane. A similar development is given by Busbridge [3] for X and Y functions simply related to these. Busbridge is able to prove only that her Fredholm operators are contracting for sufficiently large atmospheric thickness.

We improve upon Busbridge's results. Extending methods developed with Leonard in a previous paper [6], we use analytic continuation to transform our first set of Fredholm equations to different Fredholm equations with simpler kernels which are continuous and nonnegative. We show that the new Fredholm operators are contracting for all values of the atmospheric thickness. Hence, the equations can be solved by iteration, with very rapid convergence for thick atmospheres.

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For the semi-infinite atmosphere, the Fredholm equations are solved exactly to give a determination of the H function in terms of simple quadratures. This amounts to a further reduction of Fox's solution [5].

2. **The** X **and** Y **equations.** In the study of radiative transfer problems for three different types of phase functions, which describe local scattering, Chandrasekhar [4, Chapter VIII] introduces the nonlinear integral equations

$$(2.1)\qquad \begin{aligned} X(\mu) &= 1 + \mu \int_0^1 \frac{X(\mu)X(v) - Y(\mu)Y(v)}{v + \mu} \Psi(v)dv, \\ Y(\mu) &= e^{-\tau/\mu} + \mu \int_0^1 \frac{X(\mu)Y(v) - Y(\mu)X(v)}{v - \mu} \Psi(v)dv, \end{aligned}$$

$$\text{for } 0 \leqq \mu \leqq 1 \text{ and } 0 \leqq \tau < \infty.$$

The function Ψ is known as the characteristic function. For radiative transfer problems it satisfies the inequality

$$(2.2)\qquad \int_0^1 \Psi(v)dv \leqq \frac{1}{2}.$$

The function Ψ is said to be *conservative* when equality holds, *nonconservative* otherwise. We shall assume in this paper that Ψ is nonnegative and satisfies a Hölder condition on the interval $0 \leqq \mu \leqq 1$.

Busbridge [1] demonstrated existence of solutions to (2.1) by investigating the auxiliary integral equation

$$(2.3)\qquad J(x,\mu) = e^{-x/\mu} + \int_0^1 \frac{\Psi(v)}{v} \int_0^\tau e^{-|x-y|/v} J(y,\mu)dydv.$$

She showed that a solution to (2.1) is given by

$$(2.4)\qquad X_0(\mu) = J(0,\mu), \quad Y_0(\mu) = J(\tau,\mu),$$

and that X_0 and Y_0 are defined, for all complex μ, $|\mu| > 0$, to be real and nonnegative for real $\mu \neq 0$ and to be analytic in the extended complex μ-plane except near $\mu = 0$.

We have investigated [7] uniqueness of solutions to (2.1). Let the function λ be defined for z outside the interval $[-1,1]$ by

$$(2.5)\qquad \lambda(z) \equiv 1 - 2z^2 \int_0^1 \frac{\Psi(v)dv}{z^2 - v^2}.$$

Then three cases arise [1, p. 15] for z in the extended complex plane, cut along the open interval $(-1,1)$:

(2.6)
(i) λ has no zeros.
(ii) The only zeros of λ are at $\pm 1/k$, $0 \leqq k < 1$, where $k = 0$ if and only if equality holds in (2.2).
(iii) λ has a zero at ± 1.

For simplicity in the following presentation *we assume that* Ψ *never satisfies* (iii). A treatment of this case requires detailed knowledge of Ψ near $\mu = 1$.

We have shown in [7] that if (i) of (2.6) is true, then (2.4) gives the only bounded solution to (2.1). If (ii) of (2.6) holds, a one-parameter family of solutions exists.

For nonconservative Ψ $(0 < k < 1)$, all solutions to (2.1) are given in terms of the solution (2.4) by

$$X(\mu) = \left[1 + \frac{a\alpha\mu}{1-k\mu} - \frac{b\beta\mu}{1+k\mu}\right]X_0(\mu) + \left[\frac{a\beta\mu}{1-k\mu} - \frac{b\alpha\mu}{1+k\mu}\right]Y_0(\mu), \tag{2.7}$$
$$Y(\mu) = \left[1 - \frac{a\alpha\mu}{1+k\mu} + \frac{b\beta\mu}{1-k\mu}\right]Y_0(\mu) - \left[\frac{a\beta\mu}{1+k\mu} - \frac{b\alpha\mu}{1-k\mu}\right]X_0(\mu).$$

The constants α, β and k are given by

$$\lambda\left(\frac{1}{k}\right) = 0,$$
$$\alpha = 1 - \int_0^1 \frac{X_0(\mu)\Psi(\mu)}{1+k\mu}d\mu, \quad \text{and} \tag{2.8}$$
$$\beta = \int_0^1 \frac{Y_0(\mu)\Psi(\mu)}{1+k\mu}d\mu,$$

and a and b are constants constrained only by the quadratic relation

$$(a^2 - b^2)(\alpha^2 - \beta^2) - 2\alpha a k - 2\beta b k = 0. \tag{2.9}$$

For $\tau = \infty$, $Y \equiv 0$, and $\beta = b = 0$, these reduce to Busbridge's results [2].

For conservative Ψ $(k = 0)$, all solutions to (2.1) are given by

$$X(\mu) = X_0(\mu) + a\mu(X_0(\mu) + Y_0(\mu)) + b\mu[\gamma X_0(\mu) + \mu(X_0(\mu) + Y_0(\mu))], \tag{2.10}$$
$$Y(\mu) = Y_0(\mu) - a\mu(X_0(\mu) + Y_0(\mu)) - b\mu[\gamma Y_0(\mu) - \mu(X_0(\mu) + Y_0(\mu))].$$

The constant γ is given by

$$\gamma \equiv \frac{x_1 + y_1}{y_0}, \tag{2.11}$$

where

$$x_n \equiv \int_0^1 \nu^n \Psi(\nu) X_0(\nu) d\nu, \tag{2.12}$$
$$y_n \equiv \int_0^1 \nu^n \Psi(\nu) Y_0(\nu) d\nu, \quad \text{for } n = 0, 1, \cdots.$$

The constants a and b are constrained only by the relation

$$b[b\gamma^2 + 2a\gamma - 2] = 0. \tag{2.13}$$

This complete result for conservative Ψ is not given in [7], but it is easily checked by substitution in (2.1). For $b = 0$ this reduces to Chandrasekhar's result [4, p. 190].

In the next section we show that linear constraints can be imposed which together with (2.1) uniquely determine the pair of functions (2.4). This will come from an extension of these functions to complex values of μ.

For applications, μ is restricted to the interval $[0,1]$ as the cosine of an angle in $[0,\pi/2]$. However, if we define S and T from (2.3) by

$$\begin{aligned} S(\nu,\mu) &= \int_0^\tau e^{-x/\nu} J(x,\mu)dx, \\ T(\nu,\mu) &= \int_0^\tau e^{-(\tau-x)/\nu} J(x,\mu)dx, \end{aligned} \tag{2.14}$$

it is easily shown [1, Chapter 8] that

$$\begin{aligned} S(\nu,\mu) &= \frac{\mu\nu}{\mu+\nu}[X_0(\nu)X_0(\mu) - Y_0(\nu)Y_0(\mu)], \\ T(\nu,\mu) &= \frac{\mu\nu}{\nu-\mu}[X_0(\nu)Y_0(\mu) - Y_0(\nu)X_0(\mu)]. \end{aligned} \tag{2.15}$$

As functions that determine the Laplace transform of the function J, set equal to zero outside $0 \leqq x \leqq \tau$, it is natural that properties of X_0 and Y_0 in the complex plane play an important role. This also shows the connection between the methods developed in §4 and the Wiener-Hopf method [9] as applied to equation (2.3).

3. **A completion of the X and Y equations.** We shall now complete equations (2.1) in the sense that the pair of X and Y functions in (2.4) will be shown to be the unique solution to (2.1) supplemented by two linear constraints. By (2.7) and (2.10) it is sufficient to determine the solution (2.4) to know all solutions to (2.1). For applications it has to be decided which of these functions is selected by the physical problem.

Suppose a solution has been found to (2.1) for $0 \leqq \mu \leqq 1$. Then if μ in (2.1) is replaced by any complex number z which is not in the interval $[-1,1]$, the integrals are well defined and an extension of X and Y is given by

$$\begin{aligned} X(z)\left[1 - z\int_0^1 \frac{X(\nu)\Psi(\nu)}{\nu+z}d\nu\right] + Y(z)z\int_0^1 \frac{Y(\nu)\Psi(\nu)}{\nu+z}d\nu = 1, \\ Y(z)\left[1 + z\int_0^1 \frac{X(\nu)\Psi(\nu)}{\nu-z}d\nu\right] - X(z)z\int_0^1 \frac{Y(\nu)\Psi(\nu)}{\nu-z}d\nu = e^{-\tau/z}. \end{aligned} \tag{3.1}$$

As a system of linear equations in $X(z)$ and $Y(z)$, (3.1) has a unique solution where the determinant does not vanish. It is easily shown [1, §40] that the determinant is just the function λ given by (2.5). This explains the importance of the zeros of λ in (2.7) and (2.10).

For brevity we are going to introduce notation for two operators given by

$$\text{(3.2)} \qquad \begin{aligned} U(X)(z) &\equiv z \int_0^1 \frac{X(\nu)\Psi(\nu)}{\nu + z} d\nu, \\ V(X)(z) &\equiv z \int_0^1 \frac{X(\nu)\Psi(\nu)}{\nu - z} d\nu, \end{aligned}$$

for z outside the interval $[-1,1]$. For z with $-1 \leqq z \leqq 1$, the singular integrals will be understood as Cauchy principal values without further notation to indicate this. These operators map functions defined on $0 \leqq \mu \leqq 1$ into sectionally holomorphic functions in the z-plane cut along $[-1,1]$. By Plemelj's formula [8] we have, for example, for $0 < \mu < 1$,

$$\text{(3.3)} \qquad \begin{aligned} V^+(X)(\mu) &= \pi i \mu \Psi(\mu) X(\mu) + V(X)(\mu), \\ V^-(X)(\mu) &= -\pi i \mu \Psi(\mu) X(\mu) + V(X)(\mu) \end{aligned}$$

where the superscript "+" denotes the limit from the upper half-plane and the superscript "−" denotes the limit from the lower half-plane. For (3.3) to be valid, it is sufficient that ΨX satisfy a Hölder condition on $0 \leqq \mu \leqq 1$. These formulae do not hold at the endpoint $\mu = 1$.

In this notation we have the determinant of (3.1) given by

$$\text{(3.4)} \qquad \lambda = [1 - U(X)][1 + V(X)] + U(Y)V(Y)$$

for z outside $[-1,1]$. Using Plemelj's formula, we find for $0 < \mu < 1$,

$$\text{(3.5)} \qquad \begin{aligned} \lambda^+(\mu) = {} & [1 - U(X)(\mu)][1 + V(X)(\mu)] + U(Y)(\mu)V(Y)(\mu) \\ & + \pi i \mu \Psi(\mu)[X(\mu)(1 - U(X)(\mu)) + Y(\mu)U(Y)(\mu)]. \end{aligned}$$

By the definition (2.5) of λ we have

$$\text{(3.6)} \qquad \lambda^+(\mu) = \lambda_0(\mu) + \pi i \mu \Psi(\mu),$$

with

$$\text{(3.7)} \quad \lambda_0(\mu) \equiv 1 - \mu \int_0^1 \frac{\Psi(\nu) d\nu}{\mu + \nu} + \mu \int_0^1 \frac{\Psi(\nu) - \Psi(\mu)}{\nu - \mu} d\nu + \mu \Psi(\mu) \ln \frac{1 - \mu}{\mu}.$$

Equating real parts of (3.5) and (3.6), we have for $0 \leqq \mu \leqq 1$,

$$\text{(3.8)} \qquad \lambda_0 = [1 - U(X)][1 + V(X)] + U(Y)V(Y).$$

The statement of equality of the imaginary parts is simply the first equation of (2.1).

We now solve the system of linear equations (3.1) for z outside $[-1,1]$ to get

$$\lambda X = 1 + V(X) - hU(Y),$$
$$\lambda Y = h(1 - U(X)) + V(Y), \tag{3.9}$$

where h is defined by

$$h(z) = e^{-\tau/z}. \tag{3.10}$$

We have [3]

THEOREM 1. *Let X and Y be any real-valued solution of* (2.1) *for* $0 \leqq \mu \leqq 1$. *Then* (3.9) *defines the meromorphic extension of X and Y to the complex domain* $|z| > 0$. *This extension gives functions analytic in* $|z| > 0$ *except for possible poles at the zeros of λ. In addition, X and Y satisfy the singular linear integral equations*

$$\lambda_0 X = 1 + V(X) - hU(Y),$$
$$\lambda_0 Y = h[1 - U(X)] + V(Y), \quad 0 \leqq \mu < 1. \tag{3.11}$$

Proof. We certainly have a sectionally meromorphic function defined for z not in $[-1,1]$ by (3.9) as

$$X = \frac{1 + V(X) - hU(Y)}{\lambda}. \tag{3.12}$$

We want to show that indeed this function is meromorphic in $|z| > 0$; we must avoid the essential singularity of the h function at $z = 0$.

By (3.8), for $0 \leqq \mu < 1$, we have

$$\lambda_0 X - [1 + V(X) - hU(Y)] \equiv [X(1 - U(X)) - 1][1 + V(X)] + U(Y)[h + XV(Y)]; \tag{3.13}$$

by (2.1) this gives

$$\lambda_0 X - 1 - V(X) + hU(Y) = U(Y)[h + XV(Y) - Y - YV(X)] = 0. \tag{3.14}$$

This establishes the first equation in (3.11), and a similar argument will establish the second equation.

Now for $0 < \mu < 1$ we have

$$X^+(\mu) = \frac{1 + V(X)(\mu) - h(\mu)\,U(Y)(\mu) + \pi i \mu \Psi(\mu) X(\mu)}{\lambda_0(\mu) + \pi i \mu \Psi(\mu)}. \tag{3.15}$$

By (3.14) this reduces to

$$X^+(\mu) = X(\mu), \qquad 0 < \mu < 1. \tag{3.16}$$

A similar argument gives

$$X^{-}(\mu) = X(\mu), \quad 0 < \mu < 1. \tag{3.17}$$

This shows that X defined by (3.12) is continuous across $(0,1)$ and real valued on $(0,1)$. Hence (3.12) does indeed define a meromorphic extension of X, at least in the region $0 < \mathrm{Re}(z) < 1$.

To show that (3.12) defines a meromorphic function in all of the domain $\mathrm{Re}(z) > 0$, we have to investigate X near $z = 1$, where the Plemelj formula does not hold in general. By assumption on Ψ, the function λ vanishes at most at the points $\pm 1/k$ in the plane cut along $(-1,1)$. Since X is analytic in $\mathrm{Re}(z) > 0$ except possibly at 1 and $1/k$ and $\lambda(1) \neq 0$, it follows readily that $z = 1$ is a removable singularity.

We can continue X by (3.12) into $\mathrm{Re}(z) < 0$ provided it is analytic for $-1 \leqq z < 0$. We have from (3.12) and (3.2) for $0 < \mu < 1$

$$X^{+}(-\mu) = \frac{1 - U(X)(\mu) + h(-\mu)[V(Y)(\mu) + \pi i \mu \Psi(\mu) Y(\mu)]}{\lambda_0(\mu) + \pi i \mu \Psi(\mu)}. \tag{3.18}$$

By (3.11) this reduces to

$$X^{+}(-\mu) = e^{\tau/\mu} Y(\mu). \tag{3.19}$$

A similar result for $X^{-}(-\mu)$ shows X to be continuous across and real valued on $(-1,0)$. Analyticity at $z = -1$ is proved as above for $z = 1$. We have therefore shown that (3.12) defines a meromorphic function in $|z| > 0$. A similar argument applies to the function Y defined by (3.9). The proof is complete.

As a consequence of the analyticity of X and Y, we obtain from (3.19) the fact that [3,4]

$$X(-z) = e^{\tau/z} Y(z) \tag{3.20}$$

for all z, $|z| > 0$.

The solution (2.4) to (2.1) is regular in $|z| > 0$. From (3.9) we obtain the constraints necessary to specify this solution. In view of (3.20) it is sufficient to specify regularity of X and Y at the single zero $1/k$ of λ. For nonconservative Ψ $(0 < k < 1)$ we have

$$\begin{aligned} 0 &= 1 - \int_0^1 \frac{X(\nu)\Psi(\nu)}{1 - k\nu} d\nu - e^{-k\tau} \int_0^1 \frac{Y(\nu)\Psi(\nu)}{1 + k\nu} d\nu, \\ 0 &= 1 - \int_0^1 \frac{X(\nu)\Psi(\nu)}{1 + k\nu} d\nu - e^{k\tau} \int_0^1 \frac{Y(\nu)\Psi(\nu)}{1 - k\nu} d\nu. \end{aligned} \tag{3.21}$$

For conservative Ψ ($k = 0$) λ has a double zero at ∞ and we obtain the constraints

$$
\begin{aligned}
1 &= \int_0^1 [X(\nu) + Y(\nu)]\Psi(\nu)\,d\nu, \\
\tau \int_0^1 Y(\nu)\Psi(\nu)\,d\nu &= \int_0^1 [X(\nu) - Y(\nu)]\,\nu\Psi(\nu)\,d\nu.
\end{aligned}
\tag{3.22}
$$

In our previous study [7] we used the linear singular integral equations (3.11) to determine the solutions (2.7) and (2.10) with a and b arbitrary. Since all solutions to (2.1) satisfy (3.11) by Theorem 1, these nonlinear equations serve only to impose the constraints (2.9) and (2.13) on the parameters a and b. The above constraints (3.21) and (3.22) serve to specify uniquely the solution (2.4) to the linear equations (3.11) and to the nonlinear equations (2.1).

The linear equations (3.11), derived here from the nonlinear equations (2.1), arise in a more natural way in the study of a linear Boltzmann equation. This will be presented in a subsequent paper.

4. **Fredholm equations.** We now use the linear equations (3.11), and the constraints (3.21) or (3.22) to obtain Fredholm equations which can be solved easily by iteration. These equations will be especially good for computing X_0 and Y_0 functions for large values of the parameter τ. Any of the other X and Y functions given by (2.7) and (2.10) are determined by a knowledge of X_0 and Y_0.

We define functions f and g by

$$f \equiv 1 - h\,U(Y), \tag{4.1}$$

and

$$g \equiv h[1 - U(X)]. \tag{4.2}$$

These define analytic functions in the complex z-plane, cut along $[-1,0]$ and excluding a neighborhood of $z = 0$. We shall apply the theory of singular integral equations [8] to (3.11), written now as

$$
\begin{aligned}
\lambda_0 X &= f + V(X), \\
\lambda_0 Y &= g + V(Y).
\end{aligned}
\tag{4.3}
$$

It is first necessary to study the homogeneous singular equation

$$\lambda_0(\mu)N_0(\mu) = \mu \int_0^1 \frac{N_0(\nu)\Psi(\nu)}{\nu - \mu}\,d\nu, \qquad 0 \leqq \mu \leqq 1. \tag{4.4}$$

We add to the assumptions on Ψ made in §2 the following one:

$$\Psi^2 + \lambda_0^2 \neq 0. \tag{4.5}$$

We have the following:

LEMMA 1. *If* (i) *of* (2.6) *holds, the only bounded solution of* (4.4) *is* $N \equiv 0$. *If* (ii) *of* (2.6) *holds, then all solutions of* (4.4) *are proportional to the function*

$$N_0(\mu) = \left[\frac{1-\mu}{\mu}\right]^{(\theta(\mu)-1)} \frac{\exp\left[\int_0^1 \frac{\theta(t)-\theta(\mu)}{t-\mu}dt\right]}{([\lambda_0(\mu)]^2 + [\pi\mu\Psi(\mu)]^2)^{1/2}}, \tag{4.6}$$

where

$$\theta(t) = \frac{1}{\pi}\tan^{-1}\left[\frac{\pi t\Psi(t)}{\lambda_0(t)}\right], \qquad 0 \leqq \theta \leqq 1\,. \tag{4.7}$$

Proof. Following [8], we define the sectionally holomorphic function E by

$$E(z) \equiv \exp[\Gamma(z)] \tag{4.8}$$

where

$$\Gamma(z) = \int_0^1 \frac{\theta(t)dt}{t-z}, \qquad z \notin [0,1], \tag{4.9}$$

with θ defined by (4.7).

The function E is a solution to the Hilbert problem of finding a sectionally holomorphic function such that the limits, E^+ and E^- of E from the upper and lower half-planes, respectively, satisfy

$$E^+\lambda^- = E^-\lambda^+, \qquad 0 < \mu < 1. \tag{4.10}$$

The equation (4.4) has a nontrivial solution provided this Hilbert problem has a solution which vanishes at ∞, and this is possible if the index κ of the function θ is positive [8, Chapter 14]. In (4.7) the index is given by

$$\kappa = \theta(1). \tag{4.11}$$

The function λ of (2.5) is analytic in the extended complex plane cut along $[-1,1]$. By Plemelj's formula we have

$$\begin{aligned} \lambda^+(\mu) &= \lambda_0(\mu) + \pi i\mu\Psi(\mu), \\ \lambda^-(\mu) &= \lambda_0(\mu) - \pi i\mu\Psi(\mu), \qquad -1 < \mu < 1. \end{aligned} \tag{4.12}$$

By this and an application of the argument principle to λ it follows readily that the index equals one half the number of zeros of λ in the extended complex plan cut along $[-1,1]$. We have

$$\begin{aligned} \kappa &= 0 \quad \text{for (i) of (2.6)}, \\ \kappa &= 1 \quad \text{for (ii) of (2.6)}. \end{aligned} \tag{4.13}$$

We define a function N by

$$N(z) \equiv \frac{E(z)}{(1-z)^\kappa}, \tag{4.14}$$

and this also is a solution to the Hilbert problem (4.10) [8, Chapter 10]. For (ii) of (2.6), N vanishes at ∞ and determines the solution of (4.6) to (4.4) by

$$N_0(\mu) = \frac{N^+(\mu) - N^-(\mu)}{2\pi i \Psi(\mu)}, \qquad 0 \leqq \mu < 1. \tag{4.15}$$

This completes the proof.

We now define sectionally holomorphic functions ϕ_1 and ϕ_2 by

$$\begin{aligned} \phi_1(z) &\equiv N(z)\int_0^1 \frac{\Psi(t)f(t)}{N^+(t)\lambda^-(t)}\,\frac{dt}{t-z}, \\ \phi_2(z) &\equiv N(z)\int_0^1 \frac{\Psi(t)g(t)}{N^+(t)\lambda^-(t)}\,\frac{dt}{t-z}. \end{aligned} \tag{4.16}$$

We have to distinguish the two cases of (2.6); for (i) and (ii), respectively, we have

$$\begin{aligned} &\text{(i)} \qquad N(z) = E(z), \\ &\text{(ii)} \qquad N(z) = \frac{E(z)}{1-z}, \end{aligned} \tag{4.17}$$

where E is given by (4.8) and (4.9)

Again referring to Chapter 14 of [8], we have the fact the linear equations (4.3) are equivalent to the Fredholm equations on $0 \leqq \mu \leqq 1$ given by

$$\begin{aligned} X(\mu) &= AN_0(\mu) + \frac{\phi_1^+(\mu) - \phi_1^-(\mu)}{2\pi i \Psi(\mu)}, \\ Y(\mu) &= BN_0(\mu) + \frac{\phi_2^+(\mu) - \phi_2^-(\mu)}{2\pi i \Psi(\mu)}. \end{aligned} \tag{4.18}$$

The functions f and g, occuring in ϕ_1 and ϕ_2, are to be replaced by their defining relations (4.1) and (4.2). The constants A and B are arbitrary, and the function N_0 is given in the previous theorem. We merely refer to Busbridge [3] for a detailed display of these Fredholm equations which have rather complicated kernels.

We now use the fact that X and Y have analytic extensions to complex values of μ, both to evaluate the constants A and B and to obtain simpler Fredholm equations for the functions f and g than the above for X and Y. The calculation of the function f and g from the equations to be determined gives a unique determination of the functions X and Y by (4.18).

We now have

LEMMA 2. *Let X and Y be determined on $0 < \mu < 1$ by* (4.18) *for real constants A and B. Then these functions are extended analytically to* $0 < \mathrm{Re}(z) < 1$ *by*

(4.19)
$$X(z) = \frac{f(z) + z[\phi_1(z) + AN(z)]}{\lambda(z)},$$
$$Y(z) = \frac{g(z) + z[\phi_2(z) + BN(z)]}{\lambda(z)}.$$

For case (i) of (2.6), $A = B = 0$.

Proof. Clearly X and Y defined by (4.19) are analytic in the two domains $0 < \mathrm{Re}(z) < 1$, $\mathrm{Im}(z) > 0$ and $0 < \mathrm{Re}(z) < 1$, $\mathrm{Im}(z) < 0$. A simple application of Plemelj's formula shows that X is continuous across the interval $(0,1)$ and equal on this interval to the real-valued function given by (4.18). A similar argument applies to Y and completes the proof.

It will be convenient to consider the functions P and Q defined by

(4.20)
$$P(z) = f(z) + g(z),$$
$$Q(z) = f(z) - g(z).$$

Equations for P and Q, given in the following theorem, contain the linear integral operator L defined by

(4.21)
$$L(P)(z) \equiv z e^{-\tau/z} N(-z) \int_0^1 \frac{\Psi(t)P(t)}{N^+(t)\lambda^-(t)(t+z)} dt.$$

We now have the following result.

THEOREM 2. *Let f and g be defined by* (4.1) *and* (4.2) *for any real-valued solution X and Y to* (3.11). *Then for appropriate values of the constants C and D, P and Q satisfy*

(4.22)
$$P(z) = -L(P)(z) + 1 + e^{-\tau/z} + Cze^{-\tau/z}N(-z),$$
$$Q(z) = L(Q)(z) + 1 - e^{-\tau/z} + Dze^{-\tau/z}N(-z),$$

for all z not in the interval $[-1,0]$. *For z in the interval* $[0,1]$ *these are Fredholm equations in which the Fredholm operator L has a nonnegative continuous kernel.*

Proof. We have two expressions, (3.9) and (4.19), for the analytic continuation of X and Y into the complex plane. Equating these two, we find, for appropriate values of the constants A and B, that

(4.23)
$$z\int_0^1 \frac{X(v)\Psi(v)}{z-v} dv = -z[\phi_1(z) + AN(z)],$$
$$z\int_0^1 \frac{Y(v)\Psi(v)}{z-v} dv = -z[\phi_2(z) + BN(z)].$$

These expressions are analytic in the plane cut along the interval [0, 1]. Therefore, by the definitions (4.1) and (4.2) of f and g, we have for z out of the interval [− 1,0]

$$
\begin{aligned}
f(z) &= 1 - ze^{-\tau/z}[\phi_2(-z) + BN(-z)], \\
g(z) &= e^{-\tau/z} - ze^{-\tau/z}[\phi_1(-z) + AN(-z)].
\end{aligned} \tag{4.24}
$$

Adding and subtracting these equations and using (4.16) and (4.21), we obtain (4.22).

The functions P and Q are determined by evaluating (4.22) on the interval [0, 1] and solving the resulting Fredholm equations. The analytic extension of P and Q is then given by the right-hand side of (4.22). On the interval [0,1] it is easily shown for both (i) and (ii) of (2.6), that the kernel of L is a nonnegative and continuous function. The proof is complete.

Equations (4.22) on the interval [0,1] are uncoupled Fredholm equations containing the integral operator L. We propose to solve these equations by iteration. This will converge uniformly on the interval [0,1] to a solution to (4.22), provided the norm of L satisfies $\| L \| < 1$. We have

$$\| L \| \leqq \max_{0 \leqq \mu \leqq 1} \left[\mu e^{-\tau/\mu} N(-\mu) \int_0^1 \frac{\Psi(t)}{N^+(t)\lambda^-(t)(t+\mu)} dt \right]. \tag{4.25}$$

We can simplify the function in (4.25) and estimate $\| L \|$.

THEOREM 3. *For N defined by* (4.17) *and λ defined by* (2.5), *we have for* (i) *of* 2.6) *and* $z \notin [-1,0]$

$$zN(z) \int_0^1 \frac{\Psi(t)}{N^+(t)\lambda^-(t)(t-z)} dt = \frac{N(z)}{N(0)} - 1, \tag{4.26}$$

and for (ii) *of* (2.6)

$$zN(z) \int_0^1 \frac{\Psi(t)}{N^+(t)\lambda^-(t)(t-z)} dt = \frac{N(z)}{N(0)} - zN(z) - 1. \tag{4.27}$$

Therefore, from (4.25) *we obtain*

$$\| L \| \leqq e^{-\tau}\left(1 - \exp\left[-\int_0^1 \frac{\theta(t)}{t} dt \right]\right). \tag{4.28}$$

Proof. We consider two contours γ and Γ. For z outside the interval [0,1], let Γ be a circle with center at 0 and of radius $R > |z|$, and let γ be a simple closed curve interior to Γ and enclosing a region with the interval [0,1] in the interior and with z outside. Since the function N of (4.17) is analytic and nonzero in the annulus bounded by Γ and γ, we integrate the reciprocal of $N(w)(w-z)w$ around Γ and γ in a clockwise direction to obtain by Cauchy's theorem

$$(4.29)\qquad \frac{1}{2\pi i}\int_{\gamma}\frac{dw}{N(w)(w-z)w}-\frac{1}{2\pi i}\int_{\Gamma}\frac{dw}{N(w)(w-z)w}=\frac{1}{zN(z)}.$$

By (4.10) and (4.12) we conclude that for $0<\mu<1$

$$(4.30)\qquad \frac{1}{N^{+}(\mu)}-\frac{1}{N^{-}(\mu)}=-\frac{2\pi i\mu\Psi(\mu)}{N^{+}(\mu)\lambda^{-}(\mu)}.$$

We now let γ shrink to the cut along $[0,1]$ to conclude from (4.30) that

$$(4.31)\qquad \frac{1}{2\pi i}\int_{\Gamma}\frac{dw}{N(w)(w-z)w}=\frac{1}{zN(0)}-\int_{0}^{1}\frac{\Psi(\mu)d\mu}{N^{+}(\mu)\lambda^{-}(\mu)(\mu-z)}.$$

For $|z|<R$, the radius of the circle Γ, we have then

$$(4.32)\qquad \begin{aligned} zN(z)&\int_{0}^{1}\frac{\Psi(\mu)d\mu}{N^{+}(\mu)\lambda^{-}(\mu)(\mu-z)}\\ &=\frac{N(z)}{N(0)}-1-\frac{zN(z)}{2\pi i}\int_{\Gamma}\frac{dw}{N(w)(w-z)w}.\end{aligned}$$

For (i) of (2.6) and (4.17), $N(w)$ tends to 1 as $|w|$ tends to ∞, thereby establishing (4.26). For (ii) of (2.6) and (4.17), $N(w)$ vanishes like w^{-1} as $|w|$ tends to ∞, so that the integral around Γ tends to 1 as R tends to ∞. This establishes (4.27).

Since $\theta(t)/t$ is bounded and nonnegative for $0\leqq t\leqq 1$, we obtain (4.28) for both (i) and (ii) of (2.6). The proof is complete.

We have now shown that unique functions h_i, $i=1,2,3,4$, can be computed by iteration from the following four equations:

$$(4.33)\qquad \begin{aligned} &\text{(i)} && h_1(\mu)=-L(h_1)(\mu)+1+e^{-\tau/\mu},\\ &\text{(ii)} && h_2(\mu)=-L(h_2)(\mu)+\mu e^{-\tau/\mu}N(-\mu),\\ &\text{(iii)} && h_3(\mu)=L(h_3)(\mu)+1-e^{-\tau/\mu},\\ &\text{(iv)} && h_4(\mu)=L(h_4)(\mu)+\mu e^{-\tau/\mu}N(-\mu).\end{aligned}$$

If we denote the functions P and Q on the interval $[0,1]$ by P_0 and Q_0, then by (4.22)

$$(4.34)\qquad \begin{aligned} P_0(\mu)&=h_1(\mu)+Ch_2(\mu),\\ Q_0(\mu)&=h_3(\mu)+Dh_4(\mu).\end{aligned}$$

Again by (4.22), the functions P and Q for complex z are given by

$$(4.35)\qquad \begin{aligned} P(z)&=-L(P_0)(z)+1+e^{-\tau/z}+Cze^{-\tau/z}N(-z),\\ Q(z)&=L(Q_0)(z)+1-e^{-\tau/z}+Dze^{-\tau/z}N(-z).\end{aligned}$$

Our one remaining task is to relate the constraints A, B, C, and D to the X_0 and Y_0 functions given by (2.4). We do this by expressing the constraints (3.21) and (3.22) in terms of the constants.

For (i) of (2.6) we have by Lemma 2

$$A = B = C = D = 0. \tag{4.36}$$

For nonconservative Ψ $(0 < k < 1)$, we state (3.21) in terms of P and Q as

$$\begin{aligned} 1 + e^{-k\tau} - P\left(\frac{1}{k}\right) - e^{-k\tau} P\left(-\frac{1}{k}\right) &= 0. \\ 1 - e^{-k\tau} - Q\left(\frac{1}{k}\right) + e^{-k\tau} Q\left(-\frac{1}{k}\right) &= 0. \end{aligned} \tag{4.37}$$

If $P(\pm 1/k)$ and $Q(\pm 1/k)$ are inserted from (4.35), with P_0 and Q_0 as given by (4.34), then algebraic manipulations give

$$\begin{aligned} C &= \frac{1 + e^{-k\tau} - L(h_1)\left(\frac{1}{k}\right) - e^{-k\tau} L(h_1)\left(-\frac{1}{k}\right)}{\dfrac{N\left(\frac{1}{k}\right) - e^{-k\tau} N\left(-\frac{1}{k}\right)}{k} + L(h_2)\left(\frac{1}{k}\right) + e^{-k\tau} L(h_2)\left(-\frac{1}{k}\right)}, \\ D &= \frac{1 - e^{-k\tau} + L(h_3)\left(\frac{1}{k}\right) - e^{-k\tau} L(h_3)\left(-\frac{1}{k}\right)}{-\dfrac{N\left(\frac{1}{k}\right) + e^{-k\tau} N\left(-\frac{1}{k}\right)}{k} - L(h_4)\left(\frac{1}{k}\right) + e^{-k\tau} L(h_4)\left(-\frac{1}{k}\right)}. \end{aligned} \tag{4.38}$$

If in (4.38) we let k tend to 0, we have for conservative Ψ

$$\begin{aligned} C &= -1, \\ D &= \frac{\tau + 2\displaystyle\int_0^1 \frac{\psi(t)h_3(t)dt}{N^+(t)\lambda^-(t)}}{\tau + 2\left(1 - \displaystyle\int_0^1 \theta(t)dt\right) + 2\displaystyle\int_0^1 \frac{\psi(t)h_4(t)dt}{N^+(t)\lambda^-(t)}}. \end{aligned} \tag{4.39}$$

In all cases

$$A = \frac{D - C}{2} \quad \text{and} \quad B = -\frac{D + C}{2}. \tag{4.40}$$

We have given a method for computing the functions f and g and the constants A and B, all of which appear in (4.18). Since f and g are analytic in z for z satisfying $|z| > 0$ and outside $[-1,0]$, we can write ϕ_1 in (4.16) as

$$\phi_1(z) = N(z)\int_0^1 \frac{\Psi(t)}{N^+(t)\lambda^-(t)}\left[\frac{f(t)-f(z)}{t-z}+\frac{f(z)}{t-z}\right]dt$$

for $|z| > 0$ and outside the interval $[-1,1]$. By Theorem 3 we have for (i) of (2.6)

$$\phi_1(z) = \frac{f(z)}{z}\left[\frac{N(z)}{N(0)}-1\right] + N(z)\int_0^1 \frac{\Psi(t)}{N^+(t)\lambda^-(t)}\frac{f(t)-f(z)}{t-z}dt, \tag{4.41}$$

and for (ii) of (2.6),

$$\begin{aligned}\phi_1(z) &= \frac{f(z)}{z}\left[\frac{N(z)}{N(0)} - zN(z) - 1\right] \\ &\quad + N(z)\int_0^1 \frac{\Psi(t)}{N^+(t)\lambda^-(t)}\frac{f(t)-f(z)}{t-z}dt.\end{aligned} \tag{4.42}$$

Similar expressions hold for ϕ_2, in (4.16) with f replaced by g.

By an application of Plemelj's formula to ϕ_i we can compute ϕ_i^+ and ϕ_i^-, $i = 1,2$. When used in (4.18) we obtain formulae for the values of the X and Y functions on the interval $[0,1]$ determined by the solution to (4.3) which is analytic in the domain $|z| > 0$. The computations of these functions has then been reduced to the solution of Fredholm equations by iteration and to the computation of certain quadratures.

5. **The H functions.** We obtain from the results of the previous section an interesting new formula for computing H functions. This amounts to a reduction of the formula given by Fox [5], who used singular integral equation theory to study the H functions, but who made an incorrect correspondence between the mathematical and physical problems.

We consider $\text{Re}(\mu) > 0$ and let τ tend to $+\infty$ in (4.42) to find

$$h_1(\mu) = h_3(\mu) = 1,$$
$$h_2(\mu) = h_4(\mu) = 0 \text{ for } 0 \leqq \mu \leqq 1.$$

Then in (4.38) and (4.39), we let τ tend to $+\infty$ and use (4.26) and (4.27) to compute that for (ii) of (2.6)

$$A = 1 - \frac{k}{N(0)} \quad \text{and} \quad B = 0,$$

covering both the conservative and nonconservative cases. For (i) of (2.6) we have

$$A = B = 0.$$

In (4.19) we use the fact that $g \equiv 0$ to conclude that

$$Y(z) = 0 \text{ for } \text{Re}(z) > 0.$$

It is customary to denote the X functions for $\tau = \infty$ and $\text{Re}(z) > 0$ by

$$H(z) \equiv X(z).$$

From (4.41) and the fact that $f(z) = 1$, we obtain

$$\phi_1(z) = \frac{1}{z}\left[\frac{N(z)}{N(0)} - 1\right] \text{ for (i) of (2.6),}$$

and

$$\phi_1(z) = \frac{1}{z}\left[\frac{N(z)}{N(0)} - z\,N(z) - 1\right] \text{ for (ii) of (2.6).}$$

Putting these in (4.19), we find

$$H(z) = \frac{N(z)}{N(0)\lambda(z)} \text{ for (i) of (2.6),} \tag{5.1}$$

and

$$H(z) = \frac{N(z)(1-kz)}{N(0)\lambda(z)} \text{ for (ii) of (2.6).} \tag{5.2}$$

It is well known [1, p. 16] that H can be continued to $\text{Re}(z) < 0$, and that for $z \notin [-1,1]$ and $kz \neq \pm 1$

$$\lambda(z)H(z)H(-z) = 1.$$

This, together with (5.1) and (5.2) gives the following analytic expressions for the H function, analytic in $\text{Re}(z) > 0$,

$$H(z) = \exp\left[z\int_0^1 \frac{\theta(t)}{t(t+z)}dt\right] \text{ for (i) of (2.6)} \tag{5.3}$$

and

$$H(z) = \frac{1+z}{1+kz}\exp\left[z\int_0^1 \frac{\theta(t)}{t(t+z)}dt\right] \text{ for (ii) of (2.6).} \tag{5.4}$$

These formulae are useful in giving by (5.1) and (5.2) meromorphic expressions for $N(z)/\lambda(z)$. For example, this can be used with (5.3), (4.41) and (4.19) to give an analytic expression for X for (i) of (2.6) as

$$X(z) = H(z)\left\{f(z) + zN(0)\int_0^1 \frac{\Psi(t)[f(t) - f(z)]}{H(t)[\lambda_0^2(t) + (\pi t\Psi(t))^2](t-z)}dt\right\}. \tag{5.5}$$

Similar expressions hold for (ii) of (2.6) as well as for the Y function. We can also use (4.12), (5.1) and (5.2) to determine the function $N^+\lambda^-$.

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