FACTORIZATION OF DIFFERENTIABLE MAPS WITH BRANCH SET DIMENSION AT MOST $n-3$[1]

BY

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1. **Introduction.** Let M^n and N^n be separable n-manifolds (without boundary, unless otherwise specified). The *branch set* B_f of a map $f: M^n \to N^n$ is the set of points in M^n at which f fails to be a local homeomorphism. The map f is *monotone* if, for each $y \in N^n$, $f^{-1}(y)$ is a continuum, and f is *proper* if, for each compact set $X \subset N^n$, $f^{-1}(X)$ is compact.

1.1. THEOREM. *Let M^n and N^n be connected C^m manifolds ($m \geqq 3$), and let $f: M^n \to N^n$ be C^m and proper with $\dim(B_f) \leqq n-3$. Then there is a factorization $f = hg$ such that*

(1) *$g: M^n \to K^n$ is a C^m monotone map onto the C^m n-manifold K^n; and*

(2) *$h: K^n \to N^n$ is a k-to-1 C^m diffeo-covering map. Moreover,*

(3) *if $\bar{h}\bar{g}$ is another such factorization with intermediate space L^n, then there is a C^m diffeomorphism α of K^n onto L^n such that $\bar{g} = \alpha g$ and $\bar{h} = h\alpha^{-1}$.*

The differentiability condition C^m may be replaced by C^∞ or real analytic. If M^n and N^n are compact oriented manifolds, then k is the absolute value of the degree of f.

1.2. COROLLARY. *Let M^n and N^n be compact connected oriented C^3 manifolds, and let $f: M^n \to N^n$ be C^3. If N^n is simply connected and degree $f \neq \pm 1$, then $\dim(B_f) \geqq n-2$.*

1.3. *The outline of the proof of* (1.1). In (2.1) it is shown that the existence of the desired factorization is equivalent to two topological properties ((1) and (2)), and the remainder of the paper is devoted to showing that if f satisfies the hypotheses of (1.1), then it satisfies these properties. In (2.4), conclusion (1) is proved in a very special case, and (2.4) is used in (2.8) to deduce property (1) in case $n \geqq 4$ and $f(B_f) \subset f(R_{n-2}(f))$ (definition below). In (3.5) it is shown that, given any map f satisfying the hypotheses of (1.1) (with $n \geqq 4$), there is another map h such that, for each $y \in N^n$, $h^{-1}(y)$ and $f^{-1}(y)$ have the same number of components and h satisfies the hypothesis of (2.8); it follows that f satisfies condition (1). That it satisfies condition (2) is shown in (3.2). The cases $n \leqq 3$ are treated separately in (3.6), and the uniqueness in (1.1) is also given in (3.6).

The set of points $x \in M^n$ at which the Jacobian matrix of f has rank at most q is denoted by $R_q(f)$ or, if there is no ambiguity, by R_q. Coordi-

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nates are written up, e.g., x^i, f^i, and $f|X$ is the restriction of f to the set X. The interior of X is denoted by int X, the boundary by bdy X, and closure by Cl$[X]$ or $\bar{X}$. The distance between x and y is $d(x,y)$, and $S(x,\delta)$ is the sphere about x of radius δ; a map is a continuous function. Čech homology and cohomology are consistently used.

2. **The proof of (1.1) in case $f(B_f) \subset f(R_{n-2})$.**

2.1. LEMMA. *Let M^n and N^n be connected C^m n-manifolds, let $f: M^n \to N^n$ be a proper C^m map, and let $k = 1, 2, \cdots$. Suppose that, for each point $y \in N^n$,*
(1) *$f^{-1}(y)$ has exactly k components, and*
(2) *for each region U and each component V of $f^{-1}(U)$, $f(V) = U$.*
Then f has the factorization of Theorem (1.1).

For M^n compact, a map f satisfying (2) is *quasi-monotone* [24, pp. 151, 152, (8.1)].

Proof. The manifold $N^n = \bigcup_{i=1}^{\infty} X_i$, where X_i is compact and $X_i \subset X_{i+1}$ $(i = 1, 2, \cdots)$. Since f is proper, $f|f^{-1}(X_i)$ has a unique monotone-light factorization [24, p. 141]; it follows that f also is hg, where g is monotone and h is light [24, p. 130, (4.4)]. Given any point $y \in N^n$, there is a connected open set U containing y such that the k components of $f^{-1}(y)$ are in different components of $f^{-1}(U)$. By (1) and (2), $f^{-1}(U)$ has exactly k components. Thus $h^{-1}(U)$ also has k components [24, p. 138, (2.2)] and h is 1-to-1 on each. By the local compactness of M^n and the Theorem on Invariance of Domain, h is a local homeomorphism. Hence h is a k-to-1 covering map [18, p. 128]. A natural C^m structure is thus induced by h on $K^n = g(M^n)$, so that g is C^m and h is a C^m diffeo-covering map.

2.2. LEMMA. *Let $f: M^2 \to N^2$ be a nonconstant map, where M^2 and N^2 are connected compact 2-manifolds, possibly with boundary. Suppose that*

$$\dim(f(B_f)) \leqq 0, \qquad f(B_f) \cap \mathrm{bdy}(N^2) = \emptyset, \qquad f|[M^2 - f^{-1}(f(B_f))]$$

is a k-to-1 covering map, and $H^1(f^{-1}(y); Z_2) = 0$ for each $y \in N^2$. Then there exists a unique factorization $f = hg$, where g is a monotone map of M^2 onto M^2 and h is a light open map of M^2 onto N^2.

Proof. Since f is nonconstant and $\dim(f(B_f)) \leqq 0$, $f(M^2)$ meets $N^2 - f(B_f)$; from the covering property of f, it follows that f is quasi-monotone [24, p. 152, (8.1)]. Let hg be the monotone-light open factorization [24, p. 153, (8.4)] of f; it is unique [24, p. 141, (4.1)]. Since $h(g(M^2))$ is both open in N^2 and compact, and since N^2 is connected, h is onto. It suffices to prove that $g(M^2)$ is (homeomorphic to) M^2.

Each component K of bdy(M^2) is a simple closed curve; by the acyclic condition, $f(K)$ is a nondegenerate continuum. Since $\dim(f(B_f)) \leqq 0$, it follows from the covering property that $f(K) \subset \mathrm{bdy}(N^2)$; thus $\mathrm{int}(M^2) \supset f^{-1}(\mathrm{int}(N^2))$. Since $f(B_f) \cap \mathrm{bdy}(N^2) = \emptyset$, $\mathrm{int}(M^2) = f^{-1}(\mathrm{int}(N^2))$. Hence, for each $y \in g(\mathrm{int}(M^2))$, $g^{-1}(y) \in \mathrm{int}(M^2)$, so that $g|\mathrm{int}(M^2)$ is acyclic mod 2. Since h is light, $\dim(g(M^2)) \leqq 2$ [10, pp. 91-92, Theorem VI. 7]. Since any manifold is an orientable gm (generalized manifold) mod 2, it now follows that $g(\mathrm{int}(M^2))$ is a 2-gm [25, p. 22], and thus is a 2-manifold [26, pp. 271-280]. Hence, $g(M^2)$ is a 2-manifold with boundary (and, in fact, $H^i(M^2; Z_2) \approx H^i(g(M^2); Z_2)$ $(i = 0, 1, 2)$ by the Vietoris Mapping Theorem [2]).

Since B_h consists of a finite set of points [24, p. 198, (5.1)], there exists a finite set of mutually disjoint closed 2-cells E_j $(j = 1, 2, \cdots, m)$ such that $\mathrm{bdy}(E_j) \cap f(B_f) = \emptyset$, $f(B_f) \subset \bigcup_{j=1}^m \mathrm{int}(E_j)$, and each component of $h^{-1}(E_j)$ is again a closed 2-cell containing at most one point of B_h [24, p. 198]. If L is a component of $h^{-1}(E_j)$, then $g^{-1}(L)$ is a 2-manifold with boundary and $g|g^{-1}(L)$ is acyclic mod 2. By [2], $H^1(g^{-1}(L); Z_2) = 0$, so that $g^{-1}(L)$ is a closed 2-cell. Thus M^2 is homeomorphic to $g(M^2)$.

2.3. **Lemma.** *Let M^n and N^n be compact connected n-manifolds, possibly with boundary, and let $n \geqq 2$. Let $f: M^n \to N^n$ be a map with $\dim(f(B_f)) \leqq 0$, $f(B_f) \subset \mathrm{int}(N^n)$, and $f^{-1}(y)$ having at most k components for each $y \in N^n$. Then for all but at most $k + r$ points y, where $r = \dim(H^{n-1}(M^n; Z_p))$ and p is any prime, each component A of $f^{-1}(y)$ has $H^{n-1}(A; Z_p) = 0$ (and, if $n = 2$, is acyclic mod p). If $\mathrm{bdy}(N^n) \neq \emptyset$ and f is not constant, then there are at most r such exceptional points.*

Proof. Suppose that there are (at least) m points y such that

$$H^{n-1}(f^{-1}(y);\ Z_p) \neq 0.$$

If Γ is the union of these sets $f^{-1}(y)$, then $\dim(H^{n-1}(\Gamma;\ Z_p)) \geqq m$. From the exactness of the cohomology sequence, it follows that

$$\dim(H^n(M^n, \Gamma; Z_p)) \geqq m - r.$$

Now $f(\Gamma) \subset f(B_f)$, and the hypotheses imply that $f|f^{-1}(N^n - f(B_f))$ is a covering map [18, p. 128] of degree at most k (Γ may be M^n itself). Suppose that f is not constant; since $\dim(f(B_f)) \leqq 0$, each component L of $f^{-1}(N^n - f(\Gamma))$ has $f(L) \cap (N^n - f(B_f)) \neq \emptyset$. Hence each component of $f^{-1}(N^n - f(\Gamma))$ is mapped by f onto $N^n - f(\Gamma)$, so that $M^n - \Gamma$ has at most k components and $\dim(H^n(M^n, \Gamma; Z_p)) \leqq k$. Thus $m \leqq k + r$. The same conclusion is immediate if f is constant.

If $\mathrm{bdy}(N^n) \neq \emptyset$ and f is not constant, then (since $f(B_f) \subset \mathrm{int}(N^n)$) each component of $M^n - \Gamma$ meets the (nonempty) set $\mathrm{bdy}(M^n)$ so that $H^n(M^n, \Gamma; Z_p) = 0$. Thus $m \leqq r$.

2.4. **Lemma.** *Let $F: M^2 \times E^{n-2} \to N^2 \times E^{n-2}$ be a C^2 map such that $F^2(x, t) = t$, where M^2 and N^2 are compact, connected 2-manifolds (possibly with boundary), $n \geqq 3$, $\dim(B_F) \leqq n - 3$, and*

$$F(B_F) \subset F(R_{n-2}(F)) \cap (\mathrm{int}(N^2) \times E^{n-2}).$$

Suppose further that $F|[(M^2 \times E^{n-2}) - F^{-1}(F(B_F))]$ *is a* k*-to-*1 *covering map. If* $\pi_1(N^2) = 0$, *then* F *is acyclic* mod 2.

(The expression F^2 is the second coordinate of F.)

The lemma is first proved under the additional hypothesis that $H^1(F^{-1}(y,t); Z_2) = 0$ for all $(y,t) \in N^2 \times E^{n-2}$ (yielding F monotone). Then, from this case and (2.3), which shows that "most" points (y,t) satisfy this condition, the general result is finally obtained.

Proof. It follows from [5, (1.1)] that if F_t is the map $F|(M^2 \times \{t\})$, then

$$F(R_{n-2}(F)) \cap (N^2 \times \{t\}) = F_t(R_0(F_t))$$

for all $t \in E^{n-2}$, and thus [3, (1.3)] has dimension at most 0. Hence

(1) $\dim(F(B_F) \cap (N^2 \times \{t\})) \leqq 0$ for each $t \in E^{n-2}$.

Note that the branch set of F_t is contained in B_F. From (1) and the covering property of F $(k \geqq 1)$ it follows that

(2) each F_t is onto.

Suppose that D is a closed 2-cell in N^2 and, for some $s \in E^{n-2}$ and $\nu > 0$, $F(B_F) \cap ((\text{bdy}\, D) \times S(s,\nu)) = \emptyset$; let K be a component of $F^{-1}(D \times S(s,\nu))$. From the covering property of F it follows that each component of $F^{-1}(\text{bdy} D \times S(s,\nu))$ is homeomorphic to $S^1 \times E^{n-2}$, where S^1 is a circle; and from (2) that $F(K) = D \times S(s,\nu)$. As a result

(3) there is a connected 2-manifold-with-boundary L^2 and a homeomorphism μ of K onto $L^2 \times S(s,\nu)$ such that the restriction

$$\mu|(K \cap (M^2 \times \{t\})) = L^2 \times \{t\}$$

for each $t \in S(s,\nu)$.

First case. Suppose that $H^1(F^{-1}(y,t); Z_2) = 0$ *for each* $(y,t) \in N^2 \times E^{n-2}$. Thus, each component of $F^{-1}(y,t)$ is acyclic mod 2. In this case it suffices to prove that F is monotone.

Let H_tG_t be the factorization of F_t given by (2.2) (cf. (2)). Suppose that, for some $t \in E^{n-2}$, H_t has no branch point. Since H_t is a covering map [18, p. 128] and $\pi_1(N^2) = 0$, H_t is a homeomorphism; thus F_t is monotone and $k = 1$. From (1) and the covering property of F, it follows that F_s is also monotone for each $s \in E^{n-2}$. Thus,

(4) if, for some $t \in E^{n-2}$, H_t has no branch point, then F is monotone, the desired conclusion.

It follows from [24, p. 198] that H_t is topologically equivalent to a simplicial map. By [22] the number $\lambda(t)$ of branch points of H_t is at most $k\chi(N^2) - \chi(M^2)$, where χ is the Euler characteristic. Choose $\bar{t} \in E^{n-2}$ such that $\lambda(\bar{t})$ is a maximum. By (1) there exists a mutually disjoint finite family of closed 2-cells $D_j \subset \text{int}(N^2)$ $(j = 1, 2, \cdots, J)$ such that

(a) $F(B_F) \cap (\text{bdy}(D_j) \times \{\bar{t}\}) = \emptyset$,

(b) $F(B_F) \cap (N^2 \times \{\bar{t}\}) \subset \bigcup_{j=1}^{J}(\text{int}(D_j) \times \{\bar{t}\})$, and

(c) each component of $H_{\bar{t}}^{-1}(D_j \times \{\bar{t}\})$ is a closed 2-cell containing at most one branch point of $H_{\bar{t}}$ [22, p. 198]. (Note that F_t, and thus H_t, is locally a homeomorphism on a neighborhood of $F_t^{-1}(\mathrm{bdy}(N^2) \times \{t\})$.) From (c) and (2.2) applied to $F_{\bar{t}}|F_{\bar{t}}^{-1}(D_j \times \{\bar{t}\})$, each component of $F_{\bar{t}}^{-1}(D_j \times \{t\})$ is a closed 2-cell.

There exists a $\delta > 0$ such that each t in $S(\bar{t}, \delta)$ also satisfies condition (a) and (b). Let K_m ($m = 1, 2, \cdots, \lambda(\bar{t})$) be those components of $F^{-1}(D_j \times S(\bar{t}, \delta))$ ($j = 1, 2, \cdots, J$) such that $G_{\bar{t}}(K_m \cap (M^2 \times \{\bar{t}\}))$ has a branch point of $H_{\bar{t}}$ (and only one by (3) and condition (c)). By (4) we may suppose that $H_{\bar{t}}$ has a branch point, i.e., $\lambda(\bar{t}) \geqq 1$. Since $\pi_1(D_j) = 0$, the argument of (4) may be applied to the restriction $F|K_m$. Since $F|K_m$ is not monotone, $G_t(K_m \cap (M^2 \times \{t\}))$ has at least one branch point of H_t for each $t \in S(\bar{t}, \delta)$; by the maximality of $\lambda(\bar{t})$, it has precisely one ($m = 1, 2, \cdots$).

Let D_j be the 2-cell such that $F(K_1) = D_j \times S(\bar{t}, \delta)$, and for each $t \in S(\bar{t}, \delta)$ let $\alpha(t)$ be the image under H_t of the (unique) branch point of H_t in $G_t(K_1 \cap (M^2 \times \{t\}))$. The function

$$\alpha : S(\bar{t}, \delta) \to \mathrm{int}(D_j) \times S(\bar{t}, \delta)$$

is 1-to-1, and (since the union of the branch sets of the H_t is closed) α is continuous. Thus the image of α is a (tame) $(n-2)$-cell Ω. Let Ψ be $F^{-1}(\Omega) \cap K_1$, let $\hat{\Omega}$ and $\hat{\Psi}$ be the one-point compactifications of these spaces, and let ζ: $\hat{\Psi} \to \hat{\Omega}$ be the natural extension of $F|\Psi$. By our assumption, ζ is acyclic mod 2, and, by the Vietoris Mapping Theorem [2], $H_{n-2}(\hat{\Psi}; Z_2) = Z_2$. Thus $\dim\Psi \geqq n-2$ [10, p. 137, (F)], so that $\dim(B_F) \geqq n-2$, yielding a contradiction. Thus F is monotone, and hence acyclic mod 2.

Second case. Thus, we may suppose that *there exists a point* $(y, t) \in N^2 \times E^{n-2}$ *such that* $H^1(F^{-1}(y, t); Z_2) \neq 0$. By (2.3), for each $t \in E^{n-2}$ there are at most $k + r$ such points $y \in N^2$, where r is the dimension of $H^1(M^2; Z_2)$. Choose $\bar{t}$ such that the number of such points is maximal, and call the points y_i ($i = 1, 2, \cdots, m; m \geqq 1$). There exist mutually disjoint closed 2-cells D_i such that $y_i \in \mathrm{int}(D_i)$ and (by (1))

$$F(B_F) \cap (\mathrm{bdy}(D_i) \times \{t\}) = \emptyset.$$

Choose $\delta > 0$ such that

$$F(B_F) \cap (\mathrm{bdy}(D_i) \times S(\bar{t}, \delta)) = \emptyset \qquad (i = 1, 2, \cdots, m).$$

Let K_i be a component of $F^{-1}(D_i \times S(\bar{t}, \delta))$ such that

$$H^1(F^{-1}(y_i, \bar{t}) \cap K_i; Z_2) \neq 0,$$

and let $K_i(t) = K_i \cap F^{-1}(D_i \times \{t\})$. Suppose that for some i and $s \in S(\bar{t}, \delta)$, each $y \in \mathrm{int}(D_i)$ has $H^1((F|K_i)^{-1}(y, s); Z_2) = 0$. Let f be the restriction $F|K_i(s)$; by (2), f is nonconstant. Let hg be the factorization given by (2.2).

Suppose first that h has no branch points. It follows as in (4) that f is monotone, and (by(2.2)) $K_i(s)$ is a closed 2-cell. From (3), $K_i(\bar{t})$ is also a closed 2-cell, and from (2.3) applied to the restriction map $F|K_i(\bar{t})$, a contradiction of the choice of K_i results.

Thus we may suppose that there exists $x \in B_h$. There exists a closed 2-cell $E \subset D_i$ such that $h(x) \in E \times \{s\}$ and the component E' of $h^{-1}(E \times \{s\})$ containing x is a closed 2-cell containing no other point of B_h [24, p. 198]; by (1), we may suppose that $F(B_F) \cap ((\text{bdy}\, E) \times \{s\}) = \emptyset$. By (2.2) applied to $F|g^{-1}(E')$, $g^{-1}(E')$ is a closed 2-cell. Choose $\xi > 0$ such that $S(s, \xi) \subset S(\bar{t}, \delta)$ and $F(B_F) \cap ((\text{bdy}\, E) \times S(s, \xi)) = \emptyset$. Let U be the component of $F^{-1}(E \times S(s, \xi))$ containing $g^{-1}(E')$; then $U \cap (M^2 \times \{t\})$ is a closed 2-cell for each $t \in S(s, \xi)$ (by(3)). It follows from (2.3) that each component in U of $f^{-1}(y, t)$ is acyclic mod 2 for $y \in E$. By the first part of this proof, $F|U$ is monotone, contradicting the choice of E and E'.

(5) Thus, there is no such s; hence for each $t \in S(\bar{t}, \delta)$, there is at least one point $\alpha_i(t) \in \text{int}(D_i) \times \{t\}$ with $H^1(K_i \cap F^{-1}(\alpha_i(t)); Z_2) \neq 0$.

By the choice of $\bar{t}$, there is precisely one. The function

$$\alpha_i : S(\bar{t}, \delta) \to \text{int}(D_i) \times S(t, \delta)$$

is 1-to-1 and continuous. Thus, if D_i is assumed to be the closed unit disk, then there is a homeomorphism ϕ of $D_i \times S(\bar{t}, \delta)$ onto itself with $\phi^2(x, t) = t$ and $\phi(\alpha(S(\bar{t}, \delta))) = \{0\} \times S(\bar{t}, \delta)$.

Let P and Q be the one-point compactifications of K_i and $K_i \cap (\phi F)^{-1}(\{0\} \times S(\bar{t}, \delta))$, respectively. It is immediate that

$$H^{n-1}(P; Z_2) \approx H^{n-1}(P, \{p\}; Z_2),$$

where p is the added point; since K_i is homeomorphic to $L^2 \times E^{n-2}$ (by (3)), $H^{n-1}(P, \{p\}; Z_2)$ is isomorphic to $H^{n-1}(L^2 \times S^{n-2}, L^2 \times \{z\}; Z_2)$, where z is any point of S^{n-2}. By the exactness of the cohomology sequence ($n \geqq 3$), this is mapped onto $H^{n-1}(L^2 \times S^{n-2}; Z_2)$, which, by the Künneth Formula, is isomorphic to $H^1(L^2; Z_2)$. From (5) and (2.3) it follows that $H^1(L^2; Z_2) \neq 0$; thus $H^{n-1}(P; Z_2) \neq 0$.

Given any closed 2-cell $E \subset D_i - \{0\}$ with $[(\text{bdy}\, E) \times \{t\}] \cap \phi F(B_F) = \emptyset$, there exists $\xi > 0$ such that, for each $s \in S(t, \xi)$, $[(\text{bdy}\, E) \times \{s\}] \cap \phi F(B_F) = \emptyset$ also. Since $H^1((\phi F)^{-1}(y, s); Z_2) = 0$ for each $y \in D_i - \{0\}$, it follows from the first part of this proof that

(6) F is monotone on each component of $K_i \cap (\phi F)^{-1}(E \times S(t, \xi))$.

Let V be a closed 2-cell such that $V \subset \text{int}(D_i)$, $0 \in \text{int}\, V$, and

$$[(\text{bdy}\, V) \times \{t\}] \cap \phi F(B_F) = \emptyset,$$

and let W be a component of $K_i \cap (\phi F)^{-1}((D_i - \text{int}\, V) \times \{t\})$. In the factorization given by (2.2) for $f = \phi F|W$, h has no branch points (from (6)), and so is a finite-to-one covering map [16, p. 128]. Since each finite covering space of the annular region $(D_i - \text{int}\, V) \times \{t\}$ is itself, (2.2) implies that W is homeomorphic to $(D_i - \operatorname{int} V) \times \{t\}$. Since V may be chosen arbitrarily small, each component of $K_i \cap (\phi F)^{-1}((D_i - \{0\}) \times \{t\})$ is homeomorphic to $D_i - \{0\}$. It follows that $P - Q$ is homeomorphic to $(D_i - \{0\}) \times S(\bar{t}, \delta)$. Since $H^{n-1}(P; Z_2) \neq 0$, $H^{n-1}(Q; Z_2) \neq 0$ from the continuity of Čech cohomology.

Thus, $\dim Q \geqq n - 1$ [10, p. 137, (F)], and $K_i \cap (\phi F)^{-1}(\{0\} \times S(\bar{t}, \delta))$ has dimension at least n-1. On the other hand, since

$$H^1(K_i \cap (\phi F)^{-1}(0, t); Z_2) \neq 0$$

for all $t \in S(\bar{t}, \delta)$ (by (5) and the definition of ϕ), $K_i \cap (\phi F)^{-1}(\{0\} \times S(\bar{t}, \delta)) \subset B_f$, yielding a contradiction.

2.5. **Lemma.** *Let N^p be a C^∞ manifold, and let Γ^{p-q} be a C^r submanifold of N^p $(r = 1, 2, \cdots)$. Then there is a C^r diffeomorphism ψ of N^p onto itself such that $\psi(\Gamma^{p-q})$ is a C^∞ submanifold of N^{p-q} and ψ is arbitrarily near (in the fine C^r topology) the identity on N^p.*

Proof. The fine C^r topology is defined in [15, p. 25 and p. 28]. (The C^r topology of [16] is the coarse C^r topology of [15].) In [15, p. 35, (4.1)] a slight modification of the proof shows that f_1 may be chosen to approximate f in the C^r topology, i.e., the partial derivatives of f_1 of orders at most r approximate those of f.

In the proof of [15, p. 40, (4.7)] let

$$\psi(x) = x + \tilde{g}(\pi(x)) - g(\pi(x))$$

for $x \in \pi^{-1}(\mathscr{O})$, and $\psi(x) = x$ elsewhere. If δ is sufficiently small, then ψ is a C^r diffeomorphism near the identity [15, p. 29, (3.10)]; since $\psi g = \tilde{g}$, it follows that the $\psi f = h$.

The remainder of the proof is an analog of that of [15, pp. 41, 42, (4.8) and Exercise (a)], where $f_j = \psi_j f_{j-1}$ $(j = 1, 2, \cdots)$ and the diffeomorphism ψ is the (well-defined) limit, as $j \to \infty$, of the composition $\psi_j \psi_{j-1} \cdots \psi_2 \psi_1$.

Morse in [14] proved (2.5) in the special case that $N^p = E^p$, Γ^{p-q} is compact, and $q = 1$. Assuming only that N^p is a C^r manifold, Cerf [3, p. 260] proved that there exist a C^∞ manifold Q^p and a C^r diffeomorphism λ of N^p onto Q^p such that $\lambda(\Gamma^{p-q})$ is a C^∞ submanifold of Q^p.

2.6. **Remark.** The differentiability hypothesis in [22, p. 26, Theorem I. 5] can be changed from C^n to $C^{\max(n-q+1,1)}$. (In the proof, replace the theorem of A. P. Morse [22, p. 240] by Sard's Theorem [20].) Moreover, given any m $(m = 1, 2, \cdots)$ the diffeomorphism A in [22, p. 26] may be chosen C^m.

In [22, p. 27] the hypothesis C^1 suffices, and V^n need not be compact if f is proper.

The following remark follows from the proof of [22, p. 27].

2.7. **Remark.** *Let $f: M^n \to N^n$ be C^m and proper, let K^p be a compact C^m*

p-manifold, and let ρ be a C^m diffeomorphism of a region in N^n onto $K^p \times E^{n-p}$ $(m,p = 1,2,\cdots)$. If f is transverse regular [22, pp. 22-23] *on $\rho^{-1}(K^p \times \{0\})$, then there are a C^∞ p-manifold L^p, $\epsilon > 0$, and a C^m diffeomorphism σ of $L^p \times S(0,\epsilon)$ onto $f^{-1}(\rho^{-1}(K^p \times S(0,\epsilon)))$ such that $(\rho f \sigma)^2(z,r) = r$ for each $r \in S(0,\epsilon)$.*

Proof. Since f is transverse regular on $\rho^{-1}(K^p \times \{0\})$, $f^{-1}(\rho^{-1}(K^p \times \{0\}))$ is a C^m p-manifold in M^n [22, p. 23] (the hypothesis C^1 suffices). By [15, p. 42, (4.9)] and [3, p. 260] there is a C^m diffeomorphism ψ of M^n onto a C^∞ n-manifold Q^n such that $\psi(f^{-1}(\rho^{-1}(K^p \times \{0\})))$ is a C^∞ submanifold, call it L^p, of Q^n. Choose $\epsilon > 0$ so that $f\psi^{-1}$ is transverse regular on $\rho^{-1}(K^p \times \{r\})$ for each $r \in S(0,\epsilon)$ [22, p. 27]; thus [20, p. 26] $\psi f^{-1}(\rho^{-1}(K^p \times \{r\}))$ is a C^m p-manifold. By the proof of [22, p. 27], for each point x of L^p, the normal $(n-p)$-plane to L^p at x meets $\psi(f^{-1}(\rho^{-1}(K^p \times \{r\})))$ in precisely one point, call it $\tau(x,r)$; thus τ is a C^m diffeomorphism of $L^p \times S(0,\epsilon)$ onto $\psi f^{-1}(\rho^{-1}(K \times S(0,\epsilon)))$. Let $\sigma = \psi^{-1}\tau$. (For each $\bar{x} \in f^{-1}(\rho^{-1}(K^p \times S(0,\epsilon)))$, there is a C^m diffeomorphism μ of a neighborhood V of $f(\bar{x})$ onto E^n such that $\mu(V \cap \rho^{-1}(K^p \times \{r\}))$, for each $r \in E^{n-p}$, is defined by x^i constant $(i = 1,2,\cdots,n-p)$. Perhaps it is clearer that both σ and σ^{-1} are C^m if we observe [5, (1.1)] and [6, (2.3)] that there is a neighborhood U of $\bar{x}$ and a C^m diffeomorphism λ of U onto E^n such that the map $g = \mu f \lambda^{-1}$ satisfies $g^i(x^1,x^2,\cdots,x^n) = x^i$ $(i = 1,2,\cdots,n-p)$.)

2.8. Lemma. *Let M^n and N^n be connected C^3 n-manifolds, and let $f: M^n \to N^n$ be C^3 and proper, with $n \geqq 4$, $\dim(B_f) \leqq n-3$ and $f(B_f) \subset f(R_{n-2}(f))$. Then f satisfies hypotheses* (1) *and* (2) *of* (2.1).

Proof. By [19, §5], $\dim(f(R_{n-2})) \leqq n-2$, so that $\dim(f^{-1}(f(R_{n-2}))) \leqq n-2$. Since f is proper, and R_{n-2} is locally compact, $f(R_{n-2})$ is locally compact. The restriction map $f|[M^n - f^{-1}(f(R_{n-2}))]$ is a k-to-1 covering map [18, p. 128] with connected domain, so that f satisfies (2). Suppose that, for some point y in N^n, $f^{-1}(y)$ has at least $k+1$ components Y_i $(i = 1,2,\cdots,k+1)$. There is a connected open neighborhood W about y such that the components Y_i are contained in different components of $f^{-1}(W)$. A contradiction results from (2) and the covering property. Thus, for each $y \in N^n$, $f^{-1}(y)$ has at most k components.

If, for each y in N^n, there is a Euclidean coordinate neighborhood E about y such that $f^{-1}(E)$ has exactly k components, then, by (2), condition (1) is satisfied. Thus we may suppose that $N^n = E$ and $k \neq 1$, and deduce a contradiction. By [15, p. 41] we may suppose that M^n and N^n are C^∞ manifolds, and thus that $E^n = E$ $(= N^n)$.

The homomorphism f_* of $\pi_1(M^n - f^{-1}(f(R_{n-2})))$ into $\pi_1(E^n - f(R_{n-2}))$ induced by $f|[M^n - f^{-1}(f(R_{n-2}))]$ is one-to-one but not onto [9, pp. 93 and 96]. Since the semi-linear maps of the circle S^1 into $E^n - f(R_{n-2})$

generate $\pi_1(E^n - f(R_{n-2}))$ and $n \geqq 3$, the (polyhedral) embeddings of S^1 also generate $\pi_1(E^n - f(R_{n-2}))$. Let γ be an embedding such that homotopy class of γ is not in $f_*(\pi_1(M^n - f^{-1}(f(R_{n-2}))))$; each component of $f^{-1}(\gamma(S^1))$ is a topological circle (by the covering property), and each is mapped by f onto $\gamma(S^1)$ nonhomeomorphically. Moreover, there exists $\epsilon > 0$ such that, if λ is any embedding of S^1 in E^n with (uniform) distance $d(\lambda, \gamma) < \epsilon$, then $\lambda(S^1)$ has the same property.

By [8, p. 111, Theorem 2a], there is a C^∞ embedding λ of S^1 in E^n $(n \geqq 4)$ such that $d(\lambda, \gamma) < \epsilon/2$. By [8, p. 110, Corollary] there is a C^∞ embedding μ of the unit 2-disk D^2 in E^n which extends λ. If D^j denotes the unit j-disk in $E^j \subset E^n$, $D^2 \subset D^3 \subset D^n$, there is [15, p. 275] a C^∞ embedding ν of D^n into E^n which extends μ. Since $\mathrm{bdy}(D^2) \subset \mathrm{bdy}(D^3) = S^2$, $\xi = \nu | S^2$ is a C^∞ embedding which extends λ.

Let A be the (C^3) map given by [22, p. 26] (cf. 2.6), with (uniform) distance from the identity map less than $\epsilon/2$, for f and $N^{p-q} = \xi(S^2)$. By the proof of the theorem, f is transverse regular [22, p. 22-23] on $A^{-1}(\xi(S^2))$, call it X^2. Because ξ is the restriction of ν, X^2 has a tubular neighborhood Z and a C^3 diffeomorphism ρ of Z onto $X^2 \times E^{n-2}$.

By 2.7, if Z is a sufficiently small tubular neighborhood, the restriction of f to each component U of $f^{-1}(Z)$ satisfies the hypotheses of 2.4; thus the restriction $f|U$ is monotone. Because of the choice of $A^{-1}(\lambda(S^1))$, each component of $f^{-1}(A^{-1}(\lambda(S^1)))$ is a topological circle mapped by f onto the topological circle $A^{-1}(\lambda(S^1))$ nonhomeomorphically; from condition (2), $f(U) = Z$, and a contradiction results. Thus f *does* satisfy hypothesis (1) of 2.1.

3. **The proof of (1.1).** Given $U \subset E^n$ and a C^1 map $h: U \rightarrow E^1$, $D_j h$ denotes the first partial derivative of h with respect to the jth independent variable. If $n = 1$ and h is C^j, then $D^{(j)}h$ is the derivative of order j $(D^{(0)}h = h)$.

3.1. LEMMA. *Let $f: M^n \rightarrow N^n$ be C^m and proper, with $\dim(B_f) \leqq n-2$ and $m = 2, 3, \cdots$. Suppose that the Jacobian matrix (derivative map) has rank at least n-1 at every point of M^n. Then,*

(1) *for each point y in N^n, $f^{-1}(y)$ has a finite number of components, each of which is either a point or a C^m embedding of a closed interval. Moreover,*

(2) *for each open neighborhood U of y, there is a region W such that $y \in W \subset U$ and each component of $f^{-1}(W)$ which meets $f^{-1}(y)$ is mapped by f onto W.*

Proof. We may suppose that $n \geqq 2$ [18, p. 128], M^n *and* N^n are C^∞ Riemannian manifolds [15, p. 42, (4.9)], and, in fact, that $N^n = E^n$. For each point $\bar{x} \in f^{-1}(y)$, there are [5, (1.1)] C^m diffeomorphisms λ of a Euclidean neighborhood $U(\bar{x})$ of $\bar{x}$ onto E^n and μ of E^n onto itself (the latter merely interchanging dependent variables [5, (2.3)]) such that the map $g = \mu f \lambda^{-1}$ has $g^i(x^1, x^2, \cdots, x^n) = x^i$ $(i = 1, 2, \cdots, n\text{-}1)$. Since $\dim(B_f) \leqq n-2$, the Jacobian determinant of g is either non-negative or nonpositive. Thus $D_n g^n(x^1, x^2, \cdots, x^n) \geqq 0$ (say) on all of $\lambda(U(\bar{x}))$, so that the map $h: E^1 \to E^1$ defined by $g^n(x^1, x^2, \cdots, x^n) = h(x^n)$ is monotone, for all $(x^1, x^2, \cdots, x^n) \in E^n$. As a result, since f is proper, $f^{-1}(y)$ has a finite number of components, each a point, a C^m embedding of a closed interval, or a C^m embedding of the circle S^1.

Suppose that Ω is any component of $f^{-1}(y)$ and that U is any open set containing y. There is an open n-cell Y, $y \in Y$ and $\bar{Y} \subset U$, such that each component of $f^{-1}(\bar{Y})$ meets at most one component of $f^{-1}(y)$; let X be the component of $f^{-1}(Y)$ which contains Ω. There is a C^m embedding α of a 1-manifold M^1 in X such that either $M^1 = S^1$ and $\Omega = \alpha(M^1)$ or $M^1 = E^1$ and $\Omega \subset \alpha(M^1)$. By 2.5 we may suppose that $\alpha(M^1)$ is a C^∞ submanifold of X.

Let $N(\delta)$ consist of the vectors with length at most δ in the normal bundle to $\alpha(M^1)$. For δ sufficiently small the restriction of the exponential map E to $N(\delta)$ is a C^∞ diffeomorphism onto a (tubular) neighborhood of $\alpha(M^1)$. For each $x \in \Omega$, let $D(x)$ be the image under E of the $(n-1)$-disk of vectors normal to $\alpha(M^1)$ at x. From the coordinate representation at x given above, observe that any set of $(n-1)$-vectors spanning an $(n-1)$-plane transversal to $\alpha(M^1)$ at x is mapped by the Jacobian matrix of f onto a set of $n-1$ independent vectors. Thus the Jacobian matrix of the restriction map $f|D(x)$ has maximal rank at x. Moreover, given any $\bar{x} \in \Omega$, if an interval from x to $\bar{x}$ in Ω is contained in the coordinate neighborhood $U(\bar{x})$, then it follows from the form of f on $U(\bar{x})$ (the fact that $D_n g^n \geqq 0$) that $f(D(\bar{x}))$ and $f(D(x))$ have the same tangent $(n-1)$-plane at y. Thus, for all $x \in \Omega$, the sets $f(D(x))$ have the same tangent $(n-1)$-plane at y, which we may suppose is that spanned by the first $n-1$ coordinate vectors of E^n.

There is a C^∞ embedding β of S^1 in Y such that (a) $y \in \beta(S^1)$ and (b) $\beta(S^1)$ is normal to that plane at y. We have observed above that, for any such embedding β, f is transverse regular [22, pp. 22-23] to $\beta(S^1)$ at each point of Ω. In [22, pp. 24-26] (cf. 2.6), the space **H** of C^m maps A may be replaced by that subspace such that $A(y) = y$ and the first partial derivatives of A agree with those of the indentity map at y. As a result, there is a C^m embedding γ of S^1 in Y such that it satisfies conditions (a) and (b), and f is transverse regular at each point of $f^{-1}(\gamma(S^1) - \{y\})$. Thus $f|X$ is transverse regular at each point of $f^{-1}(\gamma(S^1))$. By 2.5 and [22, p. 27], we may suppose that $\gamma(S^1)$ is a C^∞ submanifold of Y.

Define, as above, a tubular neighborhood T of $\gamma(S^1)$, $T \subset Y$. Then there is a C^∞ diffeomorphism ρ of T onto $\gamma(S^1) \times E^{n-1}$ such that $\rho(z) = (z, 0)$ for each $z \in \gamma(S^1)$ (for $n = 2$, the Moebius Band cannot be embedded in E^2; for $n \geqq 3$, by the proof of [11, (1.2)]). Let L^1, $\epsilon > 0$, and σ be as given by 2.7 for $f|(X \cap f^{-1}(T))$ and $K^p = \gamma(S^1)$, and let R be the component of $X \cap f^{-1}\rho^{-1}(\gamma(S^1) \times S(0,\epsilon))$ containing Ω; we will still denote $R \cap f^{-1}\rho^{-1}(\gamma(S^1) \times \{0\})$ by L^1, a 1-sphere by 2.7.

Suppose that Ω is a 1-sphere, i.e., that $f(L^1) = \{y\}$. Given any closed interval Γ of $\gamma(S^1)$, $y \in \operatorname{int}\Gamma$, there exists a sufficiently small neighborhood $S(0,\xi) \subset S(0,\epsilon)$ such that $\rho f \sigma(L^1 \times S(0,\xi)) \subset \Gamma \times S(0,\epsilon)$. Because of the form of f on the neighborhoods $U(x)$ $(D_n g^n \geqq 0)$, it follows that $\rho f \sigma(L^1 \times \{r\})$ is a single point, for each $r \in S(0,\xi)$, contradicting the fact that $\dim(B_f) < n$. Thus Ω is not a 1-sphere, yielding conclusion (1).

From the local form of f $(D_n g^n \geqq 0)$ and the fact that no point inverse is a 1-sphere, $\rho f \sigma|(L^1 \times \{r\})$ is the composition of a monotone map and a finite-to-one covering map, and $\rho f \sigma(L^1 \times \{r\}) = \gamma(S^1) \times \{r\}$. Since Ω $(= f^{-1}(y) \cap R)$ is connected, each map $\rho f \sigma|(L^1 \times \{r\})$ is monotone; thus $f|R$ is monotone, and $f(R) = T$.

For each Ω_i in the finite set of components of $f^{-1}(y)$, there is an open tubular neighborhood T_i of y satisfying the conclusion of 2.7, $T_i \subset Y \subset U$; let R_i be the component of $f^{-1}(T_i)$ containing Ω_i. Since $T_i \subset Y$, the sets R_i are mutually disjoint. By the above paragraph, $f|R_i$ is monotone and $f(R_i) = T_i$. Let W be the component of $\bigcap_i T_i$ containing y. Then $f(R_i \cap f^{-1}(W)) = W$, and $f|(R_i \cap f^{-1}(W))$ is monotone. By [24, p. 138, (2.2)] (it is sufficient that f be proper), $R_i \cap f^{-1}(W)$ is connected, and conclusion (2) follows.

3.2. Corollary. *Let $f: M^n \to N^n$ be a C^2 proper map with $\dim(B_f) \leqq n-2$. Then f satisfies condition* (2) *of* (2.1).

Proof. Given a region $U \subset N^n - f(R_{n-2})$, let V be a component of $f^{-1}(U)$. Suppose that $f(V) \neq U$. Since $V \subset \overline{V} \cap f^{-1}(U) \subset \overline{V}$, $\overline{V} \cap f^{-1}(U)$ is connected and thus is contained in V. Since f is proper it follows that $f(V) = \mathrm{Cl}[f(V)] \cap U$. Thus there exists $y \in U \cap \operatorname{bdy}(f(V))$. Let W be the region given by 3.1, conclusion (2), for U and y. Since $y \in f(V)$, $W \subset f(V)$, and a contradiction results; thus $f(V) = U$.

Since $\dim(f(R_{n-2})) \leqq n-2$ [19, §5] and $\dim(B_f) \leqq n-2$, $\dim(f^{-1}(f(R_{n-2}))) \leqq n-2$; that f satisfies conclusion (2) of 2.1 for arbitrary regions $U \subset N^p$ follows.

3.3. Lemma. *Let E^n_+, E^n_-, and Δ be the sets in E^n defined by $x^n > 0$, $x^n < 0$, and $-\lambda < x^i < \lambda$ $(i = 1,2,\cdots,n-1)$ and $-\mu < x^n < \nu$ $(\lambda,\mu,$ and $\nu > 0)$, respectively. Let U be an open set with $\overline{\Delta} \subset U \subset E^n$, let $\epsilon > 0$, and let $f: U \to E^n$ be C^m $(m = 1,2,\cdots)$ with $f^i = x^i$ $(i = 1,2,\cdots,n-1)$ on $\overline{\Delta}$, $D_n f^n \geqq 0$ on $\overline{\Delta}$, and $D_n f^n > 0$ on $\overline{\Delta} \cap \bar{E}^n_+$. Then there is a C^m map $g: U \to E^n$ such that*

(i) *$f = g$ off Δ,*

(ii) *the (uniform) distance between each partial derivative of f with order $0,1,\cdots,m$ and the corresponding partial of g is at most ϵ,*

(iii) *the Jacobian determinant $J(g) > 0$ on $\overline{\Delta}$,*

(iv) $B_g \subset B_f$, *and*
(v) $g(\bar{\Delta}) = f(\bar{\Delta})$. *Moreover,*
(vi) *for each y in $f(\bar{\Delta})$, $\bar{\Delta} \cap f^{-1}(y)$ and $\bar{\Delta} \cap g^{-1}(y)$ each have one component.*

Proof. We may suppose that $\bar{\Delta}$ is defined by $0 < x^i < 1$ $(i = 1, 2, \cdots, n-1)$ and $-1 < x^n < 1$, that $\epsilon < 1$, and that ϵ is less than the minimum of $D_n f^n$ on $\bar{\Delta} \cap \bar{E}^n_+$. There exists a C^m map $\alpha : E^1 \to E^1$ such that $\alpha \geqq 0$, $\alpha > 0$ precisely on $(0,1)$, and $D^{(j)}\alpha < \epsilon$ $(j = 0, 1, \cdots, m)$. Let $\beta_k : E^n \to E^1$ be defined by

$$\beta_k(x^1, x^2, \cdots, x^n) = \alpha((-1)^k x^n) \prod_{i=1}^{n-1} \alpha(x^i) \qquad (k = 1, 2),$$

let $\gamma = \beta_1$ for $x^n \leqq 0$ and $\gamma = -\beta_2$ for $x^n \geqq 0$, and let

$$\delta = \int_{-1}^{x^n} \gamma(x^1, x^2, \cdots, x^{n-1}, t)\, dt.$$

Let $g^i = f^i$ $(i = 1, 2, \cdots, n-1)$, and $g^n = f^n + \delta$.

Since $\gamma > 0$ on $\Delta \cap E^n_-$ and $D_n\delta = \gamma$, $D_n g^n > 0$ on $\Delta \cap E^n_-$; since $|\gamma| < \epsilon$ on $\Delta \cap E^n_+$ and $\epsilon < D_n f^n$ on $\Delta \cap \bar{E}^n_+$, $D_n g^n > 0$ on $\Delta \cap \bar{E}^n_+$. Conclusion (iii) follows. Since

$$\delta(x^1, x^2, \cdots, x^{n-1}, -1) = \delta(x^1, x^2, \cdots, x^{n-1}, 1) = 0,$$
$$f^n(x^1, x^2, \cdots, x^{n-1}, -1) = g^n(x^1, x^2, \cdots, x^{n-1}, -1)$$

and

$$f^n(x^1, x^2, \cdots, x^{n-1}, 1) = g^n(x^1, x^2, \cdots, x^{n-1}, 1);$$

since $f^i = g^i = x^i$ $(i = 1, 2, \cdots, n-1)$, and since $D_n f^n \geqq 0$ and $D_n g^n \geqq 0$ on $\bar{\Delta}$, $g(\bar{\Delta}) = f(\bar{\Delta})$. The reader may verify that g satisfies the remaining conditions.

3.4 Lemma. *Let M^n and N^p be C^∞ manifolds, and let $\mathfrak{C}^m$ be the set of all C^m maps $g : M^n \to N^p$ $(m = 1, 2, \cdots)$. Then there are an embedding of N^p as a closed submanifold of a Euclidean space, and a complete metric ρ on $\mathfrak{C}^m$ such that:*

(1) *$\rho(g, g_j) \to 0$ implies that $g_j \to g$ in the (coarse) C^m topology; and*

(2) *if d is the metric induced on N^p by the embedding and $\rho(g, h) < 1$, then $d(g(x), h(x)) \leqq \rho(g, h)$ for all $x \in M^n$.*

Proof. The coarse C^m topology is defined in [16] (cf [15, pp. 25-28]). Let $T_1(M^n)$ be the tangent bundle of M^n, let $T_j(M^n) = T_1(T_{j-1}(M^n))$, and let $d_m g$: $T_m(M_n) \to T_m(N^p)$ be the mth derivative map. Since $T_m(M^n)$ and $T_m(N^p)$ are C^∞ manifolds, each may be embedded as a closed C^∞ submanifold of some Euclidean space [15, p. 20]. For each $x \in M^n$, replace the fiber F_x over x by the unit ball B_x (using the Euclidean metric), B_x

$\subset F_x \subset T_m(M^n)$, defining the space $B(M^n)$. For maps G and H of $B(M^n)$ into $T_m(N^p)$, let $\sigma(G,H)$ be the minimum of 1 and the least upper bound of $\{d(G(x),H(x)) : x \in B(M^n)\}$, where d is the Euclidean metric; then σ is a complete metric. Let $\rho(g,h) = \sigma(d_m g, d_m h)$. From the natural embedding of N^p in $T_m(N^p)$, conclusion (2) follows.

The coarse C^m topology on $\mathfrak{C}^m$ is, similarly, the compact open topology of the mth derivative maps of $B(M^n)$ (in fact, $T_m(M^n)$) into $T_m(N^p)$. Since the compact open topology is the topology of uniform convergence on compact sets [1, p. 485], $\rho(g,g_j) \to 0$ implies that $g_j \to g$ in the coarse C^m topology. (In fact, if M^n is compact, the topology of ρ is the (coarse = fine) C^m topology.)

Lastly we wish to show that ρ is complete. Suppose that g_j is a ρ-Cauchy sequence of $\mathfrak{C}^m$; then $d_m g_j$ is a σ-Cauchy sequence, and thus has a limit D. Since the restrictions $d_m g_j | M^n \to D | M^n$ uniformly, i.e., $g_j \to D | M^n$ uniformly, it suffices to prove that $d_m(D|M^n) = D$. In case $m = 1$ and $M^n = N^p = E^1$, the result is a straightforward consequence of the Mean-Value Theorem, and the general result follows from this special case.

The topology defined above is intermediate between the fine and coarse C^r topologies. In [3, p. 272], Cerf observes that the fine C^r topology (his $\mathscr{C}^r$ [3, p. 269]) is not metrizable, but has a complete uniform structure. The coarse C^r topology is metrizable with a complete metric (use [1, p. 490, Theorem 10] and the metrization theorem for uniform spaces), but, in general, it does not have a metric satisfying condition (2).

3.5. Lemma. *Let M^n and N^n be connected C^3 manifolds, and let $f: M^n \to N^n$ be a C^3 proper map with* $\dim(B_f) \leqq n - 3$ *and* $n \geqq 4$. *Then f satisfies condition*(1) *of* (2.1).

Proof. By [15, p. 42] we may suppose that M^n and N^n are C^∞ manifolds and that $\mathfrak{C}^3$, d, and ρ are as in 3.4. By [19, §5], $\dim(f(R_{n-2}(f))) \leqq n - 2$, and, since f is proper, $f(R_{n-2}(f))$ is locally compact. There exist compact sets Y_j and open coordinate neighborhoods U_j and V_j such that $Y_j \subset V_j$, $\overline{V}_j \subset U_j$, $\overline{U}_j$ is compact, the $\overline{U}_j$ are locally finite in $N^p - f(R_{n-2}(f))$, and $N^p - f(R_{n-2}(f)) = \bigcup_{j=1}^\infty Y_j = \bigcup_{j=1}^\infty \overline{U}_j$. Let F_j be the set of $x \in M^n$ such that $d(x, B_f) < 1/j$.

Given $h \in \mathfrak{C}^3$ and $x \in M^n$, let $r(h,x)$ be the rank of (the Jacobian matrix of) h at x; let $0 < \epsilon < 1$. A sequence of proper maps $f_i \in \mathfrak{C}^3$ $(i = 1,2,\cdots;$ $f_0 = f)$ satisfies property $\mathfrak{P}$ if

(a) f_i agrees with f_{i-1} off $F_i \cap f^{-1}(V_i)$;

(b) $f_i(f^{-1}(\overline{V}_j)) \subset U_j$ $(j = 1,2,\cdots)$;

(c) $\bigcup_{j=1}^i f^{-1}(Y_j) \cap B(f_i) = \emptyset$, where $B(f_i)$ is the branch set of f_i;

(d) for each $x \in M^n$, $r(f_i,x) \geqq r(f_{i-1},x)$, and either $f_i(x) = f(x)$ or $r(f_i,x) = n$;

(e) $B(f_i) \subset B(f_{i-1})$;

(f) $\rho(f_i, f_{i-1}) < 2^{-i}\epsilon$; and

(g) for each $y \in N^p$, $f_i^{-1}(y)$ and $f_{i-1}^{-1}(y)$ have the same number of components.

We wish to construct a sequence $\{f_i\}$ satisfying $\mathfrak{P}$. Suppose that $f_1, f_2, \cdots, f_{i-1}$ have been defined; by (e) $\dim(B(f_{i-1})) \leqq n-3$. There exists η, $0 < \eta < 2^{-i}\epsilon$, such that, if f_i is any C^3 map satisfying (a) and $\rho(f_i, f_{i-1}) < \eta$, then f_i satisfies (b) (and (f)).

Since $r(f_{i-1}, x) \geqq n-1$ for $x \in f^{-1}(\overline{V}_i)$ (by (d)) there exists ξ, $0 < \xi < \eta$, such that $h \in \mathfrak{C}^3$, $x \in f^{-1}(\overline{V}_i)$, and $\rho(h, f_{i-1}) < \xi$ implies that $r(h, x) \geqq n-1$. For each $\bar{x} \in F_i \cap f^{-1}(V_i)$, let $U[h, \bar{x}]$ and $V[h, \bar{x}]$ be the neighborhoods given by [5, (1.1)], $U[h, \bar{x}] \subset F_i \cap f^{-1}(V_i)$, with diffeomorphisms $k_1[h, \bar{x}]$ of E^n onto $U[h, \bar{x}]$ and $k_2[h, \bar{x}]$ of $V[h, \bar{x}]$ onto E^n such that $k_2 h k_1$, call it H, has the form $H^j = x^j$ $(j = 1, 2, \cdots, n-1)$. We may suppose that the image of $k_1[h, \bar{x}]$ evaluated at 0 is $\bar{x}$. Let $\Delta_{\lambda,\mu,\nu} \subset E^n$ be the set of 3.3, let $\Delta[h, \bar{x}, \lambda, \mu, \nu]$ be the $k_1[h, \bar{x}]$ image of $\Delta_{\lambda,\mu,\nu}$, and let the top of $\bar{\Delta}[h, \bar{x}, \lambda, \mu, \nu]$ (respectively, bottom; center line; $\bar{\Delta}^+[h, x, \lambda, \mu, \nu]$) be the $k_i[h, \bar{x}]$ image of the subset of $\bar{\Delta}_{\lambda,\mu,\nu}$ defined by $x^n = \nu$ (respectively, $x^n = -\mu$; $x^i = 0$, $i = 1, 2, \cdots, n-1$; $x^n \geqq 0$).

The compact set $\overline{F}_{i+1} \cap f^{-1}(Y_i)$ *is contained in the open set* $F_i \cap f^{-1}(V_i)$. Given $y \in N^n$ and a component Γ of $f_{i-1}^{-1}(y)$ such that $\Gamma \cap \overline{F}_{i+1} \cap f^{-1}(Y_i) \neq \emptyset$, we will now prove that $\Gamma \subset F_{i+1} \cap f^{-1}(Y_i)$. We may suppose that Γ is nondegenerate; thus $\Gamma \subset B(f_{i-1}) \subset B_j$ (by(e) for f_{i-1}), so that $\Gamma \subset \overline{F}_{i+1}$. By (d) for f_{i-1}, $f|\Gamma = f_{i-1}|\Gamma$, so that $f(\Gamma) = \{y\}$. Since $\Gamma \cap f^{-1}(Y_i) \neq \emptyset$, $\Gamma \subset f^{-1}(Y_i)$, yielding the desired conclusion.

Since $\dim(B(f_{i-1})) \leqq n-3$ (by (e)), $r(f_{i-1}, x) \geqq n-1$ for each $x \in f^{-1}(V_i)$ (by (d)), and f_{i-1} is proper (by (a)), it follows from 3.1 that *for each* $y \in N^n$, *each component* Γ *of* $f_{i-1}^{-1}(y) \cap \overline{F}_{i+1} \cap f^{-1}(Y_i)$ *is a point or a* C^3 *embedding of an interval*. For each $\bar{x} \in \overline{F}_{i+1} \cap f^{-1}(Y_i)$, there exist $\lambda(\bar{x}) > 0$, $\mu(\bar{x}) > 0$, and $\nu(\bar{x}) > 0$ such that

$$\bar{\Delta}[f_{i-1}, \bar{x}, \lambda(\bar{x}), \mu(\bar{x}), \nu(\bar{x})] \subset F_i \cap f^{-1}(V_i).$$

For each $h \in \mathfrak{C}^3$, let $\Delta[h, \bar{x}, \lambda(\bar{x}), \mu(\bar{x}), \nu(\bar{x})]$ be denoted by $\Delta(h, \bar{x})$.

Let Γ be the component of $f_{i-1}^{-1}(f_{i-1}(\bar{x}))$ containing $\bar{x}$; then $\Gamma \cap \bar{\Delta}(f_{i-1}, \bar{x})$ is contained in the center line of $\bar{\Delta}(f_{i-1}, \bar{x})$. If Γ is an arc and $\bar{x} \in \operatorname{int} \Gamma$, we may as well suppose, by replacing $\mu(\bar{x})$ and $\nu(\bar{x})$ by small positive numbers, that $\Gamma \cap \bar{\Delta}(f_{i-1}, \bar{x})$ is that center line. Similarly, if $\Gamma = \{\bar{x}\}$ or if Γ is an arc and $\bar{x}$ is one of the two endpoints of Γ, then Γ is disjoint from either the top or bottom of $\bar{\Delta}(f_{i-1}, \bar{x})$, say the top, and we may suppose that f_{i-1} has rank n on (a neighborhood of) the top of $\bar{\Delta}(f_{i-1}, \bar{x})$.

For each such arc component Γ, there exists a finite number of points x_m $(m = 1, 2, \cdots, L(\Gamma))$ such that the sets $\Delta(f_{i-1}, x_m)$ are a minimal cover of Γ. By renumbering the points x_m, replacing $\lambda(x_m)$, $\mu(x_m)$, and $\nu(x_m)$ by smaller numbers, and by (possibly) interchanging tops and bottoms (by a reflection of E^n), we may suppose that

(i) f_{i-1} has rank n on top of $\overline{\Delta}(f_{i-1}, x_1)$, and that

(ii) the top of $\overline{\Delta}(f_{i-1}, x_{m+1})$ is contained in $\Delta(f_{i-1}, x_m)$ $(m = 1, 2, \cdots, L(\Gamma) - 1)$.

For Γ an arc component, let $\Omega(\Gamma)$ be the union of these sets $\Delta(f_{i-1}, x_m)$; for Γ the single point $\overline{x}$, let $\Omega(\Gamma) = \Delta(f_{i-1}, \overline{x})$. The sets $\Omega(\Gamma)$ constitute and open cover of $\overline{F}_{i+1} \cap f^{-1}(Y_i)$, and thus there is a finite subcover. The corresponding sets $\Delta(f_{i-1}, x_j)$ $(j = 1, 2, \cdots, J)$ thus also constitute a finite subcover, and, for each Γ of the finite subcover, may be ordered consistent with the ordering on Γ. If $\mu' + \nu' = \mu + \nu$ $(\mu, \mu', \nu$, and $\nu' > 0)$, then $\Delta_{\lambda,\mu,\nu}$ is homeomorphic to $\Delta_{\lambda,\mu',\nu'}$ by a translation in the nth coordinate direction. Thus we may suppose that either

(i) f_{i-1} has rank n on (a neighborhood of) $\overline{\Delta}^+(f_{i-1}, x_j)$ $(j = 1, 2, \cdots, J)$ or

(ii) $\overline{\Delta}^+(f_{i-1}, x_j) \subset \Delta(f_{i-1}, x_{j-1})$ $(j = 2, 3, \cdots, J)$.

The reader may verify from the proofs of [4, (1.1)] and the rank theorem [12, p. 7, (1.8)] that, if $\rho(h_j, f_{i-1}) \to 0$ as $j \to \infty$ and $\overline{x} \in \overline{F}_{i+1} \cap f^{-1}(Y_i)$, then we may suppose that $k_1[h_j, \overline{x}] | \overline{\Delta}_{\lambda,\mu,\nu}$ approaches $k_1[f_{i-1}, \overline{x}] | \overline{\Delta}_{\lambda,\mu,\nu}$ in the C^1 topology.

Thus there exists ζ, $0 < \zeta < \xi$, such that $h_j \in C^3$ and $\rho(h_j, f_{i-1}) < \zeta$ $(j = 1, 2, \cdots, J)$ implies that

(1) the sets $\Delta(h_j, x_j)$ cover $\overline{F}_{i+1} \cap f^{-1}(Y_i)$ and each set $\overline{\Delta}(h_j, x_j) \subset F_i \cap f^{-1}(V_i)$, and

(2) for each j, either

(i) h_j has rank n on $\overline{\Delta}^+(h_j, x_j)$, or

(ii) $\overline{\Delta}^+(h_j, x_j) \subset \Delta(h_{j-1}, x_{j-1})$.

Let $h_1 = f_{j-1}$. Let h_2 be the map given by 3.3 such that $\rho(h_1, h_2) < \zeta/2$, h_2 agrees with h_1 off $\Delta(h_1, x_1)$, and h_2 has rank n on $\Delta(h_1, x_1)$. Suppose that h_{j+1} $(j = 1, 2, \cdots, m-1)$ have been given by 3.3 for h_j with $\rho(h_j, h_{j+1}) < 2^{-j}\zeta$, h_{j+1} agrees with h_j off $\Delta(h_j, x_j)$, and h_{j+1} has rank n on $\Delta(h_j, x_j)$. By (2), either (i) h_m has rank n on $\overline{\Delta}^+(h_m, x_m)$ or (ii) $\overline{\Delta}^+(h_m, x_m) \subset \Delta(h_{m-1}, x_{m-1})$; but, by the inductive hypothesis, h_m has rank n on $\Delta(h_{m-1}, x_{m-1})$. Thus, in either case, h_m has rank n on $\overline{\Delta}^+(h_m, x_m)$. Let h_{m+1} be the map given by 3.3 such that $\rho(h_{m+1}, h_m) < 2^{-m}\zeta$, h_{m+1} has rank n on $\Delta(h_m, x_m)$, and h_{m+1} agrees with h_m off that set.

Let f_i be the map h_{J+1} thus defined. Conditon $\mathfrak{P}$(a) follows from (1) and 3.3(i); $\mathfrak{P}$(c) from (1) and 3.3(ii); $\mathfrak{P}$(b) and (f) from the fact that $\zeta < \eta$; $\mathfrak{P}$(d) from 3.3(i) and (ii); $\mathfrak{P}$(e) from 3.3(iv); and $\mathfrak{P}$(g) from 3.3(i) and (vi). Thus there exists a sequence $\{f_i\}$ satisfying $\mathfrak{P}$.

Since ρ is complete, a limit map $h \in C^3$ exists, and $\rho(f, h) < \epsilon$. Let $X \subset Y$ be any compact set, and let Y consist of those points $y \in N^p$ such that $d(y, X) \leqq \epsilon$. Then Y is closed and bounded, and, since N^n is a closed subset of a Euclidean space (3.4), Y is compact. If $x \in M^n - f^{-1}(Y)$, then $d(f(x), X) > \epsilon$, so that (3.4(2)) $h(x) \notin X$. Thus $h^{-1}(X) \subset f^{-1}(Y)$, and hence is compact. Since X is an arbitrary compact set in N^n, h is proper.

Given $x \notin B_f$, choose j such that $x \notin \overline{F}_j$. By $\mathfrak{P}$(a), $f_{j-1} = h$ on a neighborhood of x; by $\mathfrak{P}$(e), $x \notin B_h$. Thus $B_h \subset B_f$. Given $z \notin f^{-1}(f(R_{n-2}(f)))$, there exists i such that $z \in f^{-1}(Y_i)$; by $\mathfrak{P}$(c), $z \notin B(f_i)$. There exists J such that $\overline{U}_j \cap \overline{U}_i = \emptyset$ for all $j \geqq J$. By $\mathfrak{P}$(a) and (b) h agrees with f_J on a neighborhood of z. By $\mathfrak{P}$(e), $z \notin B_h$. Thus $B_h \subset B_f \cap f^{-1}(f(R_{n-2}(f)))$.

From $\mathfrak{P}$(e) and 3.4(1), for all $x \in M^n$ $d(f_i(x), f_{i-1}(x)) < 2^{-i}\epsilon$ $(0 < \epsilon < 1)$; thus $d(f_i(x), h(x)) < 2^{-i}\epsilon$. From $\mathfrak{P}$(a) and $\mathfrak{P}$(b), there exists a neighborhood W of $f^{-1}(f(R_{n-2}(f)))$ such that, for all $x \in W$, $d(f(x), h(x)) < 2^{-i}\epsilon$. Since i is arbitrary, the restriction map $h | f^{-1}(f(R_{n-2}(f))) = f | f^{-1}(f(R_{n-2}(f)))$. Similarly, all partial derivatives with order at most 3 of f and h agree on $f^{-1}(f(R_{n-2}(f)))$. Thus $f^{-1}(f(R_{n-2}(f))) \subset h^{-1}(h(R_{n-2}(h)))$, so that $B_h \subset B_f \cap h^{-1}(h(R_{n-2}(h)))$.

Since $\dim B_h \leqq n - 3$ and $B_h \subset h^{-1}(h(R_{n-2}(h)))$, there exists (2.8) k such that, for each $y \in N^n$, $h^{-1}(y)$ has precisely k components. Since $h | f^{-1}(f(R_{n-2}(f))) = f | f^{-1}(f(R_{n-2}(f)))$ and $h(M^n - f^{-1}(f(R_{n-2}(f)))) \subset N^n - f(R_{n-2}(f))$ (by $\mathfrak{P}$(a) and $\mathfrak{P}$(b)), for each $y \in f(R_{n-2}(f))$, $f^{-1}(y)$ has precisely k components also. For each $y \in N^n - f(R_{n-2}(f))$, there exists i such that $y \in V_i$; there exists J such that $\overline{V}_i \cap \bigcup_{j=J+1}^{\infty} \overline{U}_j = \emptyset$, and thus ($\mathfrak{P}$(a) and $\mathfrak{P}$(b)) $h^{-1}(\overline{V}_i) = f_J^{-1}(\overline{V}_i)$ and f_J agrees with h on $h^{-1}(\overline{V}_i)$. As a result, $h^{-1}(y)$ has exactly k components. By $\mathfrak{P}$(g), $f_J^{-1}(y)$ and $f^{-1}(y)$ have the same number of components. Thus f satisfies conclusion (1) of 2.1.

3.6. **Proof of Theorem (1.1).** First, for the uniqueness, suppose that hg and $\bar{h}\bar{g}$ are two factorizations with intermediate manifolds K^n and L^n. For each point y in K^n, define $\alpha(y) \in L^n$ as the single point $\bar{g}(g^{-1}(y))$. That α is one-to-one and onto is immediate. Since $\bar{h}\alpha = h$, α is locally a diffeomorphism and thus is a diffeomorphism.

For $n = 1$ and 2, f is, by definition, a local homeomorphism; for $n = 3$, f is a local homeomorphism by [4, p. 469] and [5, p. 91]. Thus f is a k-to-1 C^m covering map [18, p. 128], and the existence of the factoring follows from (2.1). For $n \geqq 4$ it follows from (2.1), (3.2) and (3.5).

3.7. Remarks. The manifold K^n need not be C^m diffeomorphic to M^n. Let $M^n = S^7$, N^n be a C^∞ homotopy 7-sphere other than S^7, and let f be a homeomorphism of M^n onto N^n which is a C^∞ diffeomorphism except at one point [13]; we may suppose that f itself is C^∞ everywhere (e.g., by the argument of [5, p. 95, (3.3)]). In any factorization K^n is C^∞ diffeomorphic to N^n, and thus is not diffeomorphic to M^n.

In the special case that $f(B_f) \subset f(R_{n-3})$, $\pi_1(N^n) = 0$ and f is C^n, it follows from [6, (1.1)] for $p = n$, $m = 1$, and $k = n - 3$ that $\pi_1(N^n - f(R_{n-3})) = 0$.

Since $\dim(f(R_{n-3})) \leqq n-3$ [5, (1.3)] and $\dim(B_f) \leqq n-3$, $\dim(f^{-1}(f(R_{n-3})))$ $\leqq n-3$; thus $M^n - f^{-1}(f(R_{n-3}))$ is connected and the restriction map $f|[M^n - f^{-1}(f(R_{n-3}))]$ is a homeomorphism. Hence f is monotone, and (1.1) is immediate in this case.

A map f satisfying the hypothesis of (1.1) is not a branched covering in the sense of Fox [7, p. 250] unless it is actually a covering map.

In case the degree is 0, the conclusion of (1.2) can be improved.

3.8. Remark. *If M^n and N^n are compact connected oriented C^1 n-manifolds, and $f: M^n \to N^n$ is C^1 with degree* 0, *then* $\dim(B_f) \geqq n-1$.

Proof. Suppose the contrary, i.e., that f has degree 0 and $\dim(B_f)$ $\leqq n-2$. Since B_f does not separate M^n, the Jacobian of f is either nonnegative or nonpositive (well defined by [21, p. 341, Lemma 3]). By [19, §5] $\dim(f(R_{n-1})) \leqq n-1$, so there exists y with $f^{-1}(y) \subset R_n - R_{n-1}$; let x_k $(k = 1, 2, \cdots, m)$ be the points of $f^{-1}(y)$. The result follows from [21, Lemma 2 and Theorem 2].

If f is not differentiable but $f(B_f) \neq N^n$, essentially the same proof is valid. The author is grateful to R. F. Williams for suggesting this proof, which is simple than the author's version.

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