ON ISOMETRIC IMMERSIONS IN EUCLIDEAN SPACE OF MANIFOLDS WITH NON-NEGATIVE SECTIONAL CURVATURES(1)

BY

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1. This paper deals with certain isometric immersions S of a complete d-dimensional manifold $M = M^d$ in a Euclidean space $E^{d+\delta}$, $\delta > 0$:

$$\psi : M^d \rightarrow E^{d+\delta}, \qquad S = \psi(M^d). \tag{1.1}$$

One of the main results will be the following:

THEOREM (*). *Let $M = M^d$ be a complete Riemann manifold of class C^2 such that all 2-dimensional sections have non-negative curvatures. Let* (1.1) *be a C^2 isometric immersion of M^d in $E^{d+\delta}$, $\delta > 0$, such that the relative nullity function ν is a positive constant. Then S is ν-cylindrical.*

The relative nullity $\nu(m)$, $m \in M^d$ and $0 \leqq \nu(m) \leqq d$, is defined by Chern and Kuiper [2] (and for $\delta = 1$ reduces to the nullity of the second fundamental matrix); cf. §3 below.

The immersion (1.1) is said to be ν-cylindrical if M^d, ψ, and $E^{d+\delta}$ can be expressed as products: $M^d = M^{d-\nu} \times E^\nu$, $\psi = \bar{\psi} \times 1$, and $E^{d+\delta} = E^{d-\nu+\delta} \times E^\nu$, where $M^{d-\nu}$ is a complete Riemann manifold, $\bar{\psi} : M^{d-\nu} \rightarrow E^{d-\nu+\delta}$ is an isometric immersion and 1 is the identity map on E^ν. $M^{d-\nu}$ and its immersion $\bar{\psi}$ are allowed to be of class C^1.

Theorem (*) is a consequence of Lemmas 3.1 and 4.1, below.

(*) generalizes a result of O'Neill [5] who supposes that M is flat and makes the superfluous assumption that the "relative curvature of ψ is 0." For the case of hypersurfaces ($\delta = 1$), (*) is a particular case of a theorem of Sacksteder [6] which does not contain the condition that $\nu(m)$ is constant and which has a stronger conclusion. For the case of a flat M and $\delta = 1$, (*) is also contained in a result of Hartman and Nirenberg [4]. In the latter, the assumptions "M flat" and "$\delta = 1$" imply that $d - 1 \leqq \nu(m) \leqq d$. It will be clear from the proof that an analogue of (*) is correct if the assumption that "$\nu(m)$ is a positive constant" is replaced by the assumption that "$d - 1 \leqq \nu(m) \leqq d$," in which case M is necessarily flat. Professor Nirenberg has pointed out to me that this analogue can be deduced directly from our result in [4].

The problem of removing the assumption in (*) that $\nu(m)$ is a constant will remain open (when $\delta > 1$).

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The proof of (*) depends in part on a generalization of the implicit function theorem, in the large, for gradient mappings in Lemma 2 of Chern and Lashoff [3] (or, equivalently, Lemma 2 of Hartman and Nirenberg [4]). This generalization involves simultaneous gradient mappings and is given in §2.

An Appendix deals with a further generalization of this implicit function theorem.

2. Let D be an open set in a Euclidean d-dimensional space of points $u = (u^1, \cdots, u^d)$. By a *ν-dimensional plane section π_ν* of D through a point $u \in D$ is meant the connected component, containing u, of the intersection of D and a ν-dimensional plane through u.

Let $P_p(u) = (P_p^1(u), \cdots, P_p^d(u))$, where $p = 1, \cdots, \delta$, be a vector function of class C^1 on a domain D such that $u \rightarrow P_p(u)$ is a gradient mapping, i.e.,

$$\omega_p = P_p(u) \cdot du = P_p^i(u) \cdot du^i \tag{2.1}$$

is closed, so that

$$d\omega_p = 0, \qquad p = 1, \cdots, \delta. \tag{2.2}$$

Let $J_p(u) = (\partial P_p^i / \partial u^j)$, where $i, j = 1, \cdots, d$, be the $d \times d$ Jacobian matrix of the map $u \rightarrow P_p(u)$. Let $P(u) = (P_1, \cdots, P_\delta) = (P_1^1, \cdots, P_1^d, P_2^1, \cdots, P_\delta^d)$ be the $d\delta$-dimensional vector function and $J(u)$ the Jacobian matrix $(\partial P / \partial u)$ with d columns and δd rows:

$$J = \begin{pmatrix} J_1 \\ \vdots \\ J_\delta \end{pmatrix}.$$

Let $\rho(u) = \text{rank } J(u)$ and $\rho^*(u_0) = \limsup \rho(u)$ as $u \rightarrow u_0$; correspondingly, $\nu(u) = \text{nullity } J(u) = d - \rho(u)$ and $\nu^*(u_0) = d - \rho^*(u_0) = \liminf \nu(u)$ as $u \rightarrow u_0$. Finally, let D_ν be the open subset of D defined by

$$D_\nu = \{ u : \nu^*(u) \geqq \nu \}.$$

LEMMA 2.1. *Let $P_1(u), \cdots, P_\delta(u)$ be d-dimensional vector functions of class C^1 on D such that* (2.1) *satisfies* (2.2). *Let $u_0 \in D$ and $\nu(u_0)$, $\nu^*(u_0)$ have a common value ν. Then $P(u) = (P_1(u), \cdots, P_\delta(u))$ is constant on a ν-dimensional plane section $\pi_\nu(u_0)$ of D_ν through u_0. Furthermore, for all points u near u_0, $P(u) = P(u_0)$ if and only if $u \in \pi_\nu(u_0)$. Finally, $\nu(u) = \nu^*(u) = \nu$ for all $u \in \pi_\nu(u_0)$.*

As in [4], this has the following consequence:

COROLLARY 2.1. *If $\nu^*(u_0) = \nu$ at a point $u_0 \in D$, then $P(u)$ is constant on a ν-dimensional plane section $\pi_\nu(u_0)$ of D_ν through u_0. Also, $u \in \pi_\nu$ implies that $\nu^*(u) = \nu$ and that either $\nu(u) = \nu$ or $\nu(u) > \nu$ according as $\nu(u_0) = \nu$ or $\nu(u_0) > \nu$.*

While $\pi_\nu(u_0)$ is unique in Lemma 2.1, it need not be unique in Corollary 2.1. It is unique in Corollary 2.1 if $\nu^*(u) = d - 1$.

Below, repeated indices indicate summation with the following ranges for the different indices:

$$1 \leqq h, i, j, k \leqq d; \qquad 1 \leqq \alpha, \beta, \gamma \leqq \rho;$$

$$\rho + 1 \leqq \kappa, \lambda \leqq d; \qquad 1 \leqq p, q, r \leqq \delta.$$

Proof of Lemma 2.1. Let $J_{p\rho}(u)$ denote the $\rho \times \rho$ Jacobian matrix $(\partial P_p^\alpha / \partial u^\beta)$, where $\alpha, \beta = 1, \cdots, \rho = d - \nu$, in the upper left corner of J_p and let $J^\rho(u)$ denote the matrix

$$J^\rho = \begin{pmatrix} J_{1\rho} \\ \vdots \\ J_{\delta\rho} \end{pmatrix} \tag{2.3}$$

with ρ columns and $\delta\rho$ rows.

Let the j_1st, j_2nd, $\cdots, j_\nu$th columns of $J(u_0)$ be linearly independent. Let U be a $d \times d$ permutation (orthogonal) matrix such that the effect of multiplying any $d \times d$ matrix A on the right by U to give AU is to move the j_1st, $\cdots, j_\nu$th columns of A into the 1st, $\cdots, \nu$th places. Multiplication of A on the left by U^* to give U^*A moves the j_1st, $\cdots, j_\nu$th rows of A into the 1st, $\cdots, \nu$th positions, respectively.

Let Uu be renamed u and $U^*P_p(Uu)$ be called $P_p(u)$. Then $u \rightarrow P_p(u)$ satisfies (2.1), (2.2) and the first ν columns of $J(u)$ are linearly independent at $u = u_0$, hence, for u near u_0. Since $\rho(u_0) = \rho^*(u_0) = \rho$, the last $\nu = d - \rho$ columns of $J(u)$ are linear combinations of the first ρ columns for u near u_0. In particular, the last $\nu = d - \rho$ columns of $J_p(u)$ are linear combinations of the first ρ columns of $J_p(u)$ for $p = 1, \cdots, \delta$. The condition (2.2) implies that the Jacobian matrix $J_p(u)$ is symmetric and so the last ν rows of $J_p(u)$ are linear combinations of the first ρ rows. It follows that the rank of the matrix $J^\rho(u)$ in (2.3) is ρ.

Thus, $J^\rho(u)$ has ρ linearly independent rows, i.e., there exist ρ pairs of indices $(p(\alpha), i(\alpha))$, where $1 \leqq p(\alpha) \leqq \delta$, $1 \leqq i(\alpha) \leqq \rho$ and $\alpha = 1, \cdots, \rho$, such that $\det(\partial P_{p(\alpha)}^{i(\alpha)}(u) / \partial u^\beta)_{\alpha,\beta=1,\cdots,\rho} \neq 0$ at $u = u_0$. Introduce the mapping $u \rightarrow v$, defined by

$$v^\alpha = P_{p(\alpha)}^{i(\alpha)}(u) \quad \text{for } \alpha = 1, \cdots, \rho \quad \text{and } v^\kappa = u^\kappa \quad \text{for } \kappa = \rho + 1, \cdots, d, \tag{2.4}$$

so that the Jacobian $\det(\partial v / \partial u) \neq 0$ at $u = u_0$. Let v_0 correspond to u_0 and let $u = u(v)$ be the local inverse of the map (2.4).

The above description of the linear dependence of the rows of $J_p(u)$ and the assumption $\rho(u) = \rho^*(u) = \rho$ for u near u_0 imply that $P_p^i(u(v))$, for $p = 1, \cdots, \delta$ and $i = 1, \cdots, d$, is a function, say $\overline{P}_p^i(v)$, of $(v^1, \cdots, v^\rho)$. For

$$\partial \overline{P}_p^i / \partial v^\kappa = (\partial P_p^i / \partial u^j)(\partial u^j / \partial v^\kappa)$$

and if $\lambda > \rho$ and u (or v) is fixed, there are numbers $c_{\lambda\beta}$, $\beta = 1, \cdots, \rho$, such that

$$\partial P_p^i/\partial u^\lambda = c_{\lambda\beta}\partial P_p^i/\partial u^\beta \quad \text{for } i = 1, \cdots, d.$$

Hence $\partial u^\lambda/\partial v^\kappa = \delta_\kappa^\lambda$ implies that

$$\partial \bar{P}_p^i/\partial v^\kappa = (\partial P_p^i/\partial u^\beta)(\partial u^\beta/\partial v^\kappa + c_{\kappa\beta}).$$

The choice $(p, i) = (p(\alpha), i(\alpha))$ for $\alpha = 1, \cdots, \rho$, makes the left side zero and hence, $\partial u^\beta/\partial v^\kappa + c_{\kappa\beta} = 0$ for $\beta = 1, \cdots, \rho$ and $\kappa = \rho + 1, \cdots, d$. Consequently $\partial \bar{P}_p^i/\partial v^\kappa = 0$ for $p = 1, \cdots, \delta$; $i = 1, \cdots, d$; and $\kappa = \rho + 1, \cdots, d$.

Since the Pfaffian (2.1) is closed, it follows that the Pfaffian

$$\omega_{p0} = u^i dP_p^i(u) = d(u^i P_p^i) - \omega_p$$

is also closed. Introducing v as independent variable leaves ω_{p0} closed and shows that ω_{p0} is of the form

$$\omega_{p0} = (u^i \partial \bar{P}_p^i/\partial v^\alpha) dv^\alpha.$$

Corresponding to v near v_0, there is a scalar function $f_p(v^1, \cdots, v^\rho)$ of class C^1 such that $\omega_{p0} = df_p$. Hence

$$u^i \partial \bar{P}_p^i(v)/\partial v^\alpha = b_{p\alpha}(v^1, \cdots, v^\rho) \quad \text{for } p = 1, \cdots, \delta \text{ and } \alpha = 1, \cdots, \rho \tag{2.5}$$

where $b_{p\alpha} = \partial f_p/\partial v^\alpha$ is continuous.

It will be verified that the $\delta\rho$ equations (2.5) contain a set of ρ equations such that the matrix of coefficients of $u^1, \cdots, u^\rho$ is nonsingular and that the remaining equations of (2.5) are consequences of these. If this is granted for a moment, these ρ equations can be solved for $u^1, \cdots, u^\rho$ to give a result of the form

$$\begin{aligned} u^\alpha &= a^{\alpha\kappa}(v^1, \cdots, v^\rho)v^\kappa + b^\alpha(v^1, \cdots, v^\rho) \quad \text{for } \alpha = 1, \cdots, \rho, \\ u^\kappa &= v^\kappa \quad \text{for } \kappa = \rho + 1, \cdots, d. \end{aligned} \tag{2.6}$$

Since (2.6) is the inverse of (2.4), it follows that the functions $a^{\alpha\kappa}$, b^α are of class C^1.

In order to see that (2.5) can be solved to give (2.6), write $\partial \bar{P}_p^i/\partial v^\alpha$ as the sum $(\partial P_p^i/\partial u^\beta)(\partial u^\beta/\partial v^\alpha)$. Thus, if (2.5) is multiplied by $\partial v^\alpha/\partial u^\gamma$ and the result summed for $\alpha = 1, \cdots, \rho$, it follows that (2.5) is equivalent to

$$u^\alpha \partial P_p^\gamma/\partial u^\alpha + u^\kappa \partial P_p^\kappa/\partial u^\gamma = b_{p\alpha}\partial v^\alpha/\partial u^\gamma \tag{2.7}$$

for $p = 1, \cdots, \delta$ and $\gamma = 1, \cdots, \rho$, where α is a summation index over $1, \cdots, \rho$ and κ over $1, \cdots, \nu$ and $\partial P_p^\gamma/\partial u^\alpha = \partial P_p^\alpha/\partial u^\gamma$.

The normalization of $J(u)$ shows that the $\delta\rho$ equations (2.7) are equivalent to the ρ equations corresponding to $(p, \gamma) = (p(\beta), i(\beta))$ for $\beta = 1, \cdots, \rho$. The matrix of coefficients of $u^1, \cdots, u^\rho$ in these ρ equations is $(\partial P_{p(\beta)}^{i(\beta)}/\partial u^\alpha)_{\alpha,\beta=1,\cdots,\rho}$, which is nonsingular.

Since $P_p^i(u) = \bar{P}_p^i(v)$ is a function of $(v^1, \cdots, v^\rho)$, the relations (2.6) imply the "local" part of Lemma 2.1; namely, that there exists a ν-plane π_ν through u_0 such that u near u_0 is on π_ν if and only if $P(u) = P(u_0)$.

The proof of the remainder of Lemma 2.1 is similar to that of Lemma 2 in [4] and will only be indicated. Substitution of (2.6) into the first part of (2.4) and differentiation with respect to v^β gives

$$\delta_{\alpha\beta} = (\partial P_{p(\alpha)}^{i(\alpha)}/\partial u^\gamma)(v^\kappa \partial a^{\gamma\kappa}/\partial v^\beta + \partial b^\gamma/\partial v^\beta)$$

for $\alpha, \beta = 1, \cdots, \rho$; hence

$$1 = \det(\partial P_{p(\alpha)}^{i(\alpha)}/\partial u^\gamma)\det(v^\kappa \partial a^{\gamma\kappa}/\partial v^\beta + \partial b^\gamma/\partial v^\beta).$$

This shows that as $u \in D_\nu$ moves continuously from u_0 on the ν-plane (2.6), where v^α is constant, one cannot reach a first point u where

$$\det(\partial P_{p(\alpha)}^{i(\alpha)}/\partial u^\beta) = 0.$$

Thus the arguments above give Lemma 2.1.

REMARK. If $P_p(u)$ is of class C^t, $t \geqq 1$, then the change of coordinates $u \to v$ in the above proof is of class C^t.

3. Consider a piece of d-dimensional surface $S: X = X(u)$ of class C^2 in a $(d+\delta)$-dimensional Euclidean space $E^{d+\delta}$ where $u = (u^1, \cdots, u^d)$, $X = (X^1, \cdots, X^{d+\delta})$, and $X(u)$ is of class C^2 on an open connected u-set D such that rank $(\partial X^\tau/\partial u^i)$ is d.

Let $X_i = \partial X/\partial u^i, X_{ij} = \partial^2 X/\partial u^i \partial u^j$, etc. If N is a normal vector to S at a point u (i.e., $X(u)$), there is an associated $d \times d$ second fundamental matrix $(h_{ij}(u;N))$, where $h_{ij}(u;N)$ is the Euclidean scalar product $X_{ij}(u) \cdot N$. Let $\pi(u;N)$ denote the null space of $(h_{ij}(u;N))$, i.e., the set of vectors $y = (y^i, \cdots, y^d)$ satisfying $h_{ij}y^j = 0$ for $i = 1, \cdots, d$. Let $\pi(u) = \bigcap \pi(u;N)$ where the intersection is taken over all normals or, equivalently, $\pi(u) = \bigcap \pi(u;N_p)$ where the intersection is taken over a set of δ linearly independent normal vectors $N_1, \cdots, N_\delta$. The integer $\nu(u) = \dim \pi(u)$ is called the relative nullity of S at u [2]. A vector $y \neq 0$ in $\pi(u)$ will be said to be in a *trivial asymptotic direction* at u.

Let $\nu^*(u_0) = \liminf \nu(u)$, $u \to u_0$. The subset $D_\tau = \{u : \nu^*(u) \geqq \tau\}$ of D is open.

LEMMA 3.1. *Let $M = M^d$ be a d-dimensional Riemann manifold of class C^2 and $S = \psi(M)$, where $\psi: M \to E^{d+\delta}$ is a C^2 isometric immersion of M in the Euclidean space $E^{d+\delta}$. For a point $m \in M$, let $\nu(m)$ be the relative nullity of S at m (i.e., at $\psi(m)$); $\nu^*(m_0) = \liminf \nu(m)$, $m \to m_0$; and $M(\tau)$ the submanifold of M consisting of points $\{m : \nu^*(m) \geqq \tau\}$.*

(i) *Then, for any point $m_0 \in M$, there is a (not necessarily unique) maximal, totally geodesic submanifold $M^{\nu^*(m_0)}$ of $M(\nu^*(m_0))$ containing m_0, of dimension $\nu^*(m_0)$, which ψ maps isometrically onto a subset of a $\nu^*(m_0)$-plane*

π in $E^{d+\delta}$.

(ii) *The space of normal vectors to S at a point $\psi(m) \in \pi, m \in M^{\nu^*(m_0)}$, is independent of m.*

(iii) *If $m \in M^{\nu^*(m_0)}$, then $\nu^*(m) \gtreqless \nu(m)$ according as $\nu^*(m_0) \gtreqless \nu(m_0)$.*

(iv) *If $\nu(m_0) = \nu^*(m_0)$, then $M^{\nu^*(m_0)}$ is unique and m near m_0 is on $M^{\nu^*(m_0)}$ if and only if the space of normal vectors at $\psi(m)$ is the same as that at $\psi(m_0)$.*

(v) *Finally, if M is complete and $\nu(m_0) = \min \nu(m)$ for $m \in M$ (so that $M(\nu(m_0)) = M$), then $M^{\nu(m_0)}$ is complete and $\pi = \psi(M^{\nu(m_0)})$ is a $\nu(m_0)$-dimensional plane in $E^{d+\delta}$.*

The case $\delta = 1$ is contained in [4]. For arbitrary $\delta > 0$, under the additional assumption that M is flat, the last part of Lemma 3.1 is contained in [5]. For a flat M, $d - \delta \leqq \nu(m) \leqq d$ holds [2]; so that, in this case, $\delta < d$ implies that $\nu(m) > 0$ and Lemma 3.1 is not trivially true.

Proof. On introducing suitable local coordinates $u = (u^1, \cdots, u^d)$ on M at m_0 and a suitable choice of rectangular coordinates $X = (u, x) = (u^1, \cdots, u^d, x^1, \cdots, x^\delta)$ in $E^{d+\delta}$, the immersion ψ, for m near m_0, can be represented in the form

$$x^p = \phi_p(u) \quad \text{for } p = 1, \cdots, \delta, \tag{3.1}$$

where ϕ_p is a real-valued C^2 function of $u = (u^1, \cdots, u^d)$. Let $P_p(u) = (\partial\phi_p/\partial u^1, \cdots, \partial\phi_p/\partial u^d)$, i.e., $P_p^i = \partial\phi_p/\partial u^i$. Then a normal vector at u is $N_p = (P_p^1, \cdots, P_p^d, 0, \cdots, 0, 1, 0, \cdots, 0)$, where the 1 is the $(d+p)$th coordinate of N_p. The vectors $N_1, \cdots, N_\delta$ are linearly independent. This makes it clear that if $P = (P_1, \cdots, P_d)$, $J = (\partial P/\partial u)$, and $\rho(u) = \text{rank } J(u)$, then the relative nullity $\nu(u)$ is $d - \rho(u)$.

Suppose, first, that $u_0 = \psi(m_0)$ and that $\nu(u_0) = \nu^*(u_0) = \nu$. Suppose that (3.1) is defined on a u-domain D containing u_0. Then, there is a ν-plane section π_ν of D_ν through u_0 satisfying the conclusion of Lemma 2.1. Let π_ν be the intersection of D_ν and the ν-plane (2.6) on which $v = (v^1, \cdots, v^\rho)$ is a constant, $\rho = d - \nu$.

In (3.1), consider the C^1 change of local coordinates $u \to v$ on M given by (2.6). Then

$$\partial x^p/\partial v^\kappa = (\partial\phi_p/\partial u^\alpha)(a^{\alpha\kappa}) + \partial\phi_p/\partial v^\kappa.$$

Note that, by the proof of Lemma 2.1, $\partial\phi_p/\partial u^\alpha, \partial\phi_p/\partial v^\kappa$ are functions of $v^1, \cdots, v^\rho$. Hence the function x^p is of the form

$$x^p = a^{d+p,\kappa}(v^1, \cdots, v^\rho)v^\kappa + b^{d+p}(v^1, \cdots, v^\rho) \quad \text{for } p = 1, \cdots, \delta, \tag{3.2}$$

and so, (3.1) can be written as

$$X = A_\kappa(v^1, \cdots, v^\rho)v^\kappa + B(v^1, \cdots, v^\rho), \tag{3.3}$$

$X = (X^1, \cdots, X^{d+\delta})$ and

(3.4) $$B = (b^1, \cdots, b^\rho, 0, \cdots, 0, b^{d+1}, \cdots, b^{d+\delta}),$$

(3.5) $$A_\kappa = (a^{1\kappa}, \cdots, a^{\rho\kappa}, 0, \cdots, 0, 1, 0, \cdots, 0, a^{d+1,\kappa}, \cdots, a^{d+\delta,\kappa}).$$

The "1" is the κth component of A_κ. The vector functions A_κ and B are of class C^1.

The argument up to this point shows that (3.3) is valid in a neighborhood of the ν-plane section π_ν of D_ν. By the isometric property of the immersion ψ, the pre-image under ψ of a line segment in the X-space of the form: ($v^\alpha =$ const, $v^\kappa =$ linear function of t) is a geodesic. Hence a pre-image of π_ν is a totally geodesic submanifold M^ν of $M(\nu)$.

If π_ν [and/or M^ν] has a limit point $u_1 \in D_\nu$ [and/or $m_1 = \psi^{-1}(u_1) \in M(\nu)$], then the normal space at u_1 is the same as that at u_0. Hence ψ is given locally in the form (3.1). Also $\nu(u_1) = \nu^*(u_1) = \nu$ by Lemma 2.1.

This shows that M^ν has a (maximal) extension so that its boundary points, if any, are not in $M(\nu)$. This implies the lemma for the case $\nu(m_0) = \nu^*(m_0) = \nu$.

If $\nu^*(m_0) = \nu^*$ but $\nu(m_0) > \nu^*$, then there exist points $m_1, m_2, \cdots$ of M such that $m_n \to m_0$ as $n \to \infty$ and $\nu(m_n) = \nu^*(m_n) = \nu^*$. After a selection of a subsequence, if necessary, it can be supposed that the ν^*-plane section $\pi_{\nu^*}(m_n)$ of $S(\nu) = \psi(M(\nu))$ passing through $\psi(m_n)$ tends to a limiting position (in a suitable sense) and has the desired properties.

REMARK. If $M = M^d$ and its immersion ψ are of class C^{t+1}, $t \geqq 1$, then, in the local coordinates v, the immersion ψ given by (3.3) is of class C^t.

4. Of particular interest is the question as to whether or not the planes $\pi(m)$ above are parallel in $E^{d+\delta}$. A sufficient condition is given in the next lemma for the case that $\nu^*(m)$ is constant (near m_0) and $M^{\nu^*(m)}$ is complete, so that $\psi(M^{\nu^*(m)})$ is an entire $\nu^*(m)$-plane.

In this situation, the problem is reduced to the consideration of a d-dimensional surface $S: X = X(v)$ in an $E^{d+\delta}$ space of points

$$X = (X^1, \cdots, X^{d+\delta}),$$

where $X(v)$ is of the form (3.3) for small $|v^1|, \cdots, |v^\rho|$ and for arbitrary $v^{\rho+1}, \cdots, v^d$. In (3.3), B and $A_{\rho+1}, \cdots, A_d$ are $(d+\delta)$-dimensional vectors and $\partial X/\partial v^1, \cdots, \partial X/\partial v^d$ are linearly independent. The problem is to give sufficient conditions to assure that, after a suitable change of coordinates leaving the form of (3.3) unchanged, the vectors $A_\kappa(v^1, \cdots, v^\rho)$ are constant.

LEMMA 4.1. *Let $0 < \nu < d$ and $\nu + \rho = d$. Let S be a d-dimensional surface in $E^{d+\delta}$ of class C^2 having a C^1 parametric representation*

(4.1) $$S: X = A_\kappa(v^1, \cdots, v^\rho)v^\kappa + B(v^1, \cdots, v^\rho)$$

for small $|v^1|, \cdots, |v^\rho|$ and arbitrary v^κ, $\kappa = \rho + 1, \cdots, d$, such that the relative nullity $\nu(v)$ of S at v is the constant $\nu(v) = d - \rho$ and that all vectors

$y = (0, \cdots, 0, y^{\rho+1}, \cdots, y^d)$ *are in trivial asymptotic directions at* ν (*so that the normal space of* S *is independent of* $v^{\rho+1}, \cdots, v^d$; *cf. Lemma* 3.1). *In addition, suppose that all* 2-*dimensional sections of* S *have non-negative curvatures. Then there exists a* C^1 *nonsingular linear change of the* v^κ *variables depending on* $(v^1, \cdots, v^\rho)$,

$$v^\kappa = a^\kappa_\lambda(v^1, \cdots, v^\rho) w^\lambda + c^\kappa(v^1, \cdots, v^\rho), \tag{4.2}$$

such that (4.1) *becomes*

$$S: X = C_\kappa w^\kappa + D(v^1, \cdots, v^\rho), \tag{4.3}$$

where $C_{\rho+1}, \cdots, C_d$ *are constant vectors.*

Proof. Let $A_\kappa = (A^1_\kappa, \cdots, A^{d+\delta}_\kappa)$. Since $\partial X / \partial v^\kappa = A_\kappa$, the vectors $A_{\rho+1}, \cdots, A_d$ are linearly independent. After a rotation of the X-space, if necessary, it can be supposed that the ν vectors with ν components given by $(A^{\rho+1}_\kappa, \cdots, A^d_\kappa)$ are linearly independent at $v^\alpha = 0$, hence for small $|v^\alpha|$. Choose the function $a^\kappa_\lambda(v^1, \cdots, v^\rho)$ of class C^1, so that

$$A^\tau_\kappa(v^1, \cdots, v^\rho) a^\kappa_\lambda(v^1, \cdots, v^\rho) \equiv \delta^\tau_\lambda.$$

Thus, after the change of variables $v^\kappa = a^\kappa_\lambda w^\lambda$ and the renaming of w^λ back to v^λ, (4.1) has the same form, where

$$A_\kappa = (A^1_\kappa, \cdots, A^\rho_\kappa, 0, \cdots, 0, 1, 0, \cdots, A^{d+1}_\kappa, \cdots, A^{d+\delta}_\kappa) \tag{4.4}$$

and the "1" is the κth component of $A_\kappa, \kappa = \rho + 1, \cdots, d$. If, in (4.1), v^κ is replaced by $v^\kappa + c^\kappa(v^1, \cdots, v^\rho)$, then (4.1) takes the form $A_\kappa v^\kappa + (B - A_\kappa c^\kappa)$. Thus, in view of (4.4), the functions $c^\kappa(v^1, \cdots, v^\rho)$ can be chosen of class C^1, so that the κth coordinate of $B - A_\kappa c^\kappa$ is 0 for $\kappa = \rho + 1, \cdots, d$. Thus, if $B - A_\kappa c^\kappa$ is called B again, it can be supposed that

$$B = (B^1, \cdots, B^\rho, 0, \cdots, 0, B^{d+1}, \cdots, B^{d+\delta}). \tag{4.5}$$

After this normalization, it will be shown that A_κ is a constant vector *by virtue of the fact that* v^κ *is arbitrary in* (4.1).

Note that

$$X_\kappa = A_\kappa, \qquad X_\alpha = A_{\kappa\alpha} v^\kappa + B_\alpha, \tag{4.6}$$

where $A_{\kappa\alpha} = \partial A_\kappa / \partial v^\alpha$, $\alpha = 1, \cdots, \rho$, and $\kappa = \rho + 1, \cdots, d$. The vectors

$$B_1, \cdots, B_\rho \text{ and } A_{\rho+1}, \cdots, A_d \tag{4.7}$$

are linearly independent. The vectors $A_{\kappa\alpha}$ are in the span of (4.7) since the normal space of S does not depend on v^κ. In view of the normalizations (4.4) and (4.5), it is clear that $A_{\kappa\alpha}$ is in the span of the set of vectors $B_1, \cdots, B_\rho$. Hence, the analogue of the Gauss equations give

$$A_{\kappa\alpha} = \Gamma^\beta_{\kappa\alpha} B_\beta. \tag{4.8}$$

In view of the low differentiability of the parametrization of S involved, it is best to consider the equations (4.8) as defining the continuous functions $\Gamma^{\beta}_{\kappa\alpha}(v^1, \cdots, v^\rho)$, so that

$$\Gamma^{\beta}_{\kappa\alpha} = g^{\beta\gamma} A_{\kappa\alpha} \cdot B_{\gamma}, \tag{4.9}$$

where $(g^{\beta\gamma}) = (g_{\beta\gamma})^{-1}$ and $g_{\beta\gamma} = B_\beta \cdot B_\gamma$. It has to be verified that

$$\Gamma^{\beta}_{\kappa\alpha} = 0, \qquad \text{i.e., } A_\kappa \text{ is constant.} \tag{4.10}$$

Since S has local C^2 parametrizations, there exists a C^1 orthonormal basis $N^1(v^1, \cdots, v^\rho), \cdots, N^\delta(v^1, \cdots, v^\rho)$ for the normal space to S at $X(v)$. The second fundamental matrix (h^p_{ij}) corresponding to N^p is given by $h^p_{ij} = -X_i \cdot N^p_j = -X_j \cdot N^p_i$, which is consistent with the tensor character of (h^p_{ij}). The functions h^p_{ij} will be considered as functions of v^α alone (with $v^\kappa = 0$).

Since all vectors $y = (0, \cdots, 0, y^{\rho+1}, \cdots, y^d)$ are in trivial asymptotic directions, so that $0 = h^p_{ij} y^j = h^p_{i\kappa} y^\kappa$, it follows that $h^p_{i\kappa} = 0$ for $p = 1, \cdots, \delta$, $i = 1, \cdots, d$, and $\kappa = \rho + 1, \cdots, d$.

It will be shown that if Γ_κ, H^p denote the matrices $\Gamma_\kappa = (\Gamma^\beta_{\kappa\alpha}), H^p = (h^p_{\alpha\beta})$, where $\alpha, \beta = 1, \cdots, \rho$, then

$$\Gamma_\kappa H^p = (\Gamma_\kappa H^p)^* = H^p \Gamma^*_\kappa, \tag{4.11}$$

if Γ^*_κ is the transpose of Γ_κ. If the parametrization (4.1) of S is sufficiently smooth, (4.11) is a consequence of the Codazzi equations but can be deduced more simply and directly as follows: Since $N^p_\alpha = \partial N^p / \partial v^\alpha$ is a linear combination of the vectors $B_1, \cdots, B_\rho$ and $N^1, \cdots, N^\delta$, it is easy to see that the following analogue of the derivation formulae of Weingarten hold

$$N^p_\beta = -g^{\alpha\gamma} h^p_{\alpha\beta} B_\gamma + d^p_{\beta q} N^q,$$

where $0 = h^p_{\kappa\beta} = -N^p_\beta \cdot A_\kappa$ and these derivation formulae define $d^p_{\beta q}$. Multiplying these relations scalarly by $A_{\kappa\alpha}$ and using (4.9) gives

$$A_{\kappa\alpha} \cdot N^p_\beta = -\Gamma^\gamma_{\kappa\alpha} h^p_{\gamma\beta}.$$

The left side is symmetric in the indices α, β; in fact, the relations $-h^p_{\alpha\beta} = X_\alpha \cdot N^p_\beta = X_\beta \cdot N^p_\alpha$ give the identity

$$A_{\kappa\alpha} \cdot N^p_\beta v^\kappa + B_\alpha \cdot N^p_\beta \equiv A_{\kappa\beta} \cdot N^p_\alpha v^\kappa + B_\beta \cdot N^p_\alpha,$$

so that $A_{\kappa\alpha} \cdot N^p_\beta \equiv A_{\kappa\beta} \cdot N^p_\alpha$. This is equivalent to (4.11).

The fact that the relative nullity of S is identically ν implies that if $x = (x^1, \cdots, x^\rho)$ is a ρ-dimensional vector, then

$$H^p x = 0 \quad \text{for } p = 1, \cdots, \delta \text{ implies that } x = 0. \tag{4.12}$$

The condition on the curvatures of 2-sections of S is equivalent to

$$\sum_{p=1}^{\delta} \left[(H^p x \cdot x)(H^p y \cdot y) - (H^p x \cdot y)(H^p y \cdot x) \right] \geqq 0 \tag{4.13}$$

for all real vectors $x = (x^1, \cdots, x^\rho)$, $y = (y^1, \cdots, y^\rho)$.

In order to prove (4.10), it will first be shown that (4.11)—(4.13) has the following implications for $\Gamma = \Gamma_\kappa$, fixed $\kappa = \rho + 1, \cdots, d$:

(4.14) the eigenvalues of Γ^* are real;

(4.15) if $c = 0$ is an eigenvalue of Γ^*, then the corresponding elementary divisors of Γ^* are simple.

On (4.14). Let $\Gamma^* x = cx$ for some $x \neq 0$. Then $H^p \Gamma^* x = cH^p x$, so that $H^p \Gamma^* x \cdot x = cH^p x \cdot x$. Since $H^p \Gamma^*$ and H^p are symmetric matrices, it follows that c is real if $H^p x \cdot x \neq 0$ for some p. Note that (4.13) is assumed for real vectors x, y but is then valid for real vectors y and complex vectors x. Thus if $H^p x \cdot x = 0$ for $p = 1, \cdots, \delta$, it follows from (4.13) that $H^p x \cdot y = 0$ for $p = 1, \cdots, \delta$ and for all real y (hence, for all complex y). Consequently, $H^p x = 0$ for $p = 1, \cdots, \delta$. By (4.12), this implies that $x = 0$ and gives a contradiction. Thus $H^p x \cdot x \neq 0$ for some p and, consequently, c is real. This proves (4.14).

On (4.15). Let $c = 0$ be an eigenvalue of Γ^* and suppose that there is a corresponding multiple elementary divisor. Then there is a vector x such that

$$\Gamma^* x = z \neq 0, \qquad \Gamma^* z = 0.$$

Then $H^p \Gamma^* z = 0$, so that $\Gamma H^p z = 0$. Hence $H^p z \cdot \Gamma^* y = 0$ for all y and $p = 1, \cdots, \delta$. Choosing $y = x$ gives $H^p z \cdot z = 0$ for $p = 1, \cdots, \delta$. As above, this implies that $H^p z = 0$ for $p = 1, \cdots, \delta$ and hence $z = 0$. This contradiction proves (4.15).

On (4.10). Suppose that $\Gamma = \Gamma_\kappa$ is not 0 for some κ at some point $(v^1, \cdots, v^\rho)$. Then by (4.14)—(4.15), Γ^* has a nonzero, real eigenvalue, say, $-1/c \neq 0$, and an eigenvector $(c^1, \cdots, c^\rho) \neq 0$, i.e.,

$$c^\alpha (c\Gamma^b_{\kappa\alpha} + \delta_{\alpha\beta}) = 0 \quad \text{for } \beta = 1, \cdots, \rho.$$

In the second part of (4.6), choose $v^\lambda = 0$ if $\lambda \neq \kappa$ (κ fixed) and $v^\kappa = c$, so that

$$c^\alpha X_\alpha = c^\alpha (cA_{\kappa\alpha} + B_\alpha);$$

by (4.8),

$$c^\alpha X_\alpha = c^\alpha (c\Gamma^\beta_{\kappa\alpha} + \delta_{\alpha\beta}) B_\beta = 0.$$

This contradicts the linear independence of $X_1, \cdots, X_\rho$ and shows that $\Gamma_\kappa \equiv 0$. This completes the proof of Lemma 4.1.

Appendix

5. In view of the uses of Lemma 2 in [4] and of its generalization Lemma 2.1 above, it seems of interest to generalize it further. This appendix

deals with a generalization in which "gradient maps" are replaced by "involutory systems."

Let $x = (x^1, \cdots, x^d)$, $w = (w^1, \cdots, w^d)$ denote d-tuples of real numbers. The Poisson bracket (F, G) of two real-valued functions $F(x, w)$, $G(x, w)$ of class C^1 is defined to be

$$(5.1) \qquad \begin{aligned} (F, G) &= \sum_{k=1}^{d} [(\partial F/\partial x^k)(\partial G/\partial w^k) - (\partial F/\partial w^k)(\partial G/\partial x^k)] \\ &= \sum_{k=1}^{d} \partial(F, G)/\partial(x^k, w^k). \end{aligned}$$

A set $X = (X^1(x, w), \cdots, X^d(x, w))$ of d real-valued functions of class C^1 will be said to be an involutory system if the following two conditions hold:

$$(5.2) \qquad (X^i, X^j) \equiv 0 \quad \text{for } i, j = 1, \cdots, d,$$

$$(5.3) \qquad \operatorname{rank}(\partial X^i/\partial x^j, \partial X^i/\partial w^k) = d;$$

cf. [1, Chapter 6]. In (5.3), $(\partial X^i/\partial x^j, \partial X^i/\partial w^k)$ is a matrix with d rows $(i = 1, \cdots, d)$ and $2d$ columns $(j = 1, \cdots, d$ and $k = 1, \cdots, d)$.

The result to follow concerns δ involutory systems

$$X_p = (X_p^1(x, y_p), \cdots, X_p^d(x, y_p)),$$

where $p = 1, \cdots, \delta$. For a fixed $p, X_p^i(x, y_p)$ is a function of $2d$ real variables $(x, y_p) = (x^1, \cdots, x^d, y_p^1, \cdots, y_p^d)$; but in dealing with different values of $p, y = (y_1, \cdots, y_\delta) = (y_1^1, \cdots, y_1^d, y_2^1, \cdots, y_\delta^d)$ is considered as a set of $d\delta$ variables. For example, Lemma 2.1 concerns the δ involutory systems $X_p = (X_p^1, \cdots, X_p^d)$, where

$$(5.4) \qquad X_p^i(x, y_p) = P_p^i(x) - y_p^i \quad \text{for } i = 1, \cdots, d \text{ and } p = 1, \cdots, \delta.$$

Let D be an open set in the $(d + d\delta)$-dimensional $(x, y) = (x^1, \cdots, x^d, y_1^1, \cdots, y_\delta^d)$-space; $X = (X_1, \cdots, X_\delta) = (X_1^1, \cdots, X_\delta^d)$ a set of $d\delta$ functions $X_p^i(x, y_p)$ of class C^1 such that each function depends only on $2d$ variables (x, y_p) and, for a fixed $p = 1, \cdots, \delta$, the set $X_p = (X_p^1(x, y_p), \cdots, X_p^d(x, y_p))$ is an involutory system.

Let $J_p(x, y_p)$ be the $d \times d$ Jacobian matrix $J_p = (\partial X_p^i/\partial x^j)$, where $i, j = 1, \cdots, d$, and $J(x, y)$ the Jacobian matrix $J = (\partial X/\partial x)$, where $X = (X_1, \cdots, X_\delta) = (X_1^1, \cdots, X_1^d, X_2^1, \cdots, X_\delta^d)$,

$$J = \begin{pmatrix} J_1 \\ \vdots \\ J_\delta \end{pmatrix},$$

so that J has d columns and $d\delta$ rows. For $(x, y) \in D$, let

$$\rho(x, y) = \operatorname{rank} J(x, y) \quad \text{and} \quad \rho^*(x_0, y_0) = \limsup \rho(x, y)$$

as $(x, y) \to (x_0, y_0)$. For a given integer k, let S_k be the open subset of D defined by

$$S_k = \{(x, y) \in D : \rho^*(x, y) \leqq k\}$$

and $S_k(y)$ the open set in x-space given by

$$S_k(y) = \{x : (x, y) \in S_k\}.$$

LEMMA 5.1. *Let $X_p^j(x, y_p)$ be $d\delta$ real-valued functions of class C^1 on an (x, y)-domain D such that $X_p^j(x, y_p)$ depends only on $2d$ real variables and $X_p = (X_p^1(x, y_p), \cdots, X_p^d(x, y_p))$ is an involutory system for $p = 1, \cdots, \delta$. Let $(x_0, y_0) \in D$ have the property that $\rho(x_0, y_0)$, $\rho^*(x_0, y_0)$ have a common value $\rho = d - \nu$. Then*

$$X(x, y_0) = X(x_0, y_0) \tag{5.5}$$

on a unique ν-dimensional plane section $\pi_\nu(x_0)$ of $S_\rho(y_0)$ through x_0; for points x near x_0, (5.5) holds if and only if $x \in \pi_\nu(x_0)$; finally, $\rho(x, y_0) = \rho^*(x, y_0)$ for all $x \in \pi_\nu(x_0)$.

A "local" analogue of this lemma is known for the case $\delta = 1$ (under slightly stronger differentiability conditions); cf. [1, pp. 95-96].

Proof. Let $\rho = \rho(x_0, y_0) = \rho^*(x_0, y_0)$. Without loss of generality, it can be supposed that the first ρ columns of J are linearly independent (so that each of the remaining $d - \rho$ columns of J are linear combinations of the first ρ columns) for (x, y) near (x_0, y_0). It will be shown that

(†) there exists a ν-plane

$$\pi : x^\alpha = \sum_{\kappa=\rho+1}^{d} a^{\alpha\kappa} x^\kappa + b^\alpha, \qquad \alpha = 1, \cdots, \rho, \tag{5.6}$$

where $a^{\alpha\kappa}, b^\alpha$ are constants, such that π passes through x_0 and that (5.5) holds on the connected component of $\pi \cap S_\rho(y_0)$ containing x_0; furthermore, for x near x_0, (5.5) holds if and only if $x \in \pi$.

The local part of (†) will be deduced from Lemma 2.1 for arbitrary $\delta \geqq 1$ and some (essentially) known results on involutory systems.

Consider first one involutory system $X^1(x, w), \cdots, X^d(x, w)$ of class C^1 in a vicinity of a point $(x, w) = (x_0, w_0)$. Put

$$z' = X(x, w) \quad \text{and} \quad z_0' = X(x_0, w_0). \tag{5.7}$$

(a) Suppose that

$$\det(\partial X^i / \partial w^j) \neq 0 \qquad \text{at } (x_0, w_0); \tag{5.8}$$

then (5.7) has a unique solution for w of class C^1,

$$w = W(x, z') = (W^1, \cdots, W^d), \tag{5.9}$$

for (x, z') near (x_0, z'_0) and there exists a function $f(x, z')$ of class C^1 such that

$$W^i = \partial f/\partial x^i \quad \text{for } i = 1, \cdots, d. \tag{5.10}$$

(It follows that $\partial f/\partial x^i$ is of class C^1, but $\partial f/\partial z'^i$ may only be continuous.)

The proof of this is a considerably simplified version of the arguments of [1, pp. 90-91]. In order to prove (a), it is sufficient to verify that

$$\partial W^i/\partial x^j = \partial W^j/\partial x^i \quad \text{for } i, j = 1, \cdots, d. \tag{5.11}$$

To this end, substitute (5.9) into (5.7) and differentiate X^i with respect to x^j to obtain

$$0 = \partial X^i/\partial x^j + \sum_{k=1}^{d} (\partial X^i/\partial w^k)(\partial W^k/\partial x^j).$$

Multiply this relation by $\partial X^m/\partial w^j$ and add for $j = 1, \cdots, d$,

$$\sum_{j=1}^{d} \sum_{k=1}^{d} (\partial X^i/\partial w^k)(\partial W^k/\partial x^j)(\partial X^m/\partial w^j)$$
$$= -\sum_{j=1}^{d} (\partial X^i/\partial x^j)(\partial X^m/\partial w^j) = -\sum_{j=1}^{d} (\partial X^i/\partial w^j)(\partial X^m/\partial x^j),$$

where the last equality is a consequence of the fact that $X^1, \cdots, X^d$ is an involutory system. Let T, S denote the matrices

$$T = (\partial X^i/\partial w^j), \quad S = (\partial W^i/\partial x^j).$$

Then the expression on the left is the (i, m)th element of the matrix product TST^*. It follows that TST^* is a symmetric matrix. Since T is nonsingular, it is seen that S is symmetric, i.e., that (5.11) holds. This proves (a).

(b) Let $X = (X^1(x, w), \cdots, X^d(x, w))$ be an involutory system as in step (a) and let

$$r = \operatorname{rank}(\partial X^i/\partial x^j) \quad \text{at } (x_0, w_0)$$

and let the columns $\partial X/\partial x^i$ be linearly independent for $i = 1, \cdots, r$. Then

$$\det(\partial X/\partial x^1, \cdots, \partial X/\partial x^r, \partial X/\partial w^{r+1}, \cdots, \partial X/\partial w^d) \neq 0 \quad \text{at } (x_0, w_0)$$

and $\Xi(\xi, \eta) = X(x, w)$, where

$$\xi^\alpha = x^\alpha, \quad \xi^\kappa = -w^\kappa \quad \text{and} \quad \eta^\alpha = w^\alpha, \quad \eta^\kappa = x^\kappa$$
$$\text{for } \alpha = 1, \cdots, r, \ \kappa = r + 1, \cdots, d$$

is an involutory system, i.e.,

$$\sum_{k=1}^{d} \partial(\Xi^i, \Xi^m)/\partial(\xi^k, \eta^k) = 0 \quad \text{for } i, m = 1, \cdots, d.$$

This follows from considerations of [1, pp. 85-87].

Combining (a) and (b) gives:

(c) Let $X(x,w)$, z', x_0, w_0, r be as in (a), (b). Then there exists a C^1 real-valued function $f(x^{r+1},\cdots,x^d, w^1,\cdots,w^r,z')$ of $2d$ real variables such that (5.7) has a unique solution of class C^1 given by

$$x^\alpha = g^\alpha(x^{r+1},\cdots,x^d, w^1,\cdots,w^r,z') \quad \text{for } \alpha = 1,\cdots,r, \tag{5.12}$$
$$w^\kappa = h^\kappa(x^{r+1},\cdots,x^d, w^1,\cdots,w^r,z') \quad \text{for } \kappa = r+1,\cdots,d,$$

and

$$g^\alpha = \partial f/\partial w^\alpha, \qquad h^\kappa = -\,\partial f/\partial x^\kappa. \tag{5.13}$$

Thus, if

$$w'^i = \partial f(x^\kappa, w^\alpha, z')/\partial z'^i, \tag{5.14}$$

then

$$\sum_{i=1}^{d} w'^i dz'^i + \sum_{\alpha=1}^{r} x^\alpha dw^\alpha - \sum_\kappa w^\kappa dx^\kappa = df.$$

REMARK. It can be mentioned that if w' is made a function of (x,w) by inserting (5.7) into (5.14), say,

$$w' = (W^1,\cdots,W^d), \quad \text{where } W^i = \partial f/\partial z'^i \text{ at } z' = X(x,w), \tag{5.15}$$

then

$$z' = X(x,w), \qquad w' = W(x,w)$$

is a canonical transformation in the sense that

$$w'\cdot dz' - w\cdot dx \quad \text{is closed}$$

(i.e., is locally a total differential of a function of class C^1). Note, however, that $W(x,w)$ in (5.15) may only be continuous. (This is a variant of the standard deduction of a C^1 canonical transformation from an involutory system of class C^2.)

(d) Let X, z', x_0, w_0, r and f be as in (c) and put

$$F(x,w) = \sum_{\beta=1}^{n} x^\beta w^\beta + \frac{1}{2}\sum_{\lambda=r+1}^{d} (w^\lambda)^2 - f(x^\kappa, w^\alpha, z_0'), \tag{5.16}$$

then $F(x,w)$ is of class C^2 and the equation

$$X(x,w_0) = z_0' \quad [= X(x_0,w_0)] \tag{5.17}$$

is equivalent to

$$\nabla F(x,w) = \nabla F(x_0,w_0), \tag{5.18}$$

where

$$\nabla F = (\partial F/\partial x^1, \cdots, \partial F/\partial x^d, \partial F/\partial w^1, \cdots, \partial F/\partial w^d).$$

Actually, $F(x,w) = F(x,w;z_0')$ and F is a function of class C^1 in (x,w,z_0').

Assertion (d) is clear from the fact that (5.7) and (5.12) are equivalent and that (5.13), (5.16) give

$$\begin{aligned} &\partial F/\partial x^\beta = w^\beta,\ \partial F/\partial w^\beta = x^\beta - g^\beta(x^\kappa, w^\alpha, z_0') \quad \text{for } \beta = 1, \cdots, r, \\ &\partial F/\partial x^\lambda = h^\lambda(x^\kappa, w^\alpha, z_0'),\ \partial F/\partial w^\lambda = w^\lambda \quad \text{for } \lambda = r+1, \cdots, d. \end{aligned} \tag{5.19}$$

(e) Note that, at (x_0, w_0), $\operatorname{rank}(\partial(\nabla F)/\partial w) = d$ and that $\operatorname{rank}(\partial(\nabla F)/\partial x) = r$. In fact the $2d \times d$ Jacobian matrix $(\partial(\nabla F)/\partial x)$ consists of a $d \times d$ zero matrix and a $d \times d$ matrix obtained by multiplying $(\partial X/\partial x)$ by a nonsingular matrix.

(f) **Proof of the local part of** (†). Define $x' = (x_1', \cdots, x_\delta')$ by $x_p' = X(x, y_p)$. Then, by (d), there exists a function $F_p(x,y_p)$ of class C^2 such that for (x,y_p) near (x_0, y_{p0}),

$$X_p(x,y_p) = x_{p0}' \qquad [= X_p(x_0, y_{p0})] \tag{5.20}$$

is equivalent to

$$\nabla F_p(x,y_p) = \nabla F_p(x_0, y_{p0}). \tag{5.21}$$

Also, $F_p(x,y_p) = F_p(x,y_p;x_{p0}')$ is of class C^1.

Consider $F_p(x,y_p) = F_p(x,y)$ to be a function of $d + d\delta$ variables (x,y). Let $K_p(x,y)$ be the $(d+d\delta) \times (d+d\delta)$ matrix which is the Jacobian matrix $\partial(\nabla F_p)/\partial(x,y)$, where ∇F_p is the gradient of $F_p(x,y)$. Then, if

$$\rho_0(x,y) = \operatorname{rank} K(x,y), \text{ where } K(x,y) = \begin{pmatrix} K_1(x,y) \\ \vdots \\ K_\delta(x,y) \end{pmatrix}$$

and $\rho_0^*(x_0,y_0) = \limsup \rho_0(x,y)$ as $(x,y) \to (x_0,y_0)$, it follows that $\rho_0(x_0,y_0) = \rho_0^*(x_0,y_0) = \rho + d\delta$. In fact, if the first column of $K(x,y)$ is obtained by differentiating $(\nabla F_1, \nabla F_2, \cdots, \nabla F_p)$ with respect to x^1, the second with $x^2, \cdots$ and the last with y_δ^d, then the first ρ and last $d\delta$ columns of $K(x,y)$ are linearly independent. For the construction of F in (d) shows that any linear homogeneous relation between the first d columns of $K(x,y)$ is a consequence of the same relation between the columns of $J(x,y)$.

Thus, Lemma 2.1 implies that there exist constants $a^{\alpha k}$, b^α such that (5.5) holds for x near x_0 if and only if x is on the ν-plane π in (5.6). Furthermore, $a^{\alpha\kappa}$, b^α are C^1 functions of (y_0, x_0').

(g) **Completion of proof.** It remains to prove the "in the large" assertion of Lemma 5.1. To this end, note that the analogue of condition (5.3) implies that the set of $d\delta$ variables $y = (y_1^1, \cdots, y_\delta^d)$ can be divided into two sets $v = (v^1, \cdots, v^{d\delta-\rho})$ and $u = (u^1, \cdots, u^\rho)$ such that, at $(x,y) = (x_0,y_0)$,

$$\det(\partial X/\partial x^1, \cdots, \partial X/\partial x^\rho, \partial X/\partial v^1, \cdots, \partial X/\partial v^{d\delta-\rho}) \neq 0.$$

Hence, the equations $x' = X(x, y)$ can be solved locally for $x^1, \cdots, x^\rho$ and v in terms of $x', x^{\rho+1}, \cdots, x^d$, and u:

$$x^\alpha = g^\alpha(x^{\rho+1}, \cdots, x^d, u, x') \qquad \text{for } \alpha = 1, \cdots, \rho,$$
$$v^\sigma = h^\sigma(x^{\rho+1}, \cdots, x^d, u, x') \qquad \text{for } \sigma = 1, \cdots, d\delta - \rho,$$

where g^α, h^σ are of class C^1.

The fact that $\rho(x_0, y_0) = \rho^*(x_0, y_0)$ implies that $h^\sigma = h^\sigma(u, x')$ does not depend on $x^{\rho+1}, \cdots, x^d$. Also, the local part of (†) implies that g^α is linear in $x^{\rho+1}, \cdots, x^d$:

$$(5.22) \qquad \begin{aligned} x^\alpha &= \sum_{\kappa=\rho+1}^{d} a^{\alpha\kappa}(u, x')x^\kappa + b^\alpha(u, x') \quad \text{for } \alpha = 1, \cdots, \rho, \\ v^\sigma &= h^\sigma(u, x') \quad \text{for } \sigma = 1, \cdots, d\delta - \rho, \end{aligned}$$

where $a^{\alpha\kappa}$, b^α and h^σ are of class C^1. Since (5.22) is the inverse of $x' = X(x, y)$ for fixed $x^{\rho+1}, \cdots, x^d$ and u,

$$1 = \det(\partial X/\partial x^\alpha, \partial X/\partial v^\sigma) \cdot \det(\partial(x^\alpha, v^\sigma)/\partial x')$$

in obvious notation. It is clear from (5.22) that the second factor is bounded if $y = y_0$ (hence u) and $x' = x_0'$ are fixed and x is bounded. Consequently, the completion of the proof of Lemma 5.1 is similar to that of Lemma 2.1 (or of Lemma 2 in [4]).

References

1. C. Carathéodory, *Variationsrechnung und partielle Differentialgleichungen erster Ordnung,* Teubner, Leipzig, 1935.

2. S. S. Chern and N. H. Kuiper, *Some theorems on the isometric embeddings of compact Riemann manifolds in Euclidean space,* Ann. of Math. (2) **56** (1952), 422-430.

3. S. S. Chern and R. K. Lashoff, *The total curvature of immersed manifolds,* Amer. J. Math. **79** (1957), 308-318.

4. P. Hartman and L. Nirenberg, *On spherical image maps whose Jacobians do not change sign,* Amer. J. Math. **81** (1959), 901-920.

5. B. O'Neill, *Isometric immersion of flat Riemann manifolds in Euclidean space,* Michigan Math. J. **9** (1962), 199-205.

6. R. Sacksteder, *On hypersurfaces with no negative sectional curvatures,* Amer. J. Math. **82** (1960), 609-630.

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