ON THE CENTRAL DECOMPOSITION FOR POSITIVE FUNCTIONALS ON C^*-ALGEBRAS

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1. **Introduction.** Let A be a C^*-algebra with unit 1, A^* the dual Banach space of A, $\mathfrak{S}$ the set of all states ϕ on A (namely, $\phi(a^*a) \geqq 0$ for $a \in A$ and $\phi(1) = 1$), Ω the set of all extreme points of $\mathfrak{S}$.

Then $\mathfrak{S}$ is a $\sigma(A^*, A)$-compact space. Let $C(\mathfrak{S})$ be the Banach algebra of all continuous complex-valued functions on the compact space $\mathfrak{S}$ with the usual supremum norm, then the A may be topologically embedded into $C(\mathfrak{S})$; moreover, the self-adjoint portion A^s of A may be order-isomorphically embedded into the real Banach space $C_r(\mathfrak{S})$ of all continuous real-valued functions on $\mathfrak{S}$.

Any positive linear functional ψ on A extends to a positive linear functional on $C(\mathfrak{S})$ and thus, by the Riesz representation theorem, there will be some positive Radon measure μ on $\mathfrak{S}$ so that

$$(*) \qquad \psi(a) = \int \phi(a)\, d\mu(\phi) \qquad \text{for } a \in A.$$

Let $\mathfrak{M}(\psi)$ be the set of all positive Radon measures on $\mathfrak{S}$ satisfying the equality (*). In general, $\mathfrak{M}(\psi)$ consists of many different measures. An important family of measures belonging to $\mathfrak{M}(\psi)$ is the one consisting of measures concentrated on Ω. M. Tomita [11], [18] showed that if A is separable, $\mathfrak{M}(\psi)$ contains a measure μ such that $\mu(\Omega) = \mu(\mathfrak{S})$ and gave a refinement of the Mautner-Godement-Segal theorem [10], [7], [16] that an arbitrary separable $*$-representation of A can be expressed as a direct integral of irreducible $*$-representations—that is, he showed that the Mautner-Godement-Segal decomposition can be realized by a Radon measure on the compact space $\mathfrak{S}$. Moreover, at the present time, we can obtain this result of Tomita and, furthermore, show that Ω is a G_δ-set, if A is separable, from the general theory of Choquet [2], Bishop and de Leeuw [1].

Also, L. Loomis [8] gave further developments along this line.

If A is commutative, the μ such that $\mu \in \mathfrak{M}(\psi)$ and $\mu(\Omega) = \mu(\mathfrak{S})$ is unique; this implies the classical theorem that the problem of determining all cyclic unitary representations of a locally compact abelian group G can be reduced to the prob-

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lem of determining all equivalence classes of bounded positive Radon measures on the dual group $\hat{G}$.

If A is not commutative, such μ is, in general, not unique even for a finite-dimensional A. Therefore, in spite of their importance, such measures are not suitable for the decomposition theory of representations.

On the other hand, von Neumann's reduction theory (cf. [10], [12]) insures that every separable $*$-representation of A may be expressed as a direct integral of factor $*$-representations (called the central decomposition) and such a decomposition is essentially unique.

From this fact, for a separable A, we can guess the existence and the unicity of a distinguished Radon measure belonging to $\mathfrak{M}(\psi)$ which realizes the central decomposition of the $*$-representation of A constructed via ψ on the compact space $\mathfrak{S}$.

The main purpose of this paper is to show that this guess is true. We say that ϕ is primary if the $*$-representation of A constructed via ϕ is a factor representation. Let K be the set of all primary states on A. Then we shall show: if A is separable, K is measurable for all Radon measures ν on the compact space $\mathfrak{S}$ (Theorem 1) and, moreover, there is one and only one measure μ belonging to $\mathfrak{M}(\psi)$ such that $\mu(K) = \mu(\mathfrak{S})$ and the expression $\psi(a) = \int_{\mathfrak{S}} \phi(a)\, d\mu(\phi)$ realizes the central decomposition of the $*$-representation of A constructed via ψ (Theorems 2, 3 and 4).

We call such a unique μ the *central Radon measure* of ψ.

2. **Preliminaries.** Let $\mathfrak{A}$ be a C^*-algebra. The term "$*$-representation" of $\mathfrak{A}$ shall mean a homomorphism $a \to \pi_a$ of $\mathfrak{A}$ into the algebra of all bounded operators on some hilbert space $\mathfrak{H}$ such that $(\pi_a)^* = \pi_{a^*}$ for $a \in \mathfrak{A}$, and denote it by $\pi(\mathfrak{H})$. Two $*$-representations $\pi^1(\mathfrak{H}_1)$, $\pi^2(\mathfrak{H}_2)$ are said to be equivalent if there exists a unitary mapping U of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$ such that $U\pi_a^1 U^{-1} = \pi_a^2$ for $a \in \mathfrak{A}$.

Let $\mathfrak{A}^{**}$ be the second dual Banach space of $\mathfrak{A}$; then, according to a result of Sherman [17], $\mathfrak{A}^{**}$ may be regarded as a W^*-algebra, whose associated space is the dual Banach space $\mathfrak{A}^*$ of $\mathfrak{A}$, and $\mathfrak{A}$ may be regarded as a σ-dense C^*-subalgebra (the σ-topology on $\mathfrak{A}^{**}$ shall mean the weak$*$-topology on $\mathfrak{A}^{**}$) of $\mathfrak{A}^{**}$, when $\mathfrak{A}$ is canonically embedded into $\mathfrak{A}^{**}$.

Definition 1. Let G be a locally compact group, and $A(G)$ the group C^*-algebra of G (cf. 6); then the W^*-algebra $A(G)^{**}$ is called the group W^*-algebra of G and denoted by $W(G)$.

Definition 2. Let M be a W^*-algebra; then a W^*-representation $\pi(\mathfrak{H})$ of M is a continuous $*$-representation of M with the σ-topology (the σ-topology on M shall mean the weak $*$-topology on M (cf. [14])) into the algebra $B(\mathfrak{H})$, with the weak operator topology, of all bounded operators on a hilbert space $\mathfrak{H}$ such that π_1 is the identity operator on $\mathfrak{H}$ for the identity 1 of M.

We shall denote by $\pi[M]$ the image of M under π. If $\pi(\mathfrak{H})$ is a W^*-representation of a W^*-algebra M, $\pi[M]$ is a weakly closed $*$-subalgebra (called a concrete W^*-algebra) of $B(\mathfrak{H})$ (cf. 14).

Now let $\pi(\mathfrak{H})$ be a $*$-representation of a C^*-algebra $\mathfrak{A}$ such that $\{\xi \mid \pi_a\xi = 0$ for $\xi \in \mathfrak{H}$ and all $a \in \mathfrak{A}\} = (0)$ (called a nowhere trivial $*$-representation), then $\pi(\mathfrak{H})$ can be uniquelly extended to a W^*-representation of $\mathfrak{A}^{**}$ (cf. [14, Theorem in Appendix]). We shall denote also by $\pi(\mathfrak{H})$ this W^*-representation of $\mathfrak{A}^{**}$.

Conversely, let $\tilde{\pi}(\mathfrak{H})$ be a W^*representation of $\mathfrak{A}^{**}$, then its restriction on $\mathfrak{A}$ is a nowhere trivial $*$-representation $\pi(\mathfrak{H})$ of $\mathfrak{A}$. Moreover, the W^*-representation of $\mathfrak{A}^{**}$ obtained from the $*$-representation $\pi(\mathfrak{H})$ of $\mathfrak{A}$ coincides clearly with the $\tilde{\pi}(\mathfrak{H})$.

The W^*-representation of $\mathfrak{A}^{**}$ is irreducible if and only if the corresponding $*$-representation of $\mathfrak{A}$ is irreducible; two W^*-representations of $\mathfrak{A}^{**}$ are equivalent if and only if the corresponding two $*$-representations of $\mathfrak{A}$ are equivalent. Therefore the theory of $*$-representations of $\mathfrak{A}$ can be reduced to the W^*-representation theory of $\mathfrak{A}^{**}$, so that the unitary representation theory of a locally compact group G can be reduced to the W^*-representation theory of the group W^*-algebra $W(G)$.

Let M be a W^*-algebra, and $\pi(\mathfrak{H})$ a W^*-representation of M; then the kernel $\mathfrak{I} = \{a \mid \pi_a = 0,\ a \in M\}$ is a σ-closed ideal of M; hence there is a unique central projection z such that $\mathfrak{I} = M_z$ (cf. [14]). Put $1 - z = s(\pi)$; $s(\pi)$ is called the support of π. The support of π is a nonzero central projection and the restriction of π on $Ms(\pi)$ is one-to-one.

DEFINITION 3. Let $\pi^1(\mathfrak{H}_1)$, $\pi^2(\mathfrak{H}_2)$ be two W-*representations of M. If $s(\pi^1) = s(\pi^2)$, we say that $\pi^1(\mathfrak{H}_1)$ is quasi-equivalent to $\pi^2(\mathfrak{H}_2)$.

Clearly, the quasi-equivalence is a usual equivalence relation, so that by this relation we can classify W^*-representations of M into quasi-equivalence classes.

Let $\mathfrak{D}(M)$ be the family of all quasi-equivalence classes of the W^*-representations of M; then, for each element $\rho \in \mathfrak{D}(M)$, there corresponds a unique nonzero central projection $c(\rho)$ of M such that $c(\rho) = s(\pi)$ for every $\pi(\mathfrak{H}) \in \rho$.

Conversely, let z be a central nonzero projection; then M_z is a W^*-algebra, so that it has a faithful W^*-representation $\pi(\mathfrak{H})$ (cf. [13]).

Then the mapping $a \to \pi_{az}$ of M into $B(\mathfrak{H})$ is a W^*-representation of M; hence there is unique element ρ of $\mathfrak{D}(M)$ such that $c(\rho) = z$.

Therefore, there is a one-to-one correspondence between all elements of $\mathfrak{D}(M)$ and all nonzero central projections of M.

For convenience, we shall add an imaginary 0-element to $\mathfrak{D}(M)$ and we shall denote by $\mathfrak{D}'(M)$ the set $\mathfrak{D}(M) \cup (0)$; moreover, we shall make the 0-element of M correspond to this 0-element of $\mathfrak{D}'(M)$. Then there is a one-to-one correspondence between $\mathfrak{D}'(M)$ and the set Z_p of all central projections of M. Since Z_p is a complete Boolean algebra, we can canonically introduce its Boolean structure into $\mathfrak{D}'(M)$.

Also, we can regard every element ρ of $\mathfrak{D}(M)$ as a σ-continuous $*$-homomorphism of M onto the W^*-algebra $Mc(\rho)$. Therefore, within quasi-equivalence, the W^*-representation theory of M can be completely reduced to the structure theory of the W^*-algebra M.

So we shall freely use various definitions and theorems concerning the structure of W^*-algebras. For them, the reader should be referred to [3] and [14].

Let $\pi(\mathfrak{H})$ be a nowhere trivial $*$-representation of a C^*-algebra $\mathfrak{A}$. Then the support $s(\pi)$ of $\pi(\mathfrak{H})$ shall mean the support of the corresponding W^*-representation $\pi(\mathfrak{H})$ of the W^*-algebra $\mathfrak{A}^{**}$.

Let $\pi^1(\mathfrak{H}_1)$, $\pi^2(\mathfrak{H}_2)$ be two nowhere trivial $*$-representations of $\mathfrak{A}$. If the corresponding two W^*-representations of $\mathfrak{A}^{**}$ are quasi-equivalent, we say that $\pi^1(\mathfrak{H}_1)$ and $\pi^2(\mathfrak{H}_2)$ are quasi-equivalent.

3. **The decomposition theory.** Let A be a C^*-algebra with unit 1. In this section we shall always assume that A is uniformly separable. Let A^* be the dual Banach space of A and $\mathfrak{S}$ the set of all states on A; then $\mathfrak{S}$ is $\sigma(A^*, A)$-compact; let $\{a_n\}$ be a sequence of nonzero elements which is uniformly dense in the selfadjoint portion A^s of A.

For ϕ, $\psi \in \mathfrak{S}$, define

$$d(\phi, \psi) = \sum_{i=1}^{\infty} \frac{|(\phi - \psi)(a_n)|}{2^n \| a_n \|};$$

then d is a metric on $\mathfrak{S}$ which is equivalent to $\sigma(\mathfrak{S}, A)$; therefore, $\mathfrak{S}$ is considered a compact metric space with respect to $\sigma(\mathfrak{S}, A)$; hence, the compact space $\mathfrak{S}$ satisfies the second countability axiom, because a compact metric space is always separable. In this section we shall always consider the $\mathfrak{S}$ as the compact space with the topology $\sigma(\mathfrak{S}, A)$.

Let $\phi \in \mathfrak{S}$, and let V_ϕ be the invariant closed subspace under R_a, $L_a (a \in A)$ generated by ϕ; then there is a unique central projection z_ϕ such that $L_{z_\phi} A^* = V_\phi$ (cf. [14], [19]); clearly z_ϕ is the least central projection of A^{**} containing the support $s(\phi)$ of ϕ on A^{**} (cf. [14]) and, moreover, $s(\pi^\phi) = z_\phi$, where $\pi^\phi(\mathfrak{H}_\phi)$ is the $*$-representation of A constructed via ϕ.

Now we shall show

THEOREM 1. *Let K be a subset of $\mathfrak{S}$ consisting of all primary states ϕ (namely, the $*$-representation $\pi^\phi(\mathfrak{H}_\phi)$ of A is a factor representation); then K is v-measurable for all Radon measures v on the compact space $\mathfrak{S}$.*

Proof. It is clear that the primarity of ϕ is equivalent to the fact that the support $s(\pi^\phi)$ of $\pi^\phi(\mathfrak{H}_\phi)$ is a minimal central projection of A^{**}.

Now put $\mathfrak{F} = \{\phi \mid \phi \geqq 0, \ \phi(1) \leqq 1, \ \phi \in A^*\}$; then $\mathfrak{F}$ is $\sigma(A^*, A)$-compact; let $\{a_n\}$ be a sequence of elements which is uniformly dense in A. Consider the product space $\mathfrak{F} \times \mathfrak{F}$ and put

$$G_{m,n} = \{(\phi, \psi) \mid \|R_{a_m} L_{a_m^*} \phi - R_{a_n} L_{a_n^*} \psi\| = \|R_{a_m} L_{a_m^*} \phi + R_{a_n} L_{a_n^*} \psi\|, (\phi, \psi) \in \mathfrak{F} \times \mathfrak{F}\}.$$

Since

$$\left\| R_{a_m}L_{a_m^*}\phi \pm R_{a_n}L_{a_n^*}\psi \right\| = \sup_{\|a\| \leqq 1} \left| \phi(a_m^* a a_m) \pm \psi(a_n^* a a_n) \right|$$

and the function $(\phi, \psi) \to \phi(a_m^* a a_m) \pm \psi(a_n^* a a_n)$ on the compact space $\mathfrak{F} \times \mathfrak{F}$ is continuous, $\| R_{a_m}L_{a_m^*} \pm R_{a_n}L_{a_n^*}\psi \|$ is lower semi-continuous; hence, $G_{m,n}$ is a Borel set in the compact space $\mathfrak{F} \times \mathfrak{F}$; put $F = \{(\phi, \psi) \mid \phi(1) + \psi(1) = 1, (\phi, \psi) \in \mathfrak{F} \times \mathfrak{F}\}$; then F is a closed set in $\mathfrak{F} \times \mathfrak{F}$; and put $\mathfrak{F}_0 = \{\phi \mid \phi > 0, \phi(1) \leqq 1\}$; then $\mathfrak{F}_0 \times \mathfrak{F}_0$ is an open set in $\mathfrak{F} \times \mathfrak{F}$.

Now put $U = \{\mathfrak{F}_0 \times \mathfrak{F}_0\} \cap (F \cap \bigcap_{m,n=1}^{\infty} G_{m,n})$; then, clearly, U is a Borel set in $\mathfrak{F} \times \mathfrak{F}$.

Consider the mapping $(\phi, \psi) \xrightarrow{\Phi} \phi + \psi$ of F onto $\mathfrak{S}$; it is continuous, so that $\Phi(U)$ is analytic and so ν-measurable for all Radon measures ν on the compact space $\mathfrak{S}$ (cf. [3, Appendix (V)]).

If $\xi \in \Phi(U)$, then there is an element (ϕ, ψ) in U such that $\xi = \phi + \psi$; $(\phi, \psi) \in U$ implies $s(R_{a_m}L_{a_m^*}\phi) \cdot s(R_{a_n}L_{a_n^*}\psi) = 0$ for all m, n (cf. [14]) and so the closed invariant subspaces generated by ϕ and ψ, respectively, are mutually disjoint; hence $s(\pi^{\phi}) \cdot s(\pi^{\psi}) = 0$, so that $s(\pi^{\xi}) = s(\pi^{\phi}) + s(\pi^{\psi})$ is not minimal. Conversely, suppose $\xi \notin K$, then there is a nonzero central projection z of A^{**} such that $z < s(\pi^{\xi})$, then $\xi(a) = \xi(za) + \xi((s(\pi^{\xi}) - z)a)$ for $a \in A$; clearly $(L_z\xi, L_{s(\pi^{\xi})-z}\xi) \in U$, so that $\xi \in \Phi(U)$; Hence $K = \mathfrak{S} - \Phi(U)$. This completes the proof.

Problem 1. Can we conclude that K is a Borel subset in the compact space $\mathfrak{S}$?

Now let μ be a positive Radon measure on the compact space $\mathfrak{S}$. Put $\psi(a) = \int_{\mathfrak{S}} \phi(a)\, d\mu(\phi)$ for $a \in A$; then ψ is a positive linear functional on A. We shall always consider the ψ as a $\sigma(A^{**}, A^*)$-continuous positive linear functional on the W^*-algebra A^{**}, because positive linear functionals on A extend canonically to $\sigma(A^{**}, A^*)$-continuous positive linear functionals on A^{**}. Then,

Definition 4. We say μ is a central Radon measure if it satisfies the following condition:

(**) There is a σ-continuous homomorphism Φ of the center Z of the W^*-algebra A^{**} onto the W^*-algebra $L^{\infty}(\mathfrak{S}, \mu)$ of all essentially bounded μ-measurable functions on $\mathfrak{S}$ such that

$$\psi(za) = \int_{\mathfrak{S}} \Phi(z)(\phi)\, \phi(a)\, d\mu(\phi)$$

for $z \in Z$ and $a \in A$.

Now let μ be a positive central Radon measure on the compact space $\mathfrak{S}$ and put $\psi(a) = \int_{\mathfrak{S}} \phi(a)\, d\mu(\phi)$ for $a \in A$.

Consider the *-representation $\pi^{\psi}(\mathfrak{H}_{\psi})$ of A and the corresponding W^*-representation $\pi^{\psi}(\mathfrak{H}_{\psi})$ of A^{**}. We shall define a linear mapping of $\mathfrak{H}_{\psi}$ into $\mathfrak{H}_{\phi}$-valued functions on $\mathfrak{S}$ in the following way.

Let $\xi = \sum_{i=1}^{n} \pi_{c_i}^{\psi} \pi_{a_i}^{\psi} 1_{\psi} \in \mathfrak{H}_{\psi}$, where $c_i \in Z$, $a_i \in A$ and 1_{ψ} is the image of 1 in $\mathfrak{H}_{\psi}$.

Then

$$\begin{aligned}
\|\xi\|^2 &= \left\langle \sum_{i=1}^{n} \pi^{\psi}_{c_i}\pi^{\psi}_{a_i}1_{\psi},\ \sum_{i=1}^{n} \pi^{\psi}_{c_i}\pi^{\psi}_{a_i}1_{\psi} \right\rangle \\
&= \sum_{i,j=1}^{n} \langle \pi^{\psi}_{c_i a_i}1_{\psi},\ \pi^{\psi}_{c_j a_j}1_{\psi} \rangle \\
&= \sum_{i,j=1}^{n} \psi(a_j^* c_j^* c_i a_i) \\
&= \sum_{i,j=1}^{n} \psi(c_j^* c_i a_j^* a_i) \\
&= \sum_{i,j=1}^{n} \int_{\mathfrak{S}} \Phi(c_j)^*(\phi)\Phi(c_i)(\phi)\phi(a_j^* a_i)\, d\mu(\phi).
\end{aligned}$$

On the other hand, for each $\phi \in \mathfrak{S}$, we consider the $*$-representation $\pi^{\phi}(\mathfrak{H}_{\phi})$ of A. For $a \in A$, the image of a in $\mathfrak{H}_{\phi}$ is denoted by a_{ϕ}. Then the function $\phi \rightarrow a_{\phi}$ on $\mathfrak{S}$ is a $\mathfrak{H}_{\phi}$-valued function. We shall denote this function by $\tilde{a}$. The function $\phi \rightarrow \sum_{i=1}^{n} \Phi(c_i)(\phi)a_{i,\phi}$ is a $\mathfrak{H}_{\phi}$-valued function and we have

$$\|\xi\|^2 = \int_{\mathfrak{S}} \left\| \sum_{i=1}^{n} \Phi(c_i)(\phi)a_{i,\phi} \right\|^2 d\mu(\phi).$$

Hence the mapping $\xi \rightarrow \sum_{i=1}^{n} \Phi(c_i)\tilde{a}_i$ extends uniquely a unitary mapping U of $\mathfrak{H}_{\psi}$ onto the direct integral $\int_{\mathfrak{S}}\mathfrak{H}_{\phi}\, d\mu(\phi)$ of the spaces $\mathfrak{H}_{\phi}$ with respect to the measure μ (cf. [3]), because elements of the form ξ are dense in $\mathfrak{H}_{\psi}$.

Hence, under this unitary mapping, we can write $\mathfrak{H}_{\psi} = \int_{\mathfrak{S}}\mathfrak{H}_{\phi}\, d\mu(\phi)$.

Moreover, for $c \in Z$,

$$\begin{aligned}
(U\pi^{\psi}_{c}\xi)(\phi) &= U\pi^{\psi}_{c} \sum_{i=1}^{n} \pi^{\psi}_{c_i}\pi^{\psi}_{a_i}1_{\psi} = U \sum_{i=1}^{n} \pi^{\psi}_{cc_i}\pi^{\psi}_{a_i}1_{\psi} \\
&= \sum_{i=1}^{n} \Phi(cc_i)(\phi)a_{i,\phi} = \Phi(c)(\phi) \sum_{i=1}^{n} \Phi(c_i)(\phi)a_{i,\phi} \\
&= \Phi(c)(\phi)(U\xi)(\phi);
\end{aligned}$$

hence $U\pi^{\psi}_{c}U^* = \Phi(c)$.

Since $\{\pi^{\psi}_{c} \mid c \in Z\} =$ the center of the concrete W^*-algebra $\pi^{\psi}[A^{**}]$ and $\Phi(Z) = L^{\infty}(\mathfrak{S}, \mu)$, under the realization $\mathfrak{H}_{\psi} = \int_{\mathfrak{S}} \mathfrak{H}_{\phi}\, d\mu(\phi)$, the center of $\pi^{\psi}[A^{**}]$ is the set of all diagonalizable operators.

Hence we have the central decomposition (cf. [12])

$$\pi^{\psi}[A^{**}] = \int_{\mathfrak{S}} \{\pi^{\psi}[A^{**}]\}(\phi)\, d\mu(\phi),$$

and $\{\pi^{\psi}[A^{**}]\}(\phi)$ is a factor for μ-almost all $\phi \in \mathfrak{S}$. $\{\pi^{\psi}[A^{**}]\}(\phi) = \pi^{\phi}[A^{**}]$ for μ-almost all $\phi \in \mathfrak{S}$, because $\pi^{\psi}[A]$ (respectively, $\pi^{\phi}[A]$) is weakly dense in $\pi^{\psi}[A^{**}]$ (respectively, $\pi^{\phi}[A^{**}]$). Therefore we have the following theorem.

THEOREM 2. *Let μ be a positive central Radon measure on the compact space $\mathfrak{S}$ and put $\psi(a) = \int_{\mathfrak{S}}\phi(a)\,d\mu(\phi)$ for $a \in A$. Then the $*$-representation $\pi^{\psi}(\mathfrak{H}_{\psi})$ of A is expressed as a direct integral of the $*$-representations $\pi^{\phi}(\mathfrak{H}_{\phi})$ of A, with respect to the measure μ, on the compact space $\mathfrak{S}$. Moreover, this integral is the central decomposition, and so it satisfies the following properties*:

(i) *$\pi^{\phi}(\mathfrak{H}_{\phi})$ is a factor representation for μ-almost all $\phi \in \mathfrak{S}$, i.e., $\mu(K) = \mu(\mathfrak{S})$,*

(ii) *there is a Borel subset $\Delta\mu$ of the compact space $\mathfrak{S}$ such that $\Delta\mu \subset K$, $s(\pi^{\phi_1}) \cdot s(\pi^{\phi_2}) = 0$ for two arbitrary different ϕ_1, $\phi_2 \in \Delta\mu$ and $\mu(\Delta\mu) = \mu(\mathfrak{S})$.*

For the proof of property (ii) the reader is referred to [5, proof of Proposition 3].

THEOREM 3. *Suppose that μ_1, μ_2 are two positive central Radon measures on the compact space $\mathfrak{S}$ such that $\psi(a) = \int\phi(a)\,d\mu_1(\phi) = \int\phi(a)\,d\mu_2(\phi)$ for all $a \in A$; then $\mu_1 = \mu_2$.*

Proof. Let Φ_1 (respectively, Φ_2) be a σ-continuous homomorphism of Z onto $L^{\infty}(\mathfrak{S},\mu_1)$ (respectively, $L^{\infty}(\mathfrak{S},\mu_2)$) such that $\psi(za) = \int_{\mathfrak{S}} \Phi_i(z)(\phi)\,\phi(a)\,d\mu_i(\phi)$ for $z \in Z$ and $a \in A$ $(i = 1,2)$. From the discussions in the proof of Theorem 2, clearly the kernel of Φ_1 (respectively, Φ_2) $= z(1 - s(\pi^{\psi}))$ (respectively, $z(1 - s(\pi^{\psi}))$).

For a $(\geqq 0) \in A$, the continuous function $\phi \to \phi(a)$ on $\mathfrak{S}$ belongs to $L^{\infty}(\mathfrak{S},\mu_1)$ and $L^{\infty}(\mathfrak{S},\mu_2)$, respectively, so that there are positive elements c_1^a, c_2^a in $Zs(\pi^{\psi})$ such that $\phi(a) = \Phi_1(c_1^a)(\phi)$ μ_1-a.e. and $\phi(a) = \Phi_2(c_2^a)(\phi)$ μ_2-a.e.

Suppose that $c_1^a \neq c_2^a$, then there is a projection z of $Zs(\pi^{\psi})$ such that $c_1^a z < c_2^a z$ or $c_1^a z > c_2^a z$; assume that $c_1^a z < c_2^a z$, then

$$\begin{aligned}\psi(c_1^a z) &= \int_{\mathfrak{S}} \Phi_1(c_1^a)(\phi)\,\Phi_1(z)(\phi)\,d\mu_1(\phi)\\ &= \int_{\mathfrak{S}} \Phi_1(z)(\phi)\,\phi(a)\,d\mu_1(\phi)\\ &= \psi(az)\end{aligned}$$

and, analogously,

$$\psi(c_2^a z) = \psi(az);$$

hence $c_2^a z - c_1^a z = 0$, a contradiction; hence we have $c_1^a = c_2^a = c^a$. Therefore,

$$\int_{\mathfrak{S}} \phi(a_1)\phi(a_2)\cdots\phi(a_n)\,d\mu_1(\phi)$$

$$= \int_{\mathfrak{S}} \Phi_1(c^{a_1})(\phi)\Phi_1(c^{a_2})(\phi)\cdots\Phi(c^{a_n})(\phi)\,d\mu_1(\phi)$$

$$= \psi(c^{a_1}c^{a_2}\cdots c^{a_n})$$

$$= \int_{\mathfrak{S}} \Phi_2(c^{a_1}c^{a_2}\cdots c^{a_n})(\phi)\,d\mu_2(\phi)$$

$$= \int_{\mathfrak{S}} \Phi_2(c^{a_1})(\phi)\Phi_2(c^{a_2})(\phi)\cdots\Phi_2(c^{a_n})(\phi)\,d\mu_2(\phi)$$

$$= \int_{\mathfrak{S}} \phi(a_1)\phi(a_2)\cdots\phi(a_n)\,d\mu_2(\phi)$$

for $a_1, a_2, \cdots, a_n \in A$.

Therefore, by the Stone-Weierstrass Theorem,

$$\int_{\mathfrak{S}} f(\phi)\,d\mu_1(\phi) = \int_{\mathfrak{S}} f(\phi)\,d\mu_2(\phi)$$

for $f \in c(\mathfrak{S})$, where $c(\mathfrak{S})$ is the Banach algebra of all continuous functions on the compact space $\mathfrak{S}$; hence $\mu_1 = \mu_2$.

Moreover, the $*$-algebra generated by $\{c^a \mid a \in A\}$ is σ-dense in $Zs(\pi^\psi)$, because $c(\mathfrak{S})$ is σ-dense in $L^\infty(\mathfrak{S}, \mu_i)$ $(i = 1, 2)$ and Φ_i is isomorphic on $Zs(\pi^\psi)$; hence $\Phi_1 = \Phi_2$. This completes the proof.

Definition 5. From the proof of Theorem 3, the corresponding homomorphism Φ to a positive central Radon measure μ on the compact space $\mathfrak{S}$ is unique. This unique Φ is called the homomorphism of μ and denoted by Φ_μ.

By Theorem 3, the mapping $\mu \to \psi = \int \phi\,d\mu$ of the set of all positive central Radon measures on the compact space $\mathfrak{S}$ into the set of all positive linear functionals on A is one-to-one.

Now we shall show this mapping to be *onto*.

Theorem 4. *Let ψ be a positive linear functional on A; then there is a positive central Radon measure μ on the compact space $\mathfrak{S}$ such that $\psi(a) = \int \phi(a)\,d\mu(\phi)$ for $a \in A$.*

To prove Theorem 4, we shall provide some considerations.

For $\psi(\geqq 0) \in A^*$, we shall consider ψ as a $\sigma(A^{**}, A^*)$-continuous positive linear functional on A^{**} canonically.

The W^*-algebra $A^{**}s(\pi^\psi)$ has the separable associated space; therefore, there is a homogeneous type $I_{\aleph_0}$ W^*-algebra N such that $A^{**}s(\pi^\psi) \subset N$ and the center $Zs(\pi^\psi)$ of $A^{**}s(\pi^\psi) =$ the center of N.

We shall represent the N as follows: $N = L^\infty(B,Q,\omega)$, $N_* = L^1(B_*,Q,\omega)$, where B is a type $I_{\aleph_0}$-factor, N_* (respectively, B_*) is the associated space of N (respectively, B), Q is a compact space satisfying the second countability axiom and ω is a positive Radon measure on Q with $\omega(Q)=1$, and $L^\infty(B,Q,\omega)$ is the W^*-algebra of all essentially bounded weakly $*$ ω-measurable B-valued functions on the Q and $L^1(B_*,Q,\omega)$ is the Banach space of all strongly ω-integrable B_*-valued functions on the Q (cf. [14], [15]).

Let $\bar{\psi}$ be the restriction of ψ on the W^*-algebra $A^{**}s(\pi^\psi)$, then it can be extended to a σ-continuous linear functional $\tilde{\psi}$ on N (cf. [14]).

Under the above representation, we have

$$\tilde{\psi} = \int_Q \tilde{\psi}_t \, d\omega(t),$$

where the function $\tilde{\psi}_t$ on Q belongs to $L^1(B_*,Q,\omega)$ and $\tilde{\psi}_t$ is μ-almost everywhere positive (cf. [14]).

Since $\|\tilde{\psi}\| = \int_Q \|\tilde{\psi}_t\| \, d\omega(t)$, $\tilde{\psi}_t(1)$ is ω-integrable; moreover, let $F=\{t \mid \|\tilde{\psi}_t\|=0, t\in Q\}$, then the function $\chi_F(t)\cdot 1 = z \in Zs(\pi^\psi)$, where χ_F is the characteristic function of F.

Since $\int \tilde{\psi}_t(\chi_F(t)\cdot 1)\, d\omega(t) = \tilde{\psi}(z) = 0$, $\pi_z^\psi = 0$ and so $z=0$; therefore $\psi_t(1)>0$ ω-a.e.

For $a \in As(\pi^\psi) \subset A^{**}s(\pi^\psi)$, we write $a = \int_Q a(t)\, d\omega(t)$ in $L^\infty(B,Q,\omega)$; then the function $\tilde{\psi}_t(a(t))/\tilde{\psi}_t(1)$ on Q is bounded ω-measurable.

Let $\mathbb{C}$ be a commutative C^*-subalgebra of the W^*-algebra $L^\infty(Q,\omega)$ generated by the family $\{\tilde{\psi}_t(a(t))/\tilde{\psi}_t(1) \mid a \in As(\pi^\psi)\}$; then we shall show

LEMMA 1. *$\mathbb{C}$ is σ-dense in the W^*-algebra $L^\infty(Q,\omega)$.*

Proof. Suppose that there is an ω-integrable complex-valued function g on Q such that $\int_Q(\tilde{\psi}_t(a(t))/\tilde{\psi}_t(1))\, g(t)\, d\omega(t) = 0$ for all $a \in As(\pi^\psi)$. Since the function $\tilde{\psi}_t \in L^1(B_*,Q,\omega)$, the function $(g(t)/\tilde{\psi}_t(1))\,\tilde{\psi}_t$ is strongly ω-measurable and

$$\int_Q \left\| \frac{g(t)}{\tilde{\psi}_t(1)} \tilde{\psi}_t \right\| d\omega(t) = \int_Q |g(t)|\, d\omega(t) < +\infty ;$$

hence

$$\frac{g(t)}{\tilde{\psi}_t(1)} \tilde{\psi}_t \in L^1(B_*,Q,\omega)$$

and so there is an element $\tilde{f}$ of N_* such that

$$\tilde{f}(x) = \int_Q \frac{g(t)}{\tilde{\psi}_t(1)} \tilde{\psi}_t(x(t))\, d\omega(t)$$

for

$$x = \int_Q x(t)\, d\omega(t) \in L^\infty(B,Q,\omega) = N.$$

Since $\tilde{f}(As(\pi^{\psi})) = 0$, $\tilde{f}(A^{**}s(\pi^{\psi})) = 0$, so that $\tilde{f}(Zs(\pi^{\psi})) = 0$; therefore

$$\int_Q \frac{g(t)}{\tilde{\psi}_t(1)} \tilde{\psi}_t(h(t) \cdot 1)\, d\omega(t) = \int_Q h(t)\, g(t)\, d\omega(t) = 0$$

for all $h \in L^{\infty}(Q,\omega)$; hence $g(t) = 0$ ω-a.e. This completes the proof.

Let Q_1 be the spectrum space of $\mathbb{C}$, then $\omega(g) = \int_Q g(t)\, d\omega(t)$ for $g \in \mathbb{C}$ defines a positive Radon measure ω_1 on Q_1 such that $\int_Q g(t)\, d\omega(t) = \int_{Q_1} g(v)\, d\omega_1(v)$ for $g \in \mathbb{C}$.

Since $\mathbb{C}$ is σ-dense in $L^{\infty}(Q,\omega)$, the above equality implies that the isomorphism $g(t) \to g(v)$ of $\mathbb{C}$ in $L^{\infty}(Q,\omega)$ onto $\mathbb{C}$ in $L^{\infty}(Q_1, \omega_1)$ can be uniquely extended to an isomorphism Ψ of $L^{\infty}(Q,\omega)$ onto $L^{\infty}(Q_1,\omega_1)$ (cf. [14]).

Since

$$\left| \frac{\tilde{\psi}_t(a(t) - b(t))}{\tilde{\psi}_t(1)} \right| \leqq \frac{\tilde{\psi}_t(1)}{\tilde{\psi}_t(1)} \| a - b \| \ \omega\text{-a.e.}$$

and the C^*-algebra $As(\pi^{\psi})$ is uniformly separable, $\mathbb{C}$ is uniformly separable; hence Q_1 satisfies the second countability axiom (cf. [3,Appendix (I)]).

Therefore, by the theorem of von Neumann (cf. [3, Appendix (IV)]), there is a ω-null set P in Q, an ω_1-null set P_1 in Q_1 and a one-to-one measurable mapping η of $Q - P$ onto $Q_1 - P_1$ such that $\Psi(f)(\eta(t)) = f(t)$ for $f \in L^{\infty}(Q,\omega)$ and $t \in Q - P$; moreover, in our case, clearly the η is measure-preserving.

Therefore, using η, we can translate the structures $L^{\infty}(B,Q,\omega)$ and $L^1(B_*,Q,\omega)$ on the measure space (Q,ω) to the structures $L^{\infty}(B,Q_1,\omega_1)$ and $L^1(B_*,Q_1,\omega_1)$ on the measure space (Q_1, ω_1).

Then $\tilde{\psi} = \int_Q \tilde{\psi}_t\, d\omega(t) = \int_{Q_1} \tilde{\psi}_v\, d\omega_1(v)$ and $a = \int_Q a(t)\, d\omega(t) = \int_{Q_1} a(v)\, d\omega_1(v)$ for $a \in As(\pi^{\psi})$, where $v = \eta(t)$.

$$\frac{\tilde{\psi}_v(a(v))}{\tilde{\psi}_v(1)} = \frac{\tilde{\psi}_t(a(t))}{\tilde{\psi}_t(1)} = \Psi(f)(v), \quad \text{where } f(t) = \frac{\tilde{\psi}_t(a(t))}{\tilde{\psi}_t(t)};$$

since $f \in \mathbb{C}$, there is a continuous function $\xi_a(= \Psi(f))$ on Q_1 such that

$$\frac{\tilde{\psi}_v(a(v))}{\tilde{\psi}_v(1)} = \xi_a(v) \ \omega_1\text{-a.e.};$$

moreover, such a continuous function is unique—in fact, $\xi_a(v) = h(v)$ ω_1-a.e. for a continuous h on Q_1 implies $\Psi^{-1}(h) = \Psi^{-1}(\xi_a)$, so that $\xi_a = h$ in $\mathbb{C}$; hence $\xi_a(v) = h(v)$ for all $v \in Q_1$.

$$\frac{\tilde{\psi}_v((a^*a)(v))}{\tilde{\psi}_v(1)} \geqq 0 \ \omega_1\text{-a.e.}$$

implies $\xi_{a^*a}(v) \geqq 0$ for all $v \in Q_1$, and, analogously, we have $\xi_{a+b}(v) = \xi_a(v) + \xi_b(v)$,

$\xi_{\lambda a}(v) = \lambda \xi_a(v)$ and $\xi_{s(\pi^{\psi})}(v) = 1$ for all $v \in Q_1$, where $a, b \in As(\pi^{\psi})$, λ a complex number.

Now put $\phi_v(x) = \xi_{xs(\pi^{\cdot})}(v)$ for $x \in A$; then ϕ_v is a state on A for all $v \in Q_1$; moreover, suppose $v_1 \neq v_2$; then there is a function ξ_a such that $\xi_a(v_1) \neq \xi_a(v_2)$ $(a \in As(\pi^{\psi}))$, because the family $\{\xi_a \mid a \in As(\pi^{\psi})\}$ of continuous functions on Q_1 generates $\mathbb{C}$; hence, for some $x \in A$, we have $\phi_{v_1}(x) \neq \phi_{v_2}(x)$, and so $\phi_{v_1} \neq \phi_{v_2}$.

LEMMA 2. *The mapping $v \xrightarrow{p} \phi_v$ of the compact space Q_1 into the compact space $\mathfrak{S}$ is homeomorphic.*

Proof. Since the above consideration shows that p is one-to-one, it is enough to show the continuity of p.

Suppose that $v_\alpha \to v$ in Q_1; then $\xi_a(v_\alpha) \to \xi_a(v)$ for all $a \in As(\pi^{\psi})$, so that $\phi_{v_\alpha}(x) \to \phi_v(x)$ for $x \in A$. This completes the proof.

By this lemma, the Radon measure ω_1 on Q_1 may be canonically considered a Radon measure ω_1 on the compact space $\mathfrak{S}$ with the support $p(Q_1)$.

Now we shall show

Proof of Theorem 4. Put $d\mu(\phi) = \tilde{\psi}_\phi(1)\,d\omega_1(\phi)$, where $\omega_1(\phi)$ is the Radon measure on $\mathfrak{S}$ defined in the preceding discussion, and $\tilde{\psi}_\phi(1) = \tilde{\psi}_v(1)$ for $\phi = p(v)$ $(v \in Q_1)$ and $\tilde{\psi}_\phi(1) = 0$ for $\phi \notin p(Q_1)$.

Since μ is equivalent to ω_1, $L^\infty(B, \mathfrak{S}, \mu) = L^\infty(B, \mathfrak{S}, \omega_1)$. Put $\tilde{\psi} = \int_{\mathfrak{S}} \tilde{\tau}_\phi\, d\mu(\phi)$ in $L^1(B_*, \mathfrak{S}, \mu)$, then

$$\tilde{\psi} = \int_{\mathfrak{S}} \tilde{\tau}_\phi\, d\mu(\phi) = \int_{\mathfrak{S}} \tilde{\psi}_\phi \cdot \frac{1}{\tilde{\psi}_\phi(1)} \tilde{\psi}_\phi(1)\, d\omega_1(\phi);$$

hence $\tilde{\tau}_\phi = \tilde{\psi}_\phi / \tilde{\psi}_\phi(1)$ for μ-almost all $\phi \in \mathfrak{S}$.

Therefore, for $a \in A$,

$$\begin{aligned} \psi(a) &= \tilde{\psi}(as(\pi^{\psi})) = \int \frac{\tilde{\psi}_\phi((as(\pi^{\psi}))(\phi))}{\tilde{\psi}_\phi(1)}\, d\mu(\phi) \\ &= \int_{\mathfrak{S}} \xi as(\pi^{\psi})(\phi)\, d\mu(\phi) \\ &= \int_{\mathfrak{S}} \phi(a)\, d\mu(\phi). \end{aligned}$$

Moreover, for $z \in Z$, $zs(\pi^{\psi})$ belongs to the center of N; here there is a unique essentially bounded μ-measurable function f on $\mathfrak{S}$ such that $(zs(\pi_\psi))(\phi) = f(\phi) \cdot 1$ μ-a.e. Put $\Phi(z) =$ the function $f(\phi)$; then Φ is a σ-continuous homomorphism of Z onto the W^*-algebra $L^\infty(\mathfrak{S}, \mu)$.

For $z \in Z$ and $a \in A$,

$$\psi(za) = \tilde{\psi}(zas(\pi^{\psi})) = \int_{\mathfrak{S}} \tilde{\tau}_{\phi}((zas(\pi^{\psi}))(\phi))\, d\mu(\phi)$$
$$= \int_{\mathfrak{S}} \tilde{\tau}_{\phi}(\Phi(z)(\phi)(as(\pi^{\psi}))(\phi))\, d\mu(\phi)$$
$$= \int_{\mathfrak{S}} \Phi(z)(\phi)\tilde{\tau}_{\phi}((as(\pi^{\psi}))(\phi))\, d\mu(\phi)$$
$$= \int_{\mathfrak{S}} \Phi(z)(\phi)\phi(a)\, d\mu(\phi).$$

Hence, μ is a positive central Radon measure on the compact space $\mathfrak{S}$. This completes the proof.

Next, in order to make our theorem applicable for the unitary representation theory of locally compact groups, we shall consider a separable C^*-algebra $\mathfrak{A}$ without unit.

Let $\mathfrak{S}_0$ be the set of all positive linear functionals ϕ on $\mathfrak{A}$ such that $\phi \geqq 0$ and $\|\phi\| \leqq 1$, and $\mathfrak{S}_1$ the set of all positive linear functionals ϕ on $\mathfrak{A}$ such that $\|\phi\| = 1$; then $\mathfrak{S}_0$ is a $\sigma(\mathfrak{S}_0, \mathfrak{A})$-compact space.

The function $\phi \to \|\phi\|$ on the compact space $\mathfrak{S}_0$ is lower semi-continuous, so that $\mathfrak{S}_1$ is a Borel set in the compact space $\mathfrak{S}_0$. Let A be the C^*-algebra adjointing the unit 1 to $\mathfrak{A}$ and ϕ_0 be a state on A such that $\phi_0(\mathfrak{A}) = 0$.

For $\phi \in \mathfrak{S}_0$, we can uniquely extend ϕ to a positive linear functional with the same norm on A, and we shall denote it by the same notation ϕ.

Define a mapping α of $\mathfrak{S}_0$ onto $\mathfrak{S}$ as follows:

$$\alpha(\phi) = \phi + (1 - \phi(1))\phi_0.$$

Then α is a homeomorphic mapping of the compact space $\mathfrak{S}_0$ onto the compact space $\mathfrak{S}$.

Now let ψ be a bounded positive linear functional on $\mathfrak{A}$, and we shall extend uniquely ψ on A with the same norm. Then there is a unique positive central Radon measure μ on $\mathfrak{S}$ such that

$$\psi(a) = \int_{\mathfrak{S}} \phi(a)\, d\mu(\phi) \qquad \text{for } a \in A.$$

Let $\pi^{\psi}(\mathfrak{H}_{\psi})$ be the $*$-representation of A constructed via ψ; then there is a sequence of positive elements (h_n) in $\mathfrak{A}$ such that $\|h_n\| \leqq 1$, $\{\pi^{\psi}_{h_n}\}$ converges strongly to $1_{\mathfrak{H}_{\psi}}$, where $1_{\mathfrak{H}_{\psi}}$ is the identity operator on $\mathfrak{H}_{\psi}$. Therefore,

$$\lim_n \psi(1 - h_n) = \int_{\mathfrak{S}} \phi(1 - h_n)\, d\mu(\phi) = 0.$$

Since $\phi(1-h_n) \geqq 0$, there is a subsequence of (n_j) of (n) such that $\lim_j \phi(1-h_{n_j}) = 0$

μ-a.e.; hence $\sup_{\|x\| \leqq 1; x \in \mathfrak{A}} |\phi(x)| = \phi(1)| = 1$ for μ-a.e., $\phi \in \mathfrak{S}$. Therefore μ is concentrated on the Borel set $\alpha(\mathfrak{S}_1)$.

Hence we can easily prove the following theorem.

THEOREM 5. *Let $\mathfrak{A}$ be a C^*-algebra, $\mathfrak{S}_0$ the set of all positive linear functionals ϕ on $\mathfrak{A}$ such that $\phi \geqq 0$ and $\|\phi\| \leqq 1$, $\mathfrak{S}_1$ the set of all positive linear functionals ϕ on $\mathfrak{A}$ such that $\|\phi\| = 1$, K_0 the set of all primary positive linear functionals ϕ on $\mathfrak{A}$ such that $\|\phi\| \leqq 1$; then $\mathfrak{S}_1$ is a Borel subset in the $\sigma(\mathfrak{S}_0, \mathfrak{A})$-compact space $\mathfrak{S}_0$ and K_0 is measurable for all Radon measures ν on the compact space $\mathfrak{S}_0$.*

Moreover, let ψ be a bounded positive linear functional on $\mathfrak{A}$; then there is one and only one positive Radon measure μ on the compact space $\mathfrak{S}_0$ satisfying the following conditions:

(i) *$\psi(a) = \int_{\mathfrak{S}_0} \phi(a) d\mu(\phi)$ for $a \in \mathfrak{A}$.*

(ii) *There is a unique σ-continuous homomorphism Φ_μ of the center Z of the W^*-algebra $\mathfrak{A}^{**}$ onto the W^*-algebra $L^\infty(\mathfrak{S}_0, \mu)$ such that*

$$\psi(za) = \int_{\mathfrak{S}_0} \Phi_\mu(z)(\phi)\phi(a)\,d\mu(\phi) \text{ for } z \in Z \text{ and } a \in \mathfrak{A},$$

where "the ψ on $\mathfrak{A}^{**}$" is the unique canonical extension of "the ψ on $\mathfrak{A}$".

(iii) *There is a Borel subset U_ψ in the compact space $\mathfrak{S}_0$ such that $U_\psi \subseteqq K_0 \cap \mathfrak{S}_1$, $\mu(U_\psi) = \mu(\mathfrak{S}_0)$ and $s(\pi^{\phi_1}) \cdot s(\pi^{\phi_2}) = 0$ for two different $\phi_1, \phi_2 \in U_\psi$.*

We shall call this unique Radon measure μ the central Radon measure of ψ.

Finally we shall state some problems.

The problem of characterizing intrinsically central Radon measures is very interesting — in particular, it is important in the unitary representation theory of locally compact groups.

Therefore we shall put

PROBLEM 2. Is there an intrinsical characterization of central Radon measures? In particular, can we have a purely measure-theoretic characterization of those measures?

Suppose ψ is a positive linear functional on the C^*-algebra A, $\mathfrak{M}(\psi)$ the set of all positive Radon measures on the compact space $\mathfrak{S}$ such that $\psi(a) = \int_{\mathfrak{S}} \phi(a)\,d\mu(\phi)$ for $a \in A$.

Then, $\mathfrak{M}(\psi)$ contains one and only one central Radon measure μ_0.

PROBLEM 3. Is there a geometrical characterization of μ_0 in the $\mathfrak{M}(\psi)$?

PROBLEM 4. Without use of the reduction theory of von Neumann, can we show the existence of a central Radon measure in the $\mathfrak{M}(\psi)$?

Added in proof. Problem 1 is solved by J. Feldman and E. Effros.

REFERENCES

1. E. Bishop and K. de Leeuw, *The representation of linear functionals by measures on set of extreme points*, Ann. Inst. Fourier (Grenoble) **9** (1959), 305–331.

2. G. Choquet, *Existence de représentations intégrales au moyen des points extrémaux dans les cônes convexes*, C. R. Acad. Sci. Paris **243** (1956), 699–702.

3. J. Dixmier, *Les algèbres d'operateurs dans l'espace Hilbertien*, Gauthiér-Villars, Paris, 1957.

4. ———, *Dual et Quasi-dual d'une algèbra de Banach involutive*, Trans. Amer. Math. Soc. **104** (1962), 278–283.

5. J. Ernest, *A decomposition theory for unitary representations of locally compact groups*, Trans. Amer. Math. Soc. **104** (1962), 252–276.

6. J. Fell, *The dual spaces of C^*-algebras*, Trans. Amer. Math. Soc. **94** (1960), 365–403.

7. R. Godement, *Sur la théorie des representations unitaires*, Ann. of Math. (2) **53** (1951), 68–124.

8. L. Loomis, *Unique direct integral decompositions on convex sets*, Amer. J. Math. **84** (1962), 509–526.

9. G. Mackey, *The theory of group representations*, (Mimeographed notes) Univ. of Chicago, Chicago, Ill., 1955.

10. F. I. Mautner, *Unitary representations of locally compact groups*. I, Ann. of Math. (2) **51** (1950), 1–25.

11. M. Naimark, *Normed rings*, Noordhoff, Groningen, 1959.

12. J. von Neumann, *On rings of operators. Reduction theory*, Ann. of Math. (2) **50** (1949), 401–485.

13. S. Sakai, *A characterization of W^*-algebras*, Pacific J. Math. **6** (1956), 763–773.

14. ———, *The theory of W^*-algebras*, (Mimeographed notes) Yale Univ., New Haven. Conn., 1962.

15. ———, *On the reduction theory of von Neumann*, Bull. Amer. Math. Soc. **70** (1964), 393–398.

16. I. Segal, *Decompositions of operators algebras*. I, II, Mem. Amer. Math. Soc. No. 9 (1951).

17. Z. Takeda, *Conjugate spaces of operator algebras*, Proc. Japan. Acad. **30** (1954), 90–95.

18. M. Tomita, *Harmonic analysis on locally compact groups*, Math. J. Okayama Univ. **5** (1956), 133–193.

19. S. Sakai, *On the σ-weak topology of W^*-algebras*, Proc. Japan Acad. **32** (1956), 329–332.

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