CONSTANT FUNCTIONS AND LEFT INVARIANT MEANS ON SEMIGROUPS(1)

BY

THEODORE MITCHELL

1. **Introduction.** Let S be a semigroup, $m(S)$ the space of all bounded real-valued functions on S, where $m(S)$ has the supremum norm. An element $\mu \in m(S)^*$ is a *mean* on $m(S)$ if $\| \mu \| \leqq 1$ and $\mu(e) = 1$, where e denotes the constant 1 function on S. A mean μ is *left* [*right*] *invariant* if $\mu(l_s f) = \mu(f)$ $[\mu(r_s f) = \mu(f)]$ for all $f \in m(S)$ and $s \in S$, where the *left* [*right*] *translation* l_s $[r_s]$ of $m(S)$ by s is given by $(l_s f)s' = f(ss')$ $[(r_s f)s' = f(s's)]$. An *invariant mean* is a left and a right invariant mean. A semigroup that has a left invariant mean (right invariant mean) [invariant mean] is called *left amenable* (*right amenable*) [*amenable*].

Let S be a left amenable [right amenable] (amenable) semigroup, and let $f_0 \in m(S)$. Then f_0 is called *left almost* [*right almost*] (*almost*) *convergent* to the real number α, if all left invariant [right invariant] (invariant) means on f_0 have the same value, α.

G. G. Lorentz [11, Theorem 1, p. 170] has shown that if S is the semigroup of positive integers under addition, then an $f_0 \in m(S)$ is almost convergent to α if and only if the sequence f_n converges uniformly to the constant function αe, where f_n is given by $f_n = n^{-1} \sum_{i=1}^{n} r_i f_0$. M. M. Day [1, Theorem 1, p. 539] generalized this by showing that an $f_0 \in m(S)$, where S is an amenable semigroup, is almost convergent to α if and only if the constant function αe is the uniform limit of finite averages of two-sided translates of f_0. In the same spirit, K. Witz [14, Theorem 4.4] proved that if S is a right amenable semigroup with identity, an $f_0 \in m(S)$ is right almost convergent to α if and only if αe is the uniform limit of finite averages of right translates of f_0. (Left and right may be interchanged in this result.)

One of the principal theorems in the present paper supplements the above results by characterizing the values achieved by all left invariant means of $m(S)$ on an arbitrary (not necessarily left convergent) element $f_0 \in m(S)$ in terms of

Presented to the Society, June 14, 1963; received by the editors June 8, 1964.

(1) This paper consists primarily of a portion of my doctoral dissertation at the Illinois Institute of Technology. I thank my adviser, Professor Robert J. Silverman for his guidance, encouragement, valuable suggestions and questions during the course of this work. The support of this work by an Illinois Institute of Technology fellowship, and by National Science Foundation grant G-24345 is gratefully acknowledged.

pointwise convergence of finite averages of right translates of f_0 to a constant function. In addition, the other principal theorem obtains a characterization of left amenability of S in such terms. We list these principal theorems below.

THEOREM 3. *Let S be a semigroup. Then the following are equivalent*:

(a) *For every $f \in m(S)$, there exists a net of finite averages of right translates of f which converges pointwise to a constant function.*

(b) *S is left amenable.*

(c) *There exists a net, $\{T_\delta\}$, of finite averages of right translations such that for every f in $m(S)$, $\{T_\delta f\}$ converges pointwise to a constant function.*

THEOREM 4. *Let S be a left amenable semigroup, f_0 an arbitrary element in $m(S)$, and α an arbitrary real number. Then the following are equivalent*:

(a) *There exists a net of finite averages of right translates of f_0, which converges pointwise to αe.*

(b) *There exists a left invariant mean, μ, on $m(S)$ such that $\mu(f_0) = \alpha$.*

(c) $-p_R(-f_0) \leqq \alpha \leqq p_R(f_0)$.

The function p_R in Theorem 4, condition (c), is defined in §3.

In §2, we state the basic nomenclature used throughout the paper. §3 is devoted to the proof of implications (a) $\rightarrow$ (b) of Theorems 3 and 4. The major steps in the derivation exploit the w^*-continuity of translation operators on $m(S)$, the w^*-compactness of w^*-closed norm-bounded subsets of $m(S)$, and the commutativity of a left with a right translation of $m(S)$. These concepts are used to construct a suitable sublinear functional on $m(S)$, to which the Hahn-Banach extension theorem is applied. §4 is mainly concerned with completing the proofs of the principal theorems. The proof of the remaining implications of these theorems makes use of an adaptation of a concept employed by Day in [1] and [3]; that of an introversion on $m(S)$.

§5 obtains various results by the use of Theorems 3 and 4. The main topic in that section is the introduction and investigation of left thick subsets of a semigroup S, a generalization of subsets that contain a left ideal of S. A subset S' of a semigroup S is called *left thick* in S if for each finite subset $S'' \subseteq S$, there exists an $s'' \in S$ such that $S''s'' \subseteq S'$. It is shown that the left thick subsets of a left amenable semigroup S are precisely the subsets of S whose characteristic function, f_0, admits a left invariant mean μ on $m(S)$ such that $\mu(f_0) = 1$. Further, a left thick subsemigroup S' of S is left amenable if and only if S is left amenable. For finite S, several of these results on S' reduce to cases which are implicit in the work of Rosen [12].

2. **Notation.** For terms not given here or in a later section, see Day [1]. Topological terms shall follow the usage of Kelley [10]. If $b \in B$, a linear topo-

logical space, and $\mu \in B^*$, then $\mu(b)$ shall be alternatively designated by (μ, b). Let V be a subset of a topological space, then $\mathrm{CL}(V)$ designates the closure of V. Now let V be a subset of a (real) linear space, then $\mathrm{CO}(V)$ is the convex hull of V, the set of finite (weighted) averages of V.

Let S be a semigroup. Then $Q: l^1(S) \to m(S)^*$ will denote the evaluation mapping of the real semigroup algebra $l^1(S)$ into $m(S)^*$. (See [1, Definition 3, p. 521] for the product on $l^1(S)$.) For $s \in S$, I_s is the characteristic function of s defined over S. The products $I_s\theta$ and θI_s, where $\theta \in l^1(S)$, will be denoted by $s\theta$ and θs, respectively. The set of finite means $\phi \in l^1(S)$ (see [1, Definition 2, p. 513]) is designated by Φ. The symbol $\mathscr{L}$ [$\mathscr{R}$] is the set of all left [right] translations of $m(S)$ by elements of S. We denote $\Lambda = \mathrm{CO}(\mathscr{L})$, $\mathrm{P} = \mathrm{CO}(\mathscr{R})$. For $f \in m(S)$, $Z_R(f) \subseteq m(S)$ [$Z_L(f) \subseteq m(S)$] is given by

$$Z_R(f) = w^*\mathrm{CL}(\mathrm{CO}(\mathscr{R}f)) = w^*\mathrm{CL}(\mathrm{P}f),$$

$$[Z_L(f) = w^*\mathrm{CL}(\mathrm{CO}(\mathscr{L}f)) = w^*\mathrm{CL}(\Lambda f)].$$

And $K_R(f) = \{\alpha e; \alpha e \in Z_R(f)\}$, similarly $K_L(f) = \{\alpha e; \alpha e \in Z_L(f)\}$, where α ranges over the real numbers.

3. **Right stationary semigroups.** This section investigates the properties of a semigroup S that satisfies the requirement that for each f in $m(S)$, the set $K_R(f)$ [$K_L(f)$] is nonempty. Such a semigroup is called *right stationary* [*left stationary*]. The major goal of this section is to show that a right stationary semigroup is left amenable. Additionally, some information is obtained regarding the range of values attained by all left invariant means on an $f \in m(S)$.

Let X be a nonempty set, and let F be a mapping $F: X \to X$. By the *translation operator on $m(X)$ induced by F*, is meant the mapping $T_F: m(X) \to m(X)$ given by $(T_F g)x = g(Fx)$, for $x \in X$, and $g \in m(X)$.

LEMMA 1. *If X, F, and T_F are as above, then the translation operator induced by F satisfies*:

(a) $T_F e = e$,

(b) $\|T_F\| = 1$,

(c) T_F *is continuous in the w^*-topology on $m(X)$.*

Proof. (a) $(T_F e)(x) = e(Fx) = 1$, for $x \in X$.

(b) $\|T_F f\| = \sup_{x \in F(X)} |f(x)| \leqq \sup_{x \in X} |f(x)| = \|f\|$. Hence $\|T_F\| \leqq 1$. By (a), $\|T_F\| \geqq 1$, so $\|T_F\| = 1$.

(c) Let $H = F^{-1}$, that is $H(x) = \{t \in X; F(t) = x\}$, for an $x \in X$. Let $V_F: l^1(X) \to l^1(X)$ be defined by $(V_F\theta)x = \sum_{t \in H(x)} \theta(t)$, for $\theta \in l^1(X)$ and $x \in X$. If $f \in m(X)$ and $\theta \in l^1(X)$, then

$$\begin{aligned}(f, V_F\theta) &= \sum_{x \in X} f(x) \sum_{t \in H(x)} \theta(t)\\ &= \sum_{x \in X} \sum_{t \in H(x)} f(x)\theta(t)\\ &= \sum_{t \in X} \sum_{x = Ft} f(x)\theta(t) = \sum_{t \in X} f(F(t))\theta(t)\\ &= (T_F f, \theta).\end{aligned}$$

So $V_F^* = T_F$. But the adjoint of a linear operator on a Banach space is w^*-continuous, by [4, Theorem 2, p. 18], hence T_F is continuous in the w^*-topology on $m(X)$.

Corollary 1. *If S is a semigroup, and if either $T \in \mathrm{P}$ or $T \in \Lambda$, then*:
(a) $Te = e$,
(b) $\| T \| = 1$,
(c) T is *w^*-continuous on $m(S)$.*

Proof. The operator $l_{s'}$ is the translation operator on $m(S)$ induced by the set map $F: S \to S$, where $F(s) = s's$. Hence Lemma 1 applies to $T \in \mathscr{L}$. But the convex hull of any family of operators that has properties (a), (b), and (c), also possesses these properties, A similar argument applies to $T \in \mathrm{P}$.

If we define $\partial^* l_{s'}: l^1(S) \to l^1(S)$, for an $s' \in S$ by $^*l_{s'}\theta = s'\theta$, for $\theta \in l^1(S)$, then $(^*l_{s'})^* = l_{s'}$. This may be verified by comparing the formula for $s'\theta$ with the V_F in the proof of Lemma 1, or a result of Day can be used. From [1, (D), p. 522], $l_s^*(Q\theta) = Q(s\theta)$ for all $s \in S$, $\theta \in l^1(S)$. So for $f \in m(S)$,

$$(l_s f, \theta) = (Q(\theta), l_s f) = (l_s^*(Q\theta), f) = (Q(s\theta), f) = (f, s\theta),$$

which provides an additional proof of Corollary 1(c).

Lemma 2. *Let S be a semigroup, let $f, g \in m(S)$, and let α be a real number. Then*:
(a) $Z_R(\alpha f) = \alpha Z_R(f)$,
(b) $Z_R(f)$ *is convex*,
(c) $g \in Z_R(f) \to \| g \| \leqq \| f \|$,
(d) $Z_R(f)$ *is w^*-compact*,
(e) $Z_R(f + g) \subseteq Z_R(f) + Z_R(g)$,
(f) $T \in \mathrm{P} \to T(Z_R(f)) \subseteq Z_R(f)$,
(g) $g \in Z_R(f) \to Z_R(g) \subseteq Z_R(f)$.

Proof. (a) The space $m(S)$ is a linear topological space in the w^*-topology, by [4, Corollary 1, p. 17]. And

$$Z_R(\alpha f) = w^*\mathrm{CL}(\mathrm{CO}(\mathscr{R}(\alpha f))) = w^*\mathrm{CL}(\mathrm{CO}(\alpha(\mathscr{R} f))).$$

But if A is a subset of a linear topological space, then

$$\mathrm{CL}(\mathrm{CO}(\alpha A)) = \alpha \mathrm{CL}(\mathrm{CO} A),$$

by [6, Lemma 4, II, III, p. 415]. Hence,

$$w^*\mathrm{CL}(\mathrm{CO}(\alpha(\mathscr{R}f))) = \alpha w^*\mathrm{CL}(\mathrm{CO}(\mathscr{R}f)) = \alpha Z_R(f).$$

(b) The closure of a convex set in a linear topological space is convex [6, Theorem 1, a, p. 413]. And $Z_R(f) = w^*\mathrm{CL}(\mathrm{P}f)$, the w^*-closure of a convex set.

(c) By Corollary 1(b), if $g \in \mathrm{P}f$, then $\|g\| \leqq \|f\|$. So $\mathrm{P}f \subseteq \|f\|U$, where U is the norm-closed unit ball of $m(S)$. So,

$$Z_R(f) = w^*\mathrm{CL}(\mathrm{P}f) \subseteq w^*\mathrm{CL}(\|f\|U),$$

by monotonicity of the set map CL in a topology, But $w^*\mathrm{CL}(\|f\|U) = \|f\|U$, since a norm-closed ball in a conjugate space of a Banach space is w^*-closed [4, Theorem 3, p. 40]. Hence $Z_R(f) \subseteq \|f\|U$.

(d) A norm-bounded w^*-closed subset of a conjugate space of a Banach space is w^*-compact [4, Corollary 3, p. 41]. But $Z_R(f)$ is w^*-closed by its definition, and norm-bounded by (c).

(e) For $T \in \mathscr{R}$, $T(f+g) = Tf + Tg$. Hence $\mathscr{R}(f+g) \subseteq \mathscr{R}f + \mathscr{R}g$. Then

$$Z_R(f+g) = w^*\mathrm{CL}(\mathrm{CO}(\mathscr{R}(f+g))) \subseteq w^*\mathrm{CL}(\mathrm{CO}(\mathscr{R}f + \mathscr{R}g))$$

by monotonicity of the set maps w^* CL, and CO. But if A, B are subsets of a linear topological space, and $\mathrm{CL}(\mathrm{CO}(A))$ is compact, then

$$\mathrm{CL}(\mathrm{CO}(A+B)) = \mathrm{CL}(\mathrm{CO}(A)) + \mathrm{CL}(\mathrm{CO}(B)),$$

by [6, Lemma 4, II, IV, p. 415]. Thus by (d),

$$w^*\mathrm{CL}(\mathrm{CO}(\mathscr{R}f + \mathscr{R}g)) = w^*\mathrm{CL}(\mathrm{CO}(\mathscr{R}f)) + w^*\mathrm{CL}(\mathrm{CO}(\mathscr{R}g)) = Z_R(f) + Z_R(g),$$

which shows (e).

(f) If A is a subset of a topological space X, and F is a continuous mapping, $F: X \to X$, then $F(\mathrm{CL}(A)) \subseteq \mathrm{CL}(F(A))$, by [6, Lemma 16, d, p. 13]. Then by Corollary 1(c), it follows that for $T \in \mathrm{P}$,

$$T(Z_R(f)) = T(w^*\mathrm{CL}(\mathrm{P}f)) \subseteq w^*\mathrm{CL}(T\mathrm{P}f).$$

Since P is a semigroup of operators, $T\mathrm{P} \subseteq \mathrm{P}$. Hence $T\mathrm{P}f \subseteq \mathrm{P}f$. Taking the w^*-closure, we obtain

$$w^*\mathrm{CL}(T\mathrm{P}f) \subseteq w^*\mathrm{CL}(\mathrm{P}f) = Z_R(f).$$

(g) From (f), it follows that if $g \in Z_R(f)$, then $\mathrm{P}g \subseteq Z_R(f)$. Taking the w^*-closure, we obtain

$$Z_R(g) = w^*\mathrm{CL}(\mathrm{P}g) \subseteq w^*\mathrm{CL}(Z_R(f)) = Z_R(f),$$

which completes the proof.

LEMMA 3. *Let X be a nonempty set. A norm-bounded net, $\{f_\delta\}$ where $f_\delta \in m(X)$, converges pointwise to $f_0 \in m(X)$ if and only if $\{f_\delta\}$ converges w^* to f_0.*

Proof. Let $w^*\lim_\delta (f_\delta) = (f_0)$. Since $I_{x'} \in l^1(X)$, then for all $x' \in X$,

$$\lim_\delta (f_\delta(x')) = \lim_\delta (f_\delta, I_{x'}) = (f_0, I_{x'}) = f_0(x').$$

Hence $\{f_\delta\}$ converges pointwise to f_0.

Let $\{f_\delta\}$ converge pointwise to f_0, and let there exist a real number $\alpha \geqq 0$, such that $\|f_\delta\| \leqq \alpha$ for all $\delta \in \Delta$, the directed set. Then for any $\beta > 0$, and for any $\theta \in l^1(X)$, there exists a finite subset, $X' \subseteq X$, such that $\sum_{x \in X-X'} |\theta(x)| \leqq \beta$, because of convergence of $\sum_{x \in X} |\theta(x)|$ to $\|\theta\|$. But pointwise convergence of $\{f_\delta\}$ implies uniform convergence over any finite subset of X, in particular over X'. So there exists $\delta_0 \in \Delta$ such that for $\delta \geqq \delta_0$, $|f_\delta(x) - f_0(x)| \leqq \beta$, for all $x \in X'$. Hence for $\delta \geqq \delta_0$,

$$\begin{aligned}
|((f_\delta - f_0), \theta)| &= \Big|\sum_{x \in X} (f_\delta(x) - f_0(x))\theta(x)\Big| \\
&\leqq \sum_{x \in X'} |f_\delta(x) - f_0(x)| \cdot |\theta(x)| + \sum_{x \in X - X'} |f_\delta(x) - f_0(x)| \cdot |\theta(x)| \\
&\leqq \sum_{x \in X'} \beta |\theta(x)| + \sum_{x \in X - X'} (|f_\delta(x)| + |-f_0(x)|) \cdot |\theta(x)| \\
&\leqq \beta \|\theta\| + \sum_{x \in X - X'} 2\alpha |\theta(x)| \leqq \beta(\|\theta\| + 2\alpha).
\end{aligned}$$

So $\{f_\delta\}$ converges w^* to f_0, which shows Lemma 3.

Let S be a semigroup, and let $f \in m(S)$. Then $\mathrm{P}(f)$ is a norm-bounded set by Corollary 1(b). It follows from the correspondence shown above between pointwise convergence and w^*-convergence of norm-bounded nets in $m(S)$, that the set $Z_R(f)$ can also be described as the set of elements of $m(S)$ that are limits of pointwise convergent nets of elements of $\mathrm{P}(f)$. We shall be concerned with the constant functions that are so obtained, that is, with the set $K_R(f)$.

LEMMA 4. *Let S be a right stationary semigroup, and let f, $g \in m(S)$. Then:*
(a) *The set $\{\alpha; \alpha e \in K_R(f)$, where α is real$\}$ is a closed interval, $[\alpha_1, \alpha_2]$, where $\alpha_1 \leqq \alpha_2$,*
(b) $K_R(f + g) \subseteq K_R(f) + K_R(g)$.

Proof. (a) This follows from Lemma 2, parts (b), (c), and (d). And since $K_R(f)$ is nonempty, then $\alpha_1 \leqq \alpha_2$.

(b) Let $\gamma e \in K_R(f + g)$, where γ is a real number. Since $K_R(f + g) \subseteq Z_R(f + g)$, then by Lemma 2(e), there exist functions $h \in Z_R(f)$ and $k \in Z_R(g)$ such that $h + k = \gamma e$. Since S is right stationary, then there exists a constant function, αe, such that $\alpha e \in Z_R(h)$. So, there exists a net, $\{T_\delta\}$, where $T_\delta \in \mathrm{P}$, such that $w^*\lim_\delta(T_\delta h) = \alpha e$. But $k = \gamma e - h$, so

$$w^*\lim_\delta (T_\delta k) = w^*\lim_\delta T_\delta(\gamma e - h) = \gamma e - \alpha e = \beta e,$$

where $\beta = \gamma - \alpha$. Thus $\beta e \in Z_R(k)$, and since $k \in Z_R(g)$, then by Lemma 2(g), we have $\beta e \in Z_R(g)$. Hence $\beta e \in K_R(g)$. Similarly, since $\alpha e \in Z_R(h)$, and $h \in Z_R(f)$, then by Lemma 2(g), we have $\beta e \in K_R(f)$. Then (b) follows, since $\gamma e = \alpha e + \beta e$, which proves Lemma 4.

Let S be a right stationary semigroup, and let $f \in m(S)$. Then by $p_R(f)$ $[p_L(f)]$ is meant

$$p_R(f) = \max\{\alpha; \alpha e \in K_R(f)\} \; [p_L(f) = \max\{\alpha; \alpha e \in K_L(f)\}],$$

where α is real. The maximum, rather than the supremum, may be used because of Lemma 4(a).

LEMMA 5. *Let S be a right stationary semigroup, let $f, g \in m(S)$, and let α be a real number. Then*:

(a) $|p_R(f)| \leqq \|f\|$,

(b) *the set* $\{\beta; \beta e \in K_R(f)\}$, *where* β *is real, is the closed interval* $[-p_R(-f), p_R(f)]$,

(c) $\alpha \geqq 0 \rightarrow p_R(\alpha f) = \alpha p_R(f)$,

(d) $p_R(f + g) \leqq p_R(f) + p_R(g)$,

(e) $-p_R(-e) = p_R(e) = 1$.

Proof. (a) This follows from Lemma 2(c).

(b) By Lemma 4(a), we have $\{\beta; \beta e \in K_R(f)\} = [\alpha_1, \alpha_2]$, where $\alpha_1 \leqq \alpha_2$. By definition, $p_R(f) = \alpha_2$. From Lemma 2(a), it follows that $K_R(-f) = -K_R(f)$. So $\{\beta; \beta e \in K_R(-f)\} = [-\alpha_2, -\alpha_1]$. Hence $p_R(-f) = -\alpha_1$, so $-p_R(-f) = \alpha_1$.

(c) This follows from Lemma 2(a).

(d) This follows from Lemma 4(b).

(e) From Corollary 1(a), we have $Z_R(e) = \{e\}$. Hence from (b),

$$p_R(e) = 1 = -p_R(-e).$$

THEOREM 1. *Let S be a right stationary semigroup, f_0 an arbitrary element of $m(S)$, and α an arbitrary real number. If there exists a net of finite averages of right translates of f_0 which converges pointwise to the constant function αe, then there exists a left invariant mean, μ, on $m(S)$ such that $\mu(f_0) = \alpha$.*

Proof. A net, $\{T_\delta\}$ where $T_\delta \in \mathrm{P}$, exists such that $\{T_\delta f_0\}$ converges pointwise to αe, if and only if $\alpha e \in K_R(f_0)$, by Lemma 3 and Corollary 1(b). So

$$-p_R(-f_0) \leqq \alpha \leqq p_R(f_0),$$

by Lemma 5(b). And p_R is a sublinear functional on $m(S)$, by Lemma 5, parts (c), (d). Hence by the Hahn-Banach theorem [4, Theorem 1, p. 9], there exists

$\mu \in m(S)^*$ such that $\mu(f_0) = \alpha$, and $\mu(f) \leqq p_R(f)$ for all $f \in m(S)$. Then $\|\mu\| \leqq 1$ by Lemma 5(a), and $\mu(e) = 1$ by Lemma 5(e). Hence μ is a mean on $m(S)$.

We have only to show that μ is left invariant. We shall show that for any $f \in m(S)$, and any $l_{s'} \in \mathscr{L}$, that $K_R(f - l_{s'}f) = \{\mathbf{0}\}$, where $\mathbf{0} = 0e$, the constant zero function on S. Let $\gamma e \in K_R(f - l_{s'}f)$, and suppose $\gamma \neq 0$. Then there exists a net, $\{T_\delta\}$ where $T_\delta \in \mathrm{P}$, such that

$$w^*\lim_{\delta}(T_\delta(f - l_{s'}f)) = \gamma e.$$

Since $T_\delta f \in Z_R(f)$, then by w^*-compactness of $Z_R(f)$ (Lemma 2(d)), there exists a subnet, $\{T_\eta\}$ of $\{T_\delta\}$, and a function $g \in Z_R(f)$ such that $w^*\lim_\eta(T_\eta f) = g$. Since $\{T_\eta(f - l_{s'}f)\}$ is a subnet of the w^*-convergent subnet $\{T_\delta(f - l_{s'}f)\}$, then

$$w^*\lim_{\eta}(T_\eta(f - l_{s'}f)) = \gamma e.$$

Thus

$$\begin{aligned} \gamma e &= w^*\lim_{\eta}(T_\eta(f - (f - l_{s'}f)) \\ &= w^*\lim_{\eta}(T_\eta f) - w^*\lim_{\eta}(T_\eta(l_{s'}f)) \\ &= g - w^*\lim_{\eta}(l_{s'}(T_\eta f)) = g - l_{s'}\left(w^*\lim_{\eta}(T_\eta f)\right) \\ &= g - l_{s'}g. \end{aligned} \tag{1}$$

The third equality above follows from the fact that a left translation commutes with a right translation(2), so $T_\eta l_{s'} = l_{s'}T_\eta$; and the fourth equality follows from the w^*-continuity of $l_{s'}$ (Corollary 1(c)). By substitution of $(s')^n$, where the exponent n is a positive integer, in (1), we obtain $g((s')^n) - g((s')^{n+1}) = \gamma$. So,

$$\begin{aligned} \gamma k &= \sum_{n=1}^{k} \gamma = \sum_{n=1}^{k} (g((s')^n) - g((s')^{n+1})) \\ &= g(s') - g((s')^{k+1}), \end{aligned}$$

where k is any positive integer. So $g((s')^{k+1}) = g(s') - \gamma k$. Thus g is unbounded on S, since we can make k arbitrarily large, and since we supposed that $\gamma \neq 0$. But this is a contradiction because $g \in Z_R(f) \subseteq m(S)$. But S is right stationary, so $K_R(f - l_{s'}f) = \{\mathbf{0}\}$.

Then by Lemma 2(a)

$$K_R(l_{s'}f - f) = -K_R(f - l_{s'}f) = -\{\mathbf{0}\} = \{\mathbf{0}\}.$$

---

(2) I initially attempted to prove that if S is right stationary, then S is right amenable, which is false as a counterexample later will indicate. The crucial observation that the incomplete proof could be amended to show that S is left amenable by use of the commutativity of a left with a right translation is due to R. J. Silverman.

So $p_R(l_{s'}f-f)=p_R(f-l_{s'}f)=0$, for $f\in m(S)$, and for $l_{s'}\in\mathscr{L}$. So by domination of μ by p_R, we have $\mu(l_{s'}f)-\mu(f)\leqq 0$, and $\mu(f)-\mu(l_{s'}f)\leqq 0$, thus $\mu(f)=\mu(l_{s'}f)$, for all $f\in m(S)$ and all $l_{s'}\in\mathscr{L}$.

COROLLARY 2. *If S is a right stationary semigroup, then S is left amenable.*

Proof. Let $f_0\in m(S)$. Since $K_R(f_0)$ is nonempty, then by Lemma 3 and Corollary 1(b), there exists the required net of finite averages of right translates of f_0 which converges pointwise to αe, for some real α. The result follows from Theorem 1.

The commutativity of a left with a right translation was essential to the proof of Theorem 1, hence Corollary 2. If we try to show that "S is right stationary implies S is right amenable" the part of the proof that breaks down is equation (1) Hence we cannot show that "$\mathbf{0}$ is the *unique* constant function in $K_R(f-r_{s'}f)$." And indeed, a counterexample to both statements in quotes is provided by the two element semigroup $S=\{a,b\}$, where $aa=ba=a$, and $ab=bb=b$.

4. **Left amenable semigroups.** This section is concerned with proving the converses to Theorem 1 and Corollary 2, and hence completing the characterization of left invariant means on semigroups in terms of constant functions. It is shown, that if a semigroup is left amenable, it has a property that is formally stronger than being right stationary, though in fact equivalent to it. This converse, Theorem 2, together with Theorem 1 and Corollary 2 then yield the principal theorems of this paper, Theorems 3 and 4.

Let S be a semigroup, and let $\mu\in m(S)^*$. By the *left* [*right*] *introversion on $m(S)$ induced by μ* is meant a mapping, $\mu_l\colon m(S)\to m(S)$, defined by $(\mu_l f)s=\mu(l_s f)$ $[(\mu_r f)s=\mu(r_s f)]$ for $f\in m(S)$ and $s\in S$, (c.f. [1, p. 540]).

THEOREM 2. *Let S be a semigroup, f_0 an arbitrary element of $m(S)$, α a real number and μ a left invariant mean on $m(S)$. If $\mu(f_0)=\alpha$, then there exists a net, $\{T_\delta\}$, of finite averages of right translations such that:*

(a) *For any $f\in m(S)$, the net $\{T_\delta f\}$ converges pointwise to a constant function.*

(b) *The net $\{T_\delta f_0\}$ converges pointwise to αe.*

Proof. Let $\nu=Q(I_s)$, for $s\in S$. Then the left introversion, ν_l, induced by ν satisfies $\nu_l=r_s$, the right translation by s, [1, p. 528, bottom 2 lines]. Also from the definition of a left introversion, it follows that if μ_1, μ_2, μ_3 are elements of $m(S)^*$ such that $\mu_1=\beta\mu_2+\gamma\mu_3$, where β and γ are real numbers, then

$$(\mu_1)_l=\beta(\mu_2)_l+\gamma(\mu_3)_l.$$

Let $\phi\in\Phi$. Then the finite mean, ϕ, may be expressed as $\phi=\sum_{s\in S}\phi(s)I_s$. Let $\theta=Q(\phi)$. Then $\theta_l=\sum_{s\in S}\phi(s)r_s$. So θ_l is a finite average of right translations on $m(S)$.

But $Q(\Phi)$ is w^*-dense in the set of means on $m(S)$ [1, (D), p. 513]. So there exists a net, $\{\theta_\delta\}$ where $\theta_\delta = Q(\phi_\delta)$ and $\phi_\delta \in \Phi$, such that $\{\theta_\delta\}$ is w^*-convergent to the left invariant mean, μ. Thus for any $f \in m(S)$,

$$\lim_\delta ((\theta_\delta)_l f)(s) = \lim_\delta (\theta_\delta, l_s f) = (\mu, l_s f) = \mu(f),$$

for all $s \in S$. Hence $\{(\theta_\delta)_l f\}$ converges pointwise to the constant function βe, where $\beta = \mu(f)$. And since $\mu(f_0) = \alpha$, then $\{(\theta_\delta)_l\}$ is the required net of finite averages of right translations.

The principal theorems can now be given.

THEOREM 3. *Let S be a semigroup. Then the following are equivalent:*

(a) *For every $f \in m(S)$, there exists a net of finite averages of right translates of f which converges pointwise to a constant function.*

(b) *S is left amenable.*

(c) *There exists a net, $\{T_\delta\}$, of finite averages of right translations such that for every f in $m(S)$, $\{T_\delta\ \}$ converges pointwise to a constant function.*

(a) $\rightarrow$ (b) It follows from Lemma 3, Corollary 1(b), and condition (a) that S is right stationary. Hence (b) follows by Corollary 2.

(b) $\rightarrow$ (c) This follows from Theorem 2(a).

(c) $\rightarrow$ (a) Condition (c) is formally stronger than (a).

THEOREM 4. *Let S be a left amenable semigroup, f_0 an arbitrary element in $m(S)$, and α an arbitrary real number. Then the following are equivalent:*

(a) *There exists a net of finite averages of right translates of f_0, which converges pointwise to αe.*

(b) *There exists a left invariant mean, μ, on $m(S)$ such that $\mu(f_0) = \alpha$.*

(c) $-p_R(-f_0) \leqq \alpha \leqq p_R(f_0)$.

Proof. By Theorem 3, S is right stationary. So by Lemma 5(b), Corollary 1(b), and Lemma 3, it follows that conditions (a) and (c) are equivalent.

(a) $\rightarrow$ (b) This follows from Theorem 1.

(b) $\rightarrow$ (a) This follows from Theorem 2(b).

COROLLARY 3. *Let S be a left amenable semigroup, f_0 an arbitrary element of $m(S)$, and α an arbitrary real number. Then the following are equivalent:*

(a) *f_0 is left almost convergent to α.*

(b) $-p_R(-f_0) = p_R(f_0) = \alpha$.

COROLLARY 4. *Let S be a left amenable semigroup. Then*

$$p_R(f) = \max_\mu \{\mu(f)\},$$

where μ is taken over all left invariant means on $m(S)$.

REMARKS. (a) Theorems 3 and 4, and Corollaries 3 and 4, remain true if the words left and right are transposed along with p_R and p_L.

(b) If S is a finite semigroup, Theorem 3, conditions (a) and (c); and Theorem 4, condition (a) do not require nets of finite averages of translations, but can use a single finite average of translations. For example, Theorem 3, condition (c) becomes:

3(c′) There exists a finite average of right translations, T, such that for every $f \in m(S)$, Tf is a constant function.

This follows from the fact that if S is finite, then P is a compact subset of the bounded operators on $m(S)$, in the uniform operator topology, say. Hence the net $\{T_\delta\}$; $T_\delta \in$ P, of Theorem 3, condition (c), has a cluster point, $T \in$ P.

(c) If S is a countably infinite semigroup, then the nets in Theorem 3, condition (a); and Theorem 4, condition (a) may be replaced by sequences. This follows from the fact that if B is a Banach space, then the w^*-topology of the closed unit sphere of B^* is metrizable if and only if B is separable [6, Theorem 1, p. 426]. If S is countable, then $l^1(S)$ is separable, so $Z_R(f)$ is metrizable in the w^*-topology, hence is first countable in that topology.

(d) Theorems 3 and 4, and Corollaries 3 and 4, remain true for left invariant means over certain subspaces, $X \subseteq m(S)$, rather than $m(S)$ itself. Let $X \subseteq m(S)$, where S is a semigroup, have the following properties:

(1) X is a linear subspace of $m(S)$,

(2) $e \in X$,

(3) $\mathscr{R}X \subseteq X$ and $\mathscr{L}X \subseteq X$,

(4) X is w^*-closed.

We say that $\mu \in X^*$ is a *left invariant mean* on X if $\|\mu\| \leqq 1$, $\mu(e) = 1$, and $\mu(f) = \mu(l_s f)$ for all $l_s \in \mathscr{L}$. Then if we substitute, "$f \in X$" for "$f \in m(S)$," "X has a left invariant mean" for "S is left amenable," then Theorem 3, so modified, remains valid. Similar substitutions in Theorem 4, and Corollary 3 and 4, and the dual results indicated in remark (b) result in valid statements. The proof of the modified statements goes through as before.

5. **Consequences of the characterization.** This section derives various results by the use of Theorems 3 and 4. A new proof is given of Day's generalization [2, Theorem 1, p. 586] of the Markov-Kakutani fixed point theorem (Theorem 5). An extension of a result by Day [1, (c′), p. 521] on weak convergence of finite means to left invariance is obtained (Theorem 6). The remainder of the section is primarily concerned with the left thick subsets of a semigroup S.

A map, $T: L \to L'$, where L and L' are real linear spaces, is called *affine* if $T(\alpha x + (1 + \alpha)y) = \alpha T(x) + (1 - \alpha)T(y)$, for $x, y \in L$ and $0 \leqq \alpha \leqq 1$.

THEOREM 5. *Let K be a compact convex subset of a locally convex linear topological space, X, and let S be a semigroup, under functional composition, of continuous affine transformations of K into itself. If S, when regarded as an* *abstract semigroup, is left amenable, then K contains a common fixed point of the family S.*

Proof. Since S is left amenable, then by Theorem 3, there exists a net, $\{T_\gamma\}$ where $T_\gamma \in \mathrm{P}$, such that for any $f \in m(S)$, $\{T_\gamma f\}$ converges pointwise to a constant function. For any T_γ, there exists $\phi_\gamma \in \Phi$ such that $T_\gamma = \sum_{s \in S} \phi_\gamma(s) r_s$. Let the map $J_\gamma \colon K \to K$ be given by $J_\gamma k = \sum_{s \in S} \phi_\gamma(s) s(k)$, for $k \in K$. For the remainder of the proof, let y be a specific point in K. By compactness of K, there exists a subnet, $\{J_\delta\}$ of $\{J_\gamma\}$, such that $\{J_\delta y\}$ converges to some $y_0 \in K$. And the associated subnet, $\{T_\delta\}$ of $\{T_\gamma\}$, satisfies that for any $f \in m(S)$, $\{T_\delta f\}$ converges pointwise to a constant function, since $\{T_\delta f\}$ is a subnet of $\{T_\gamma f\}$.

We will show that for any $s_0 \in S$, that $s_0 y_0$ is the required common fixed point. (It can be shown by counterexample that y_0 itself need not be one.) For each $\mu \in X^*$, define as in [2, p. 587], a real valued function, f_μ on S by $f_\mu(s) = \mu(sy)$, for $s \in S$. Since μ is continuous over the compact set K, then $f_\mu \in m(S)$. So for $s' \in S$,

$$\begin{aligned}
(T_\delta f_\mu)s' &= \sum_{s \in S} \phi_\delta(s)((r_s f_\mu)s') \\
&= \sum_{s \in S} \phi_\delta(s) f_\mu(s's) = \sum_{s \in S} \phi_\delta(s)\mu(s'sy) \\
&= \mu\left(\sum_{s \in S} \phi_\delta(s)s'sy\right) = \mu\left(s'\left(\sum_{s \in S} \phi_\delta(s)sy\right)\right) \\
&= \mu(s'J_\delta y),
\end{aligned}$$

by linearity of μ and affineness of s', in steps 4 and 5, respectively. Then

$$\begin{aligned}
\lim_\delta (T_\delta f_\mu)s' &= \lim_\delta (\mu(s'(J_\delta y))) = \mu\left(s'\left(\lim_\delta (J_\delta y)\right)\right) \\
&= \mu(s'y_0),
\end{aligned}$$

by continuity of μ and s', in step 2. Since for any $s_0 \in S$, $\lim_\delta (T_\delta f_\mu)ss_0 = \lim_\delta (T_\delta f_\mu)s_0$ (recall that for every $f \in m(S)$, $\{T_\delta f\}$ converges pointwise to a constant function), then $\mu(ss_0y_0) = \mu(s_0y_0)$, for all $s \in S$ and all $\mu \in X^*$. Since X is locally convex, then X^* is total over X [4, Theorem 2, p. 14], so $s(s_0y_0) = s_0y_0$, for all $s \in S$, which completes the proof.

An alternate proof of Theorem 5 is also given by Glicksberg [7, p. 98], and for a special case of the space X, by Heyneman [9, 4.3.1, p. 1340].

By use of Theorem 3, Day's result on convergence of means to left invariance may be extended. The inclusion of condition (a) in the equivalence below is new.

THEOREM 6. *Let S be a semigroup. Then the following are equivalent:*

(a) *For each $f_0 \in m(S)$, there exists a net, $\{\phi_\eta\}$, of finite means such that for every $s \in S$, $\lim_\eta (f_0, s\phi_\eta - \phi_\eta) = 0$.*

(b) *S is left amenable.*

(c) *There exists a net,* $\{\phi_\delta\}$, *of finite means such that for all* $f \in m(S)$, *and all* $s \in S$, $\lim_\delta(f, s\phi_\delta - \phi_\delta) = 0$.

Proof. (b)$\leftrightarrow$(c) This is [1, (c′), p. 521].

(c)$\to$(a) Condition (c) is formally stronger than (a).

(a)$\to$(b) By w^*-compactness of the set of means on $m(S)$ there exists a subnet, $\{\phi_\delta\}$, of $\{\phi_\eta\}$ such that $\{Q\phi_\delta\}$ is w^*-convergent to some mean, μ, on $m(S)$. And by w^*-continuity of l_s^* on $m(S)^*$, $w^*\lim_\delta(l_s^*(Q\phi_\delta)) = l_s^*\mu$, for all $s \in S$. Then

$$\begin{aligned} 0 &= \lim_\eta (f_0, s\phi_\eta - \phi_\eta) = \lim_\delta (f_0, s\phi_\delta - \phi_\delta) \\ &= \lim_\delta (Q(s\phi_\delta - \phi_\delta), f_0) = \lim_\delta (l_s^*(Q\phi_\delta) - Q\phi_\delta, f_0) \\ &= (l_s^*\mu, f_0) - (\mu, f_0). \end{aligned}$$

So $(l_s^*\mu, f_0) = (\mu, f_0)$ for all $s \in S$. Recall that if $\phi \in \Phi$, then $(Q\phi)_l \in \mathrm{P}$, from the proof in Theorem 2. Then for every $s \in S$,

$$\begin{aligned} \lim_\delta (((Q\phi_\delta)_l f_0)(s)) &= \lim_\delta (Q\phi_\delta, l_s f_0) \\ &= \lim_\delta (l_s^*(Q\phi_\delta), f_0) = (l_s^*\mu, f_0) = (\mu, f_0). \end{aligned}$$

Thus $\{(Q\phi_\delta)_l\}$ is a net in P such that $\{(Q\phi_\delta)_l f_0\}$ converges pointwise to the constant function $(\mu, f_0)e$. Then S satisfies condition (a) of Theorem 3, hence (b) follows by Theorem 3, which shows Theorem 6.

Let S' be a subset of S' a semigroup. Then S' is called *left thick* in S if for each finite subset $S'' \subseteq S$, there exists an $s'' \in S$ such that $S''s'' \subseteq S'$.

REMARKS. (a) If S' is a left thick subset of S, then the s'' above may be chosen to be in S'. For let $S'' = \{s_1, s_2, \cdots, s_n\}$. Consider the finite set

$$S''' = \{s_1 s_1, s_2 s_1, \cdots, s_n s_1, s_1\}.$$

There exists $s''' \in S$ such that $S'''s''' \subseteq S'$, since S' is left thick in S. Thus

$$S'' s_1 s''' \subseteq S', \text{where } s_1 s''' \in S'.$$

(b) Let S_1 be a left thick subset of a semigroup S. If $S_1 \subseteq S_2 \subseteq S$, then S_2 is left thick in S.

(c) Let $S_1 \subseteq S_2 \subseteq S$, where S_2 and S are semigroups. If S_1 is left thick in S_2, and S_2 is left thick in S, then S_1 is left thick in S.

(d) If V is a left ideal of a semigroup S, then V is left thick in S.

(e) A left thick subsemigroup S' of a semigroup S, need not be or contain a left ideal of S. For example, let S' be the positive integers under addition, and let S be the integers under addition. Then S' is left thick in S but contains no left ideal of S.

More generally, let $F(n)$ be the free group on n generators $\{g_1, g_2, \cdots, g_n\}$, and $F^+(n)$ the subsemigroup of reduced words with the property that for each i, the sum of the exponents of g_i in the word, is positive. Then $F^+(n)$ is left thick in $F(n)$, and contains no left ideal of $F(n)$.

(f) Let S' be a left thick subset of a semigroup S. If v is a homomorphism on S into a semigroup, then $v(S')$ is left thick in $v(S)$.

(g) Let S' be a subset of a finite semigroup S. Then S' is left thick in S if and only if S' contains a left ideal of S. The "if" part follows from (b) and (d). For the converse, since S is finite, there exists $s \in S$ such that $Ss \subseteq S'$. But Ss is a left ideal of S.

(h) A recent result by C. Wilde and K. Witz clarifies the relationship between left ideals and left thick subsets of a semigroup S. Let S' be a subset of S, let $\beta(S)$ be the Stone-Čech compactification of S where S has the discrete topology, and $\mathrm{CL}(S')$ is the closure of S' in $\beta(S)$. Then S' is left thick in S if and only if $\mathrm{CL}(S')$ contains a left ideal of $\beta(S)$ [13, Lemma 5.1].

THEOREM 7. *Let S' be a subset of S, a left amenable semigroup. Let f_0 be the characteristic function of S'. Then the following are equivalent*:

(a) *S' is left thick in S.*

(b) *There exists a left invariant mean, μ, on $m(S)$ such that $\mu(f_0) = 1$.*

Proof. (a) $\rightarrow$ (b) Let Γ be the family of all finite subsets of S, ordered upwards by inclusion. Then Γ is a directed set. For convenience, we will refer to a finite subset of S as S_γ, for $\gamma \in \Gamma$, rather than as γ itself. Then by (a), for any $S_\gamma \subseteq S$, there exists $s(\gamma) \in S$ such that $S_\gamma s(\gamma) \subseteq S'$. Let $T_\gamma = r_{s(\gamma)}$. Then for any $s \in S$,

$$\lim_\gamma (T_\gamma f_0)s = \lim_\gamma f_0(ss(\gamma)) = 1.$$

So $T_\gamma f_0$ converges pointwise to e, hence (b) follows by Theorem 4.

(b) $\rightarrow$ (a) Suppose (a) is not true. Then there exists a finite subset $S'' \subseteq S$ such that for every $s' \in S$, $S''s' \nsubseteq S'$. Hence for each $s' \in S$, there exists $s \in S''$ such that $f_0(ss') = 0$. Let N be the cardinality of S''. Then for any $s' \in S$,

$$\sum_{s \in S''} (r_{s'} f_0)s = \sum_{s \in S''} f_0(ss') \leqq N - 1.$$

And for $T \in \mathrm{P}$, then $T = \sum_{s' \in S} \phi(s') r_{s'}$ for some $\phi \in \Phi$. So

$$\begin{aligned}\sum_{s \in S''} (Tf_0)s &= \sum_{s \in S''} \sum_{s' \in S} \phi(s') f_0(ss') \\ &= \sum_{s' \in S} \phi(s') \sum_{s \in S''} f_0(ss') \\ &\leqq \sum_{s' \in S} \phi(s')(N-1) = N - 1.\end{aligned}$$

So for each $T \in \mathrm{P}$, there exists $s \in S''$ such that $(Tf_0)s \leqq (N-1)/N$. Since S'' is finite, no net, $\{T_\gamma f_0\}$ where $T_\gamma \in \mathrm{P}$, can converge pointwise to e. Hence (b) is not true, by Theorem 4, if (a) is not true. So (b)$\rightarrow$(a), which completes the proof.

If S is the semigroup of positive integers, E, under addition, an invariant mean μ, on $m(E)$ is called a *Banach limit.*

COROLLARY 5. *Let f_0 be a sequence consisting entirely of zeros and ones. Then there exists a Banach limit, μ, such that $\mu(f_0) = 1$, if and only if for every positive integer, n, there exists a consecutive block of at least n ones in f_0.*

Proof. Let E designate the semigroup of positive integers under addition. It is well known that E is amenable. Let $E' = \{i \in E; f_0(i) = 1\}$. Then f_0 is the characteristic function of E'. Let, for $n \in E$, $E_n = \{j \in E; 1 \leqq j \leqq n\}$. Then E' is left thick in E, if and only if for each $E_n \subseteq E$, there exists $i \in E$ such that $E_n + i \subseteq E'$. This holds since the E_n are finite subsets of E, and are such that any finite subset, $E'' \subseteq E$, is contained in some E_n. But $E_n + i$ is a consecutive block of n ones. The result follows by Theorem 7.

LEMMA 6. *Let $\{S_\gamma\}$ be the family of finite subsets of a semigroup S, directed upwards by inclusion. Then these are equivalent:*

(a) *For each $f_0 \in m(S)$ and for each $S_\gamma \subseteq S$, there exists a net, $\{T_{\gamma\eta}\}$ where $T_{\gamma\eta} \in_l \mathrm{P}$, such that the restriction of $\{T_{\gamma\eta} f_0\}$ to S_γ converges pointwise, with respect to η, to a constant function on S_γ.*

(b) *S is left amenable.*

Proof. (b)$\rightarrow$(a) This follows from implication (b)$\rightarrow$(a) of Theorem 3.

(a)$\rightarrow$(b) There are two cases. Let S be finite, then S is some S_γ. Then condition (b) above follows by implication (a)$\rightarrow$(b) of Theorem 3.

Now let S be infinite for the remainder of the proof. For any finite nonempty $S_\gamma \subseteq S$, let the cardinality of S_γ be N_γ, a positive integer. Then for any $f_0 \in m(S)$, there exists an η', and a real number α_γ such that

$$|(T_{\gamma\eta'} f_0)s - \alpha_\gamma| \leqq 1/N_\gamma \quad \text{for all } s \in S_\gamma,$$

since pointwise convergence on a finite set, S_γ, implies uniform convergence on S_γ. Designate this $T_{\gamma\eta'}$ by T_γ. Since $1/N_\gamma \leqq 1$, and $\|T_\gamma f_0\| \leqq \|f_0\|$ by Corollary 1(b), then $|\alpha_\gamma| \leqq \|f_0\| + 1$ for all α_γ. By compactness of the closed real interval $[-\|f_0\| - 1, \|f_0\| + 1]$, there exists a subnet, $\{\alpha_\delta\}$ of $\{\alpha_\gamma\}$, such that $\{\alpha_\delta\}$ converges to a real number, α. Let $\{T_\delta\}$ be the corresponding subnet of $\{T_\gamma\}$. Then for $s \in S$,

$$\begin{aligned} |(T_\delta f_0)s - \alpha| &= |(T_\delta f_0)s - \alpha_\delta + (\alpha_\delta - \alpha)| \\ &\leqq |(T_\delta f_0)s - \alpha_\delta| + |\alpha_\delta - \alpha| \leqq (1/N_\delta) + |\alpha_\delta - \alpha|. \end{aligned}$$

Since S is infinite, and $\{S_\delta\}$ is a subnet of $\{S_\gamma\}$, then N_δ becomes arbitrarily large.

Hence

$$\lim_{\delta} |(T_\delta f_0)s - \alpha| \leqq \lim_{\delta} ((1/N_\delta) + |\alpha_\delta - \alpha|) = 0,$$

for $s \in S$. So $\{T_\delta f_0\}$ converges pointwise to the constant function αe. Thus (b) follows by Theorem 3.

LEMMA 7. *Let S be a semigroup, and let $\{S_\gamma\}$ be any system of subsets of S such that $\{S_\gamma\}$ forms a set directed upwards by inclusion, and such that $\bigcup_\gamma S_\gamma = S$. Then these are equivalent:*

(a) *For each $f_0 \in m(S)$, and for each $S_\gamma \subseteq S$, there exists a net, $\{T_{\gamma\eta}\}$ where $T_{\gamma\eta} \in \mathrm{P}$, such that the restriction of $\{T_{\gamma\eta} f_0\}$ to S_γ converges pointwise with η to a constant function on S_γ.*

(b) *S is left amenable.*

Proof. (b)$\rightarrow$(a) This follows from the implication (b)$\rightarrow$(c) of Theorem 3.

(a)$\rightarrow$(b) Let S' be any finite subset of n elements of S. Since $\bigcup_\gamma S_\gamma = S$, then for each $s_i \in S'$, there exists $S_{\gamma(i)}$ such that $s_i \in S_{\gamma(i)}$. Since the $\{S_\gamma\}$ forms a directed set by inclusion, there exists S_γ such that $S_\gamma \supseteq S_{\gamma(i)}$ for $i = 1, 2, \cdots, n$. So $S_\gamma \supseteq S'$. Then condition (a) above, implies condition (a) of Lemma 6. Hence by Lemma 6, there follows (b), which completes the proof.

The next result is a generalization of a theorem by Dixmier [5, Theorem 2(δ), p. 215], and by Day [3, Corollary 8, p. 287].

THEOREM 8. *Let S be a semigroup, and let $\{S_\gamma\}$ be any system of subsets of S such that $\{S_\gamma\}$ forms a set directed upwards by inclusion, and such that $\bigcup_\gamma S_\gamma = S$. Then these are equivalent:*

(a) *For each $S_\gamma \subseteq S$, there exists a left amenable subsemigroup, $S'_\gamma \subseteq S$, and an element $s(\gamma) \in S$, such that $S_\gamma s(\gamma) \subseteq S'_\gamma$.*

(b) *S is left amenable.*

Proof. (b)$\rightarrow$(a) For each S_γ, let $S'_\gamma = S$, and let $s(\gamma) \in S$ be arbitrary. Then (a) follows.

(a)$\rightarrow$(b) For each of the left amenable semigroups, S'_γ, let $\{T'_{\gamma\eta}\}$ be the net whose existence is asserted by Theorem 3, condition (c). For each $T'_{\gamma\eta}$, there exists $\phi_{\gamma\eta} \in \Phi(S'_\gamma)$, the set of finite means on $m(S'_\gamma)$, such that $T'_{\gamma\eta} = \sum_{s \in S'} \phi_{\gamma\eta}(s) r_s$. Let $T_{\gamma\eta}$, where $T_{\gamma\eta}$ is a finite average of right translations on $m(S)$, be defined by the same expression, but where the r_s are now regarded as right translations on $m(S)$, rather than on $m(S'_\gamma)$. Then for $f_0 \in m(S)$, we have for each S'_γ, $\lim_\eta (T_{\gamma\eta} f_0)s = \alpha$ for $s \in S'_\gamma$, where α is a real number depending on S_γ and f_0, but not on $s \in S_\gamma$. Hence

$$\lim_{\eta} (r_{s(\gamma)} T_{\gamma\eta} f_0)s = \lim_{\eta} (T_{\gamma\eta} f_0)(ss(\gamma)) = \alpha.$$

So $\{r_{s(\gamma)} T_{\gamma\eta}\}$ is the required net in condition (a) of Lemma 7, thus (b) follows by Lemma 7, which proves the theorem.

COROLLARY 6. *Let S be a semigroup such that for each s, $s' \in S$, there exists $s'' \in S$ so that $ss'' = s's''$ Then S is left amenable.*

Proof. First we show that S satisfies property (a) below:

(a) For every finite subset $S'' \subseteq S$, there exists $s'' \subseteq S$ such that $S''s'' = \{s''\}$.

We proceed by induction on the number n of elements in S''. For $n = 1$, $S'' = \{s\}$. By hypothesis there exists $s' \in S$ such that $(s)s' = (ss)s'$. Letting $s'' = ss'$, then $ss'' = s''$, proving (a) for $n = 1$.

Suppose (a) holds for all subsets of S that contain n elements. Let S'' contain $n+1$ elements. Then $S'' = S_N \cup \{s\}$ where $s \in S''$ and S_N contains n elements. By supposition, there exists $s' \in S$ such that $S_N s' = \{s'\}$. By hypothesis, there exists $s'' \in S$ such that $(s')s'' = (ss')s''$. Since (a) is shown for $n = 1$, there exists $s''' \in S$ such that $(s's'')s''' = s'''$. It follows by direct computation that $S''(s's''s''')$ $= \{s's''s'''\}$, which completes the proof of (a).

Let $\{S_\gamma\}$ be the family of finite subsets of S directed upwards by inclusion. By (a), for each $S_\gamma \subseteq S$, there exists $s(\gamma) \in S$ such that $S_\gamma s(\gamma) = \{s(\gamma)\}$. Let $S'_\gamma = \{(s(\gamma))^n; n = 1, 2, \cdots\}$. Then S'_γ is an Abelian semigroup, hence left amenable. Since $S_\gamma s(\gamma) \subseteq S'_\gamma$, then the result follows by Theorem 8.

THEOREM 9. *Let S' be a left thick subsemigroup of a semigroup S. Then S' is left amenable if and only if S is left amenable.*

Proof. (a) Let S be left amenable. Let f_0 be the characteristic function of S'. By Theorem 7, there exists a left invariant mean μ on $m(S)$ such that $\mu(f_0) = 1$. By Day [1, Theorem 2, p. 518], S' is left amenable.

(b) Let S' be left amenable. Then S is left amenable by Theorem 8.

Implication (b) of Theorem 9 also follows from a result of Granirer [8, Corollary 5.1, p. 43].

By combining Theorem 9 with remark (d) of this section, there follows directly the known result that if V is a left ideal of a semigroup S, then V is left amenable if and only if S is left amenable.

A direct proof of this result can also be simply obtained. Let $\pi: m(S) \to m(V)$ be the projection map $(\pi f)s = f(s)$ for $f \in m(S)$ and $s \in V$. Let s' be any element of the left ideal V. Let $T_{s'}: m(V) \to m(S)$ be given by $(T_{s'}g)(s) = g(ss')$ for $g \in m(V)$ and $s \in S$. Then the following can be shown by direct computation. If μ' and λ are left invariant means on $m(V)$ and $m(S)$ respectively, then $\mu = \pi^*\mu'$ and $\lambda' = T_{s'}^*\lambda$ are left invariant means on $m(S)$ and $m(V)$ respectively.

BIBLIOGRAPHY

1. M. M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544.

2. ———, *Fixed-point theorems for compact convex sets*, Illinois J. Math. **5** (1961), 585–590.

3. ———, *Means for the bounded functions and ergodicity of the bounded representations of semigroups*, Trans. Amer. Math. Soc. **69** (1950), 276–291.

4. ———, *Normed linear spaces*, 2nd. ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer, Berlin, 1962.

5. J. Dixmier, *Les moyennes invariantes dans les semi-groupes et leurs applications*, Acta Sci. Math. Szeged **12** (1950), 213–227.

6. N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Pure and Applied Mathematics No. 7, Interscience, New York, 1958.

7. I. Glicksberg, *On convex hulls of translates*, Pacific J. Math. **13** (1963), 97–113.

8. E. Granirer, *On amenable semigroups with a finite-dimensional set of invariant means*. I, Illinois J. Math. **7** (1963), 32–48.

9. R. G. Heyneman, *Duality in general ergodic theory*, Pacific J. Math. **12** (1962), 1329–1341.

10. J. L. Kelly, *General topology*, Van Nostrand, Princeton, N. J., 1955.

11. G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190.

12. W. G. Rosen, *On invariant means over compact semigroups*, Proc. Amer. Math. Soc. **7** (1956), 1076–1082.

13. C. Wilde and K. G. Witz, *Invariant means and the Stone-Čech compactification*, Pacific J. Math. (to appear).

14. K. G. Witz, *Applications of a compactification for bounded operator semigroups*, Illinois J. Math. **8** (1964), 685–696.

State University of New York at Buffalo,
Buffalo, New York