ON THE SPECTRA OF SEMI-NORMAL OPERATORS

BY
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1. **Introduction.** Let $\mathfrak{H}$ denote a Hilbert space of elements $f, g, \cdots$, with the norm $\|f\| = (f,f)^{1/2}$. There will be considered only bounded operators, that is, linear transformations T defined on the whole of $\mathfrak{H}$ and satisfying $\|T\| = \sup \|Tf\| < \infty$, where $\|f\| = 1$. The spectrum of T will be denoted by $\mathrm{sp}(T)$, while the closure of the value domain of T, that is, the closure of the set of complex numbers (Tf,f) where $\|f\| = 1$, will be denoted by $W(T)$. It is known (Hausdorff-Toeplitz; cf. Stone [12, p. 131]) that $W(T)$ is a closed convex set and always contains $\mathrm{sp}(T)$.

An operator T will be called semi-normal if

$$TT^* - T^*T \equiv D \geqq 0 \text{ or } D \leqq 0. \tag{1.1}$$

If T is semi-normal with the Cartesian representation

$$T = H + iJ, \text{ where } H = (T + T^*)/2 \text{ and } J = (T - T^*)/2i, \tag{1.2}$$

then (1.1) holds if and only if

$$HJ - JH = iC, \quad \text{where } C \geqq 0 \text{ or } C \leqq 0 \text{ (with } D = 2C). \tag{1.3}$$

In particular T is normal if C (or D) is 0.

By an isolated part σ of $\mathrm{sp}(T)$ is meant a subset of $\mathrm{sp}(T)$ which lies at a positive distance from its complementary part $\mathrm{sp}(T) - \sigma$; see Riesz and Sz-Nagy [10, pp. 418 ff.]. It is known that if σ is an isolated part of $\mathrm{sp}(T)$, then there exists a "parallel projection" $P = P_\sigma$, a bounded operator, not necessarily self-adjoint, satisfying $P^2 = P$ and such that both $P\mathfrak{H}$ and $(I - P)\mathfrak{H}$ are invariant under T. Moreover $\mathrm{sp}(T') = \sigma$, where $T' = T/P\mathfrak{H}$ denotes the restriction of T to the space $P\mathfrak{H}$. In case T is semi-normal, so also is T'; cf. Berberian [1, p. 161, problem 10].

In case A is a self-adjoint operator with the spectral resolution

$$A = \int \lambda dE(\lambda), \tag{1.4}$$

then the set $\mathfrak{H}_a$ of elements f in $\mathfrak{H}$ for which $\|E(\lambda)f\|^2$ is an absolutely continuous function of λ is known to be a subspace of $\mathfrak{H}$; see Halmos [2, p. 104].

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Ordinary one and two dimensional Lebesgue measure of a corresponding Borel set S of the line or plane will be denoted respectively by $\mu_1(S)$ and $\mu_2(S)$. If S is a Borel set of the real line then the spectral family $\{E(\lambda)\}$ of (1.4) assigns a (self-adjoint) projection measure $E(S)$; see Halmos [2, pp. 58 ff].

In §2 there will be stated several results for semi-normal operators T which represent generalizations of corresponding results for normal operators. §3 is concerned with estimates for $\|D\|$ (see (1.1)) involving the areas of the sets $W(T)$ and $\mathrm{sp}(T)$. Further results on the nature of the spectrum of T are given in §§4 and 5. Some remarks on absolute continuity of the real and imaginary parts of T are made in §6. §§7–13 contain the proofs of the theorems. The last two §§14 and 15 are devoted to a few applications of the results to Toeplitz matrices and singular integral operators.

2. THEOREM I. *Let T of* (1.2) *be semi-normal, so that* (1.1) *or* (1.3) *holds* (i) *If $x_0 \in \mathrm{sp}(H)$ there exists some real number y_0' and a sequence $\{h_n\}$ of unit vectors for which $(H - x_0I)h_n \to 0$ and $(J - y_0'I)h_n \to 0$ as $n \to \infty$ so that, in particular, $x_0 + iy_0' \in \mathrm{sp}(T)$. Similarly, if $y_0 \in \mathrm{sp}(J)$ there exists some real number x_0' and a sequence $\{j_n\}$ of unit vectors for which $(H - x_0'I)j_n \to 0$ and $(J - y_0I)j_n \to 0$ as $n \to \infty$ so that, in particular, $x_0' + iy_0 \in \mathrm{sp}(T)$.* (ii) *If x_0 and y_0 are real and if $x_0 + iy_0 \in \mathrm{sp}(T)$ then $x_0 \in \mathrm{sp}(H)$ and $y_0 \in \mathrm{sp}(J)$.*

It follows from the above theorem that the spectra of the real and imaginary parts respectively of a semi-normal operator are precisely the sets of real numbers obtained by projecting the spectrum of T onto the x- and y-axes. This result for normal operators is known and can be deduced, for instance, from the spectral resolution formula.

There follows immediately the

COROLLARY 1 OF THEOREM I. *If T is semi-normal and if* $\mathrm{sp}(T)$ *is real then T is self-adjoint.*

Another consequence is

COROLLARY 2 OF THEOREM I. *If T is semi-normal then the set $W(T)$ is the smallest closed convex set containing the spectrum of T.*

In order to prove Corollary 2, note that for a self-adjoint operator A, the set $W(A)$ is always the closed segment of the real axis joining the maximum and minimum points of $\mathrm{sp}(A)$. In addition, if θ is real, then

$$T_\theta = Te^{i\theta} \tag{2.1}$$

is also semi-normal. Since $\mathrm{sp}(T_\theta) = e^{i\theta}\mathrm{sp}(T)$ and $W(T_\theta) = e^{i\theta}W(T)$, it follows from Theorem I that $W(T)$ is contained in every closed rectangle of the complex plane which contains $\mathrm{sp}(T)$. Thus $W(T)$ is contained in the intersection of all such rectangles, that is, $W(T)$ is contained in the least closed convex set containing $\operatorname{sp}(T)$. Since, even for arbitrary T, $\operatorname{sp}(T)$ is always a subset of $W(T)$, the proof of the corollary is complete.

In case T is normal the assertion of Corollary 2 is known (Toeplitz).

3. **Areas of** $W(T)$ **and** $\operatorname{sp}(T)$. Let T be arbitrary and define the function $M(x)$ on $-\infty < x < \infty$ by

$$M(x) = \begin{cases} \sup \operatorname{Im}(z) - \inf \operatorname{Im}(z), \text{ where } z \in \operatorname{sp}(T) \text{ and } x = \operatorname{Re}(z), \\ 0 \text{ if } x \notin \operatorname{Re}(\operatorname{sp}(T)). \end{cases} \tag{3.1}$$

Thus, for $x \in \operatorname{Re}(\operatorname{sp}(T))$, $M(x)$ is the distance between the upper and lower boundaries of $\operatorname{sp}(T)$ over x. For every real θ define T_θ by (2.1) and let the function $M_\theta(x)$ correspond to T_θ as $M(x)$ $(= M_0(x))$ does to $T(= T_0)$.

THEOREM II. *Let T be semi-normal, so that* (1.1) *holds. Then for every real θ,*

$$\pi \| D \| \leqq \int M_\theta(x)\, dx. \tag{3.2}$$

More generally, if $H_\theta = \operatorname{Re}(T_\theta)$ has the spectral resolution

$$H_\theta = \int \lambda \, dE_\theta(\lambda), \tag{3.3}$$

and if S denotes any Borel set of the real axis, then

$$\pi \| E_\theta(S) D E_\theta(S) \| \leqq \int_S M_\theta(x)\, dx. \tag{3.4}$$

That, in fact, relation (3.4) implies (3.2) follows from the observation that if $S = (-\infty, \infty)$ then $E_\theta(S) = I$.

In case S is a Borel set of measure 0, relation (3.4) implies that $E_\theta(S) D E_\theta(S) = 0$. Since D is semi-definite, then $DE_\theta(S) = 0$, a result proved in [5]. See also the remarks of §6 below.

In order to clarify the assertion of Theorem II a few consequences will be noted. First there follows the

COROLLARY 1 OF THEOREM II. *If T satisfies* (1.1) *then*

$$\pi \| D \| \leqq \mu_2(W(T)). \tag{3.5}$$

In order to prove (3.5) let θ be fixed. It is clear from the definition of $M_\theta(x)$ and the fact that $\operatorname{sp}(T)$ is contained in $W(T)$ that, for $x \in \operatorname{Re}(\operatorname{sp}(T_\theta))$, $M_\theta(x)$ is not greater than the distance between those points of the upper and lower boundaries of the set $W(T_\theta)$ which lie over the point x of the real axis. Thus the right side of (3.2) is not greater than the area of $W(T)$, and (3.5) follows.

It can be noted that the above corollary implies Corollary 1 of Theorem I.

COROLLARY 2 OF THEOREM II. *Let T satisfy* (1.1). *Suppose that for some fixed θ the set $e^{i\theta}\operatorname{sp}(T)$ has the property that, except possibly for a set of real values x of measure* 0, *the set $S_x = \{z : z \in e^{i\theta}\operatorname{sp}(T) \text{ and } \operatorname{Re}(z) = x\}$ is either a closed interval, or a single point, or the empty set. Then*

$$\pi \| D \| \leqq \mu_2(\operatorname{sp}(T)). \tag{3.6}$$

The proof follows from (3.2) if it is noted that $\mu_2(e^{i\theta}\operatorname{sp}(T)) = \mu_2(\operatorname{sp}(T))$ and that, in the present case, for almost all x, $M_\theta(x) = \mu_1(S_x)$ for $x \in \operatorname{Re}(e^{i\theta}\operatorname{sp}(T))$ and $M_\theta(x) = 0$ otherwise.

The restriction imposed on $\operatorname{sp}(T)$ by the hypothesis of the preceding corollary is that there should exist some direction, determined by a line L, with the property that almost all sections of $\operatorname{sp}(T)$, obtained by intersections of $\operatorname{sp}(T)$ with lines parallel to L, should be intervals or points.

The inequalities (3.5) and (3.6) are optimal in the sense that there exist semi-normal operators T which are not normal and for which both (3.5) and (3.6) become equalities. In fact, if T is isometric but not unitary, then $T^*T = I$ while TT^* is singular. It is easily verified that $\|D\| = 1$. Also both $\operatorname{sp}(T)$ and $W(T)$ are the closed unit disk $|z| \leqq 1$ (see, e.g., [6, p. 1650]) and so equality holds in (3.5) and (3.6).

Whether (3.6) must hold for all semi-normal operators will remain undecided. In fact, the question will remain open as to whether

$$\mu_2(\operatorname{sp}(T)) > 0 \tag{3.7}$$

holds for all semi-normal, but not normal, operators. That (3.7) is satisfied in certain special cases however was shown by Putnam [6], Stampfli [11].

4. **Isolated parts of** $\operatorname{sp}(T)$. Let T satisfy (1.1) and let

$$\Omega = \text{smallest subspace of } \mathfrak{H} \text{ reducing } T \text{ and containing } \mathfrak{R}_D, \tag{4.1}$$

where $\mathfrak{R}_D$ denotes the range of D. Thus, the orthogonal complement $\Omega^\perp$ of Ω is the largest subspace of $\mathfrak{H}$ reducing T and contained in the null space of D; or, equivalently, $\Omega^\perp$ is the largest subspace reducing T on which T is normal. It will be supposed that T is not normal on $\mathfrak{H}$, so that $\Omega \neq 0$. The assertion of the next theorem will relate to the operator T on Ω and it can therefore be supposed that $\Omega = \mathfrak{H}$.

THEOREM III. *Consider the semi-normal operator T as an operator on the space $\mathfrak{H} = \Omega(\neq 0)$, so that there do not exist any nontrivial subspaces reducing T on which T is normal. For each real θ, let T_θ be defined by* (2.1).

(i) *If S is a Borel set on the real axis, then*

$$M_\theta(x) = 0 \text{ a.e. on } S \text{ implies } E_\theta(S) = 0, \tag{4.2}$$

where $E_\theta(\lambda)$ is defined by (3.3).

(ii) *Let* σ *be any isolated part of* $\mathrm{sp}(T)$ *with the parallel projection* P *(see* §1) *and let* $T' = T/P\mathfrak{H}$ *denote the restriction of* T *to the subspace* $P\mathfrak{H}$. *Let* $H'_\theta = \mathrm{Re}(T_\theta)$, *where* $T'_\theta = T'e^{i\theta}$, *and suppose that* H'_θ *has the spectral resolution*

$$H'_\theta = \int \lambda \, dE'_\theta(\lambda). \tag{4.3}$$

Then

$$M'_\theta(x) = 0 \text{ a.e. on } S \text{ implies } E'_\theta(S) = 0, \tag{4.4}$$

where $M'_\theta(x)$ *corresponds to* T' *as* $M_\theta(x)$ *does to* T.

The above theorem has various implications concerning the nature of the spectrum of a semi-normal operator T. Since a normal operator is also semi-normal and since any closed bounded set is the spectrum of some normal operator, it is clear that the investigation of $\mathrm{sp}(T)$ when T is semi-normal should be restricted to the case $\mathfrak{H} = \Omega$ as in Theorem III.

Corollary I of Theorem III. *Let* T *be semi-normal, suppose* $\mathfrak{H} = \Omega$ $(\neq 0)$ *as in Theorem* III, *and let* σ *denote any isolated part of* $\mathrm{sp}(T)$. *Let* Q *denote any open strip of the complex plane bounded by two parallel lines and such that the set* $\sigma \cap Q$ *is not empty. Then* $\sigma \cap Q$ *is not a subset of any set* N *with the following property: for some* θ, *the strip* $Qe^{i\theta}$ *is perpendicular to the x-axis, intersects the x-axis in an open interval* (α, β), *and the set* $Ne^{i\theta}$ *is given by*

$$Ne^{i\theta} = \{x, f(x)) \colon \alpha < x < \beta, \ f(x) \text{ single-valued}\}. \tag{4.5}$$

In fact, if the assertion were false, then $\sigma \cap Q$ would be a nonempty subset of some set N of the type described. Since $\sigma = \mathrm{sp}(T')$ (cf. §1), then $\sigma e^{i\theta}$ is the spectrum of $T'_\theta = T'e^{i\theta}$, while

$$\sigma e^{i\theta} \cap Qe^{i\theta} \text{ is not empty} \tag{4.6}$$

and

$$(\rho e^{i\theta} \cap Qe^{i\theta}) \text{ is a subset of } Ne^{i\theta}. \tag{4.7}$$

But (4.5) and (4.7) imply that $M'_\theta(x) = 0$ on (α, β) and so, by (4.4), $E'_\theta((\alpha, \beta)) = 0$. According to Theorem I this implies that the set $\mathrm{Re}(\mathrm{sp}(T'_\theta)) \cap (\alpha, \beta)$ is empty, in contradiction with (4.6). This proves the corollary.

It is seen that the above corollary implies that when $\mathfrak{H} = \Omega$, no isolated part of $\mathrm{sp}(T)$ is contained in a segment (cf. Corollary 1 of Theorem I) or, for instance, in a *proper* subset of the boundary of a rectangle or a circle. On the other hand, the possibility that an isolated part of $\mathrm{sp}(T)$ might consist of the entire boundary of a rectangle or circle is not ruled out. Actually, it will remain undecided whether such a situation is possible, or more generally, whether or not an isolated part of $\mathrm{sp}(T)$ (assuming $\mathfrak{H} = \Omega$) must have a positive two dimensional Lebesgue measure (cf. the end of §3). However, there will be proved the following somewhat curious result.

5. Theorem IV. *Let T of* (1.2) *satisfy* (1.1). *Then either*

$$\text{both } \mathrm{sp}(H) \text{ and } \mathrm{sp}(J) \text{ contain an interval,} \tag{5.1}$$

or (3.6) *holds.*

Of course, if T is normal, then $D = 0$ and (3.6) certainly holds, while the assertion (5.1) may be false. On the other hand, if $D \neq 0$, not only the general validity of (3.6) but also that of (5.1) will remain undecided. However, there do exist estimates similar to (3.6) for the real and imaginary parts of T and in which the two dimensional measure is replaced by one dimensional measure. In fact it was shown in [9] that whenever T of (1.2) satisfies (1.1), then

$$\pi \| D \| \leqq 2 \| J \| \mu_1(\mathrm{sp}(H)) \tag{5.2}$$

and

$$\pi \| D \| \leqq 2 \| H \| \mu_1(\mathrm{sp}(J)). \tag{5.3}$$

A result similar to Theorem IV is

Theorem V. *Let T of* (1.2) *satisfy* (1.1) *and suppose that $\mathfrak{H} = \Omega$ ($\neq 0$) where Ω is defined by* (4.1), *so that T possesses no nontrivial reducing subspaces on which it is normal. If* $\mathrm{sp}(T)$ *has zero area, that is, if*

$$\mu_2(\mathrm{sp}(T)) = 0, \tag{5.4}$$

then there exist two open sets whose closures are respectively the sets $\mathrm{sp}(H)$ *and* $\mathrm{sp}(J)$.

As noted above it is conceivable that the assertion of Theorem V is vacuous in the sense that the hypothesis (5.4) may never hold (when $D \neq 0$). Also, it will remain undecided whether the assertion of Theorem V always holds even without the assumption (5.4).

6. **Remarks.** Concerning the spectrum of semi-normal operators, it can be mentioned that the real and imaginary parts, H_θ and J_θ, of T_θ are absolutely continuous on the space Ω defined by (4.1); [9]. In particular, if Z is a Borel set of zero Lebesgue measure, necessarily $E_\theta(Z) = 0$. In this connection, see (3.4) and (4.2).

7. **Proof of (i) of Theorem I.** Since iT is also semi-normal and has the Cartesian form $iT = (-J) + iH$, it is clearly sufficient to prove only the first part of (i). It will be clear from the proof that there is no loss of generality in supposing that $D \geqq 0$.

Let $x_0 \in \mathrm{sp}(H)$. Then there exists a sequence $\{f_n\}$ satisfying

$$(H - x_0 I)f_n \to 0, \qquad \|f_n\| = 1, \tag{7.1}$$

hence also,

$$J(H - x_0 I)f_n \to 0. \tag{7.2}$$

But, by (1.3),

$$(H - x_0 I)J - J(H - x_0 I) = iC, \tag{7.3}$$

and so, by (7.1), $(f_n, (H - x_0 I)Jf_n) - (f_n, J(H - x_0 I)f_n) = i\| C^{1/2}f_n \|^2 \to 0$. Hence $Cf_n = C^{1/2}(C^{1/2}f_n) \to 0$ and so, by (7.2) and (7.3), $(H - x_0 I)Jf_n \to 0$. Similarly, if Jf_n is now identified with the previous f_n, then $(H - x_0 I)J^2 f_n \to 0$ and, in like manner, $(H - x_0 I)p(J)f_n \to 0$, where $p(J)$ denotes any polynomial in J. Hence if $\phi(\lambda)$ denotes any continuous function on $-\infty < \lambda < \infty$ and if $\phi(J)$ is defined by the usual functional calculus, then $\phi(J)$ can be approximated uniformly by polynomial operators $p(J)$ and so

$$(H - x_0 I)\phi(J)f_n \to 0, \qquad \|f_n\| = 1. \tag{7.4}$$

Next, let J have the spectral resolution

$$J = \int \lambda \, dF(\lambda), \tag{7.5}$$

and suppose that $\mathrm{sp}(J)$ is contained in the interior of $\Delta_1 = [c,d]$. Then $\|F(\Delta_1)f_n\| = 1$ for all n. Clearly, for at least one of the intervals $\Delta = [c, \frac{1}{2}(c+d)]$ or $\Delta = [\frac{1}{2}(c+d), d]$, say $\Delta = \Delta_2$,

$$\| F(\Delta_2)f_n^{(2)} \| \geqq 1/2 \qquad (n = 1,2,\cdots), \tag{7.6}$$

where $\{f_n^{(2)}\}$ is a subsequence of $\{f_n^{(1)}\}$, with $f_n^{(1)} = f_n$. Continuing this process one obtains intervals $\Delta_1, \Delta_2, \cdots$ with the properties that for each fixed $k = 1,2,\cdots$, Δ_{k+1} is contained in Δ_k, the length of Δ_k is $(d-c)/2^{k-1}$, $\{f_n^{(k+1)}\}$ is a subsequence of $\{f_n^{(k)}\}$ and

$$\| F(\Delta_k)f_n^{(k)} \| \geqq \frac{1}{2^{k-1}} \qquad (k, n = 1,2,\cdots). \tag{7.7}$$

Let y_0' denote the real number determined by the nested sequence of intervals $\{\Delta_k\}$, so that

$$c_k, d_k \to y_0' \text{ as } k \to \infty, \text{ where } \Delta_k = [c_k, d_k]. \tag{7.8}$$

For each $k = 1,2,\cdots$, choose $\gamma_k > 0$ so that

$$\gamma_k \to 0 \text{ as } k \to \infty, \tag{7.9}$$

and define the continuous function $\phi_k(\lambda)$ on $-\infty < \lambda < \infty$ as the function whose graph is the real axis from $-\infty$ to $(c_k - \gamma_k, 0)$, the three segments joining $(c_k - \gamma_k, 0)$ to $(c_k, 1)$ to $(d_k, 1)$ to $(d_k + \gamma_k, 0)$ and the real axis from $(d_k + \gamma_k, 0)$ to ∞.

Clearly,

$$0 \leqq \frac{1}{2^{k-1}} \leqq \| F(\Delta_k) f_n^{(k)} \| \leqq \| \phi_k(J) f_n^{(k)} \| \leqq \| f_n^{(k)} \|. \tag{7.10}$$

On putting $g_{kn} = \phi_k(J) f_n^{(k)} / \| \phi_k(J) f_n^{(k)} \|$, it is seen that $\| g_{kn} \| = 1$ and, from (7.4) that for each fixed k,

$$(H - x_0 I) g_{kn} \to 0 \text{ as } n \to \infty. \tag{7.11}$$

On the other hand, it is clear from the definition of the g_{kn} that

$$\| (J - y_0' I) g_{kn} \| \leqq d_k - c_k + \gamma_k. \tag{7.12}$$

It now follows from (7.11) and (7.12), together with (7.8) and (7.9), that a subsequence $\{m_k\}$ of the positive integers can be chosen so that for $h_k = g_{k\, m_k}$, both $(H - x_0 I) h_k \to 0$ and $(J - y_0' I) h_k \to 0$ hold as $k \to \infty$, as was to be shown. This completes the proof of (i) of Theorem I.

8. **Proof of (ii) of Theorem I.** Let $q = x_0 + i y_0 \in \mathrm{sp}(T)$. It will be shown that $x_0 \in \mathrm{sp}(H)$. (The argument that $y_0 \in \mathrm{sp}(J)$ is similar.) Again it can be supposed without loss of generality that $D \geqq 0$. If $T_z = T - zI$, then it is seen from (1.2) that

$$T_z T_z^* = (H - xI)^2 + (J - yI)^2 + C, \text{ where } z = x + iy. \tag{8.1}$$

In case $T_q T_q^*$ is singular, there exists a sequence $\{f_n\}$ of unit vectors satysfying $(T_q T_q^* f_n, f_n) \to 0$. Since $C \geqq 0$, this implies by (8.1) that $(H - x_0 I) f_n \to 0$ and $(J - y_0 I) f_n \to 0$, and (ii) is proved. In case $T_q T_q^* > 0$ then necessarily $T_q^* T_q$ is singular and it follows from [6, p. 1650], that there exists a whole disk about q lying in $\mathrm{sp}(T)$. Let then y_0' be defined as the maximum value y with the property that, for $z = x_0 + iy$, T_z is singular. Clearly, $r = x_0 + i y_0' \in \mathrm{sp}(T)$ and $T_r T_r^*$ must be singular. Consequently, it follows as before that $x_0 \in \mathrm{sp}(H)$ (as well as $y_0' \in \mathrm{sp}(J)$). This completes the proof of (ii).

9. **Proof of Theorem II.** Let the real part H of the semi-normal operator T have the spectral resolution

$$H = \int \lambda \, dE(\lambda), \tag{9.1}$$

and let $\Delta = (a, b]$ denote a half-open interval of the λ-axis. For the projection $E(\Delta) = E(b) - E(a)$ and for an arbitrary operator A, put $A_\Delta = E(\Delta) A E(\Delta)$. Then A_Δ leaves invariant the Hilbert space $\mathfrak{H}_\Delta = E(\Delta)\mathfrak{H}$; $\mathrm{sp}(A_\Delta)$ will denote the spectrum of A_Δ as an operator on $\mathfrak{H}_\Delta$.

It was shown in [9] that (1.1)–(1.3) imply (5.3), that is,

$$\pi \| C \| \leqq \| H \| \mu_1(\mathrm{sp}(J)). \tag{9.2}$$

Now, it follows from (1.2) and (1.3) that $T_\Delta = H_\Delta + iJ_\Delta$ and

$$H_\Delta J_\Delta - J_\Delta H_\Delta = iC_\Delta. \tag{9.3}$$

Since $C_\Delta \geqq 0$ or $C_\Delta \leqq 0$ according as $C \geqq 0$ or $C \leqq 0$, then T_Δ is semi-normal on $\mathfrak{H}_\Delta$. If μ is arbitrary, then (9.3) implies that

$$(H_\Delta - \mu I_\Delta)J_\Delta - J_\Delta(H_\Delta - \mu I_\Delta) = iC_\Delta. \tag{9.4}$$

Hence, if μ is chosen to be the midpoint of Δ and if d denotes the length of Δ, then $\| H_\Delta - \mu I_\Delta \| \leqq d/2$, and it follows from a relation similar to (9.2) but in which H and J are replaced by $H_\Delta - \mu I_\Delta$ and J_Δ, that

$$(2\pi)^{1/2} \| C^{1/2} E(\Delta) f \| \leqq [d\mu_1(\mathrm{sp}(J_\Delta))]^{1/2} \| E(\Delta) f \|. \tag{9.5}$$

Let $y_0 \in \mathrm{sp}(J_\Delta)$. Then by Theorem I, there exists some x_0 and a sequence $\{f_n\}$ of unit vectors in $\mathfrak{H}_\Delta$, thus $\| f_n \| = 1$ and $f_n = E(\Delta)f_n$, for which

$$(H - x_0 I) f_n \to 0 \tag{9.6}$$

and

$$E(\Delta)(J - y_0 I) f_n \to 0. \tag{9.7}$$

(Note that $E(\Delta)HE(\Delta) = HE(\Delta)$, also that $x_0 \in \mathrm{sp}(H_\Delta)$ and hence $x_0 \in \Delta^*$, the closure of Δ.)

Now, relation (9.7) implies that

$$((J - y_0 I) f_n, f_n) \to 0, \tag{9.8}$$

from which it follows that there exists some pair of real numbers y_1 and y_2 satisfying

$$y_1 \leqq y_0 \leqq y_2 \quad \text{and} \quad y_1, y_2 \in \mathrm{sp}(J). \tag{9.9}$$

(The possibility $y_1 = y_2$ is allowed.)

Next, it will be shown that there exists a point $y_1' \leqq y_0$ for which $x_0 + iy_1' \in \mathrm{sp}(T)$. To this end, note that if $(J - y_0 I) f_n \to 0$ as $n \to \infty$, then in fact y_0' can be chosen to be y_0. Consequently it can be supposed that

$$\limsup \| (J - y_0 I) f_n \| > 0, \qquad n \to \infty. \tag{9.10}$$

As in §7, suppose that $\mathrm{sp}(J)$ is contained in the interior of $[c, d]$ so that by (9.9), $c < y_0$. If $\Delta_1 = [c, y_0]$, it follows from (9.8) and (9.10) that

$$\limsup \| F(\Delta_1)(J - y_0 I) f_n \| > 0, \qquad n \to \infty, \tag{9.11}$$

where $F(\lambda)$ is defined by (7.5). (In fact, if (9.11) were false, it would follow from (9.8) that $\int | \lambda - y_0 | d \| F f_n \|^2 \to 0$ and hence $\int (\lambda - y_0)^2 d \| F f_n \|^2 \to 0$, in contradiction with (9.10).) Hence there exists a subsequence $\{g_n\}$ of $\{f_n\}$ for which

$$(H - x_0 I)g_n \to 0 \tag{9.12}$$

and

$$\| F(\Delta_1)g_n \| > \text{const.} > 0. \tag{9.13}$$

The argument of §7 can now be applied so as to yield a point y_1' belonging to Δ_1, hence $y_1' \leqq y_0$, and a sequence $\{h_n\}$ of unit vectors for which

$$(H - x_0 I)h_n \to 0 \text{ and } (J - y_1' I)h_n \to 0, \qquad n \to \infty. \tag{9.14}$$

(Note that the present Δ_1 plays the role of $[c, d]$ in the argument of §7.) Thus, $z_1 = x_0 + iy_1' \in \mathrm{sp}(T)$; a similar argument shows that $z_2 = x_0 + iy_2' \in \mathrm{sp}(T)$ for some $y_2' \geqq y_0$.

Consequently, whenever $y_0 \in \mathrm{sp}(J)$, there exists a number x_0 in the closure Δ^* of Δ and a pair y_1', y_2' for which

$$y_1' \leqq y_0 \leqq y_2'; \; x_0 + iy_1' \text{ and } x_0 + iy_2' \text{ in } \mathrm{sp}(T). \tag{9.15}$$

This implies

$$\mu_1(\mathrm{sp}(J_\Delta)) \leqq I(\Delta^*), \tag{9.16}$$

where $I(\delta)$ denotes the interval function defined by $I(\delta) = 0$ if $\delta \cap \mathrm{Re}(\mathrm{sp}(T))$ is empty and $I(\delta) = \sup \mathrm{Im}(z) - \inf \mathrm{Im}(z)$ where $z \in \mathrm{sp}(T)$ and $\mathrm{Re}(z) \in \delta$.

Relation (9.5) now yields

$$(2\pi)^{1/2} \| C^{1/2} E(\Delta) f \| \leqq [dI(\Delta^*)]^{1/2} \| E(\Delta) f \|. \tag{9.17}$$

If $(c, d]$ contains $\mathrm{sp}(H)$ and if $P: c = c_0 < c_1 < \cdots < c_N = d$ is a partition of $(c, d]$ into subintervals $\Delta_k = (c_{k-1}, c_k]$ then $I = \sum_{k=1}^N E(\Delta_k)$ and $\| C^{1/2} f \| = \| C^{1/2} \sum_{k=1}^N E(\Delta_k) f \| \leqq \sum_{k=1}^N \| C^{1/2} E(\Delta_k) f \|$. An application of the Schwarz inequality to (9.17) then implies by virtue of $\| f \|^2 = \sum_{k=1}^N \| E(\Delta_k) f \|^2$,

$$(2\pi)^{1/2} \| C^{1/2} f \| \leqq \sum_{k=1}^{N} d_k I(\Delta_k^*)^{1/2} \| f \|, \tag{9.18}$$

where d_k is the length of Δ_k. If $F(x)$ is the function defined on $(c, d]$ by $F(x) = I(\Delta_k^*)$ on Δ_k, then (9.18) becomes

$$(2\pi)^{1/2} \| C^{1/2} f \| \leqq \left(\int_c^d F(x)\, dx \right)^{1/2} \| f \|. \tag{9.19}$$

Next, choose a sequence of partitions $\{P_n\}$ with the property that P_{n+1} is a refinement of P_n and such that the lengths of the intervals of P_n tend to zero as $n \to \infty$. Let $F_n(x)$ correspond to P_n as $F(x)$ does to P. It is clear from the definition of $M(x)$ ($= M_0(x)$) in §3 and the fact that $\mathrm{sp}(T)$ is a closed set that $I(\Delta^n) \to M(x)$ as $n \to \infty$, whenever $\{\Delta^n\}$ is any sequence of intervals containing x for which $\Delta^n \to x$ as $n \to \infty$. Consequently, $F_n(x) \to M(x)$ as $n \to \infty$ for all x on $(c, d]$, except possibly for those numbers x in the (denumerable) set of partitioning points. Since $0 \leqq F_n(x) \leqq \text{const.}$, it then follows from (9.19) and from Lebesgue's term by term integration theorem that (3.2) holds with $\theta = 0$. Since the same argument also applies to T_θ, relation (3.2) is seen to hold for any real θ.

Finally, by considering coverings of S by denumerable unions of the type $\sum \Delta_k$, where $\Delta_k = (a_k, b_k]$ and the Δ_k are pairwise disjoint, a similar argument leads to

$$(2\pi)^{1/2} \| C^{1/2}E(S)f \| \leqq \left(\int_S M_\theta(x)\, dx \right)^{1/2} \| f \|, \tag{9.20}$$

for any Borel set S of the real line. Since, for any operator A, $\| A \|^2 = \| A^*A \|$ and since $E(S)CE(S) = (C^{1/2}E(S))^*(C^{1/2}E(S))$, relations (9.20) and $D = 2C$ yield (3.4). This completes the proof of Theorem II.

10. **Proof of (i) of Theorem III.** For convenience it can be supposed that $\theta = 0$. A similar argument with T replaced by T_θ takes care of the general case. It follows from (3.4) that if $M_\theta(x) = 0$ a.e. on S then $C^{1/2}E(S) = 0$, where $E(\lambda)$ is defined by (9.1), and hence

$$CE(S) = 0. \tag{10.1}$$

Multiplication of (1.3) on the left by H leads to $H^2J - HJH = iHC$ while multiplication on the right leads to $HJH - JH^2 = iCH$, hence

$$H^2J - JH^2 = i(HC + CH). \tag{10.2}$$

Since, by (10.1), $Cf = 0$ whenever $f \in \mathfrak{H}_S = E(S)\mathfrak{H}$ and since also $Hf \in \mathfrak{H}_S$, then (10.2) implies that $(H^2J - JH^2)f = 0$ for $f \in \mathfrak{H}_S$. Similarly $(H^nJ - JH^n)f = 0$ and hence $p(H)Jf = Jp(H)f$ for $f \in \mathfrak{H}_S$ and for any polynomial $p(H)$. On choosing a sequence of polynomials $p_n(H)$ converging (strongly) to $E(S)$, one obtains

$$E(S)Jf = JE(S)f. \tag{10.3}$$

Consequently, $\mathfrak{H}_S$ is invariant under J (as well as under H). Thus T is reduced by $\mathfrak{H}_S$ and, by (10.1), T is normal on $\mathfrak{H}_S$. Consequently, from the assumption $\mathfrak{H} = \Omega$ and the definition (4.1) of Ω, $\mathfrak{H}_S = 0$, that is, $E(S) = 0$. This completes the proof of (i) of Theorem III.

11. **Proof of (ii) of Theorem III.** As above, it can be supposed that $\theta = 0$. Let T' have the Cartesian representation

$$T' = H' + iJ' \qquad \left(H' = \int \lambda\, dE'(\lambda) \right) \tag{11.1}$$

on the Hilbert space $\mathfrak{H}' = P\mathfrak{H}$. Since $M_0'(x) = 0$ a.e. on S, then, as above, T' is reduced by, and is normal on, $E'(S)\mathfrak{H}' = \mathfrak{H}_S'$. Consequently, T leaves invariant $\mathfrak{H}_S'$ (and $T' = T/P\mathfrak{H}$ is normal on $\mathfrak{H}_S'$). If $D \leqq 0$, then T is hyponormal on $\mathfrak{H}$ and consequently T is reduced by $\mathfrak{H}'_S$; Berberian [1, p. 161, problem 9]. In case $D \geqq 0$, then T^* is hyponormal and the preceding argument implies that T^* (hence also T) is reduced by $\mathfrak{H}'_S$. In any case, T is reduced by, and is normal on $\mathfrak{H}'_S$. As before, this implies $\mathfrak{H}'_S = 0$, that is, $E'(S) = 0$.

12. **Proof of Theorem IV.** The proof begins with the relations (9.5)–(9.7). Suppose that neither endpoint of $\Delta = (a, b]$ belongs to $\operatorname{sp}(H)$. Then it will be shown that (9.7) can be replaced by the stronger relation

$$(J - y_0 I) f_n \to 0. \tag{12.1}$$

Since $x_0 \in \operatorname{sp}(H_\Delta)$ then x_0 belongs to the closure of Δ, and since the endpoints of Δ do not belong to $\operatorname{sp}(H)$, it is clear that x_0 is an interior point of Δ. In addition, it follows from (9.6) (cf. (7.4)) that for $g_n = (J - y_0 I) f_n$,

$$\| (H - x_0 I) g_n \|^2 = \int (\lambda - x_0)^2 \, d \| E(\lambda) g_n \|^2 \to 0. \tag{12.2}$$

Since x_0 is interior to Δ, relations (12.2) and (9.7) imply that $g_n \to 0$, that is, (12.1). (That x_0 be an interior point of Δ is crucial here.) It then follows from (9.6) and (12.1) that $z_0 = x_0 + i y_0 \in \operatorname{sp}(T)$.

Thus, whenever $y_0 \in \operatorname{sp}(J_\Delta)$ and the endpoints of Δ do not belong to $\operatorname{sp}(H)$, there exist some $z_0 \in \operatorname{sp}(T)$ with $\operatorname{Im}(z_0) = y_0$ and $\operatorname{Re}(z_0) = x_0 \in \Delta$. If now the Δ-strip: $\{x \in \Delta,\ y \text{ arbitrary}\}$ is subdivided into a finite or an infinite number of rectangles by horizontal segments, then it is clear that $d\mu_1(\operatorname{sp}(J_\Delta))$ is not greater than any sum S_Δ of the areas of those rectangles containing points of $\operatorname{sp}(T)$. Hence, by (9.5),

$$(2\pi)^{1/2} \| C^{1/2} E(\Delta) f \| \leqq S_\Delta^{1/2} \| E(\Delta) f \|. \tag{12.3}$$

Now, in order to prove Theorem IV, suppose that (5.1) fails to hold, so that either $\operatorname{sp}(H)$ or $\operatorname{sp}(J)$ fails to contain an interval. There is no loss of generality in supposing that $\operatorname{sp}(H)$ does not contain an interval. Let $(\alpha, \beta]$ contain $\operatorname{sp}(H)$, hence, by Theorem I, $(\alpha, \beta]$ contains $\operatorname{Re}(\operatorname{sp}(T))$. Suppose also that α and β do not belong to $\operatorname{sp}(H)$, and consider subdivisions of $(\alpha, \beta]$: $(\alpha, \beta] = \sum \Delta_k$, consisting either of a finite or of an *infinite* (denumerable) union of disjoint subintervals Δ_k of the type Δ and for which no endpoints of the Δ_k lie in $\operatorname{sp}(H)$. Then, by (12.3), $(2\pi)^{1/2} \| C^{1/2} E(\Delta_k) f \| \leqq S_{\Delta k}{}^{1/2} \| E(\Delta_k) f \|$ and therefore, by the Schwarz inequality,

$$(2\pi)^{1/2} \| C^{1/2} f \| \leqq \left(\sum S_{\Delta_k} \right)^{1/2} \| f \|. \tag{12.4}$$

Hence,

$$2\pi \| C \| \leqq \sum S_{\Delta_k}. \tag{12.5}$$

Since $\operatorname{sp}(H)$ does not contain an interval it is clear that $\mu_2(\operatorname{sp}(T)) = \inf_{P, S_{\Delta_k}} \sum S_{\Delta_k}$, where only those partitions P (finite or infinite) are allowed with points not belonging to $\mathrm{sp}(H)$. The desired relation (3.6) now follows from (12.5).

13. **Proof of Theorem V.** It is sufficient to prove the theorem for H. It will be shown that the set consisting of the union of all open intervals contained in $\mathrm{sp}(H)$ is dense in $\mathrm{sp}(H)$. If the assertion were false, there would exist some closed interval K containing a point of $\mathrm{sp}(H)$ in its interior, in particular,

$$E(K) \neq 0, \tag{13.1}$$

with the property that no subinterval of K belongs to $\mathrm{sp}(H)$.

It follows from the argument of §12 however that

$$(2\pi)^{1/2}\| C^{1/2}E(K)f \| \leqq [\mu_2\{z: z \in \mathrm{sp}(T) \text{ and } \mathrm{Re}(z) \in K\}]^{1/2} \, \| E(K)f \|. \tag{13.2}$$

Since (5.4) implies that the right side of (13.2) is 0, it follows that $C^{1/2}E(K) = 0$, hence

$$CE(K) = 0. \tag{13.3}$$

As in §10 (cf. (10.1)), it follows that T is reduced by, and is normal on, $E(K)\mathfrak{H}$. Since $\mathfrak{H} = \Omega$, then $E(K) = 0$, in contradiction with (13.1), and the proof of Theorem V is complete.

14. **Toeplitz matrices.** Let $\{c_n\}$ for $n = 1, 2, \cdots$, be a sequence of complex numbers for which the power series

$$f(z) = \sum_{n=1}^{\infty} c_n z^n \tag{14.1}$$

is bounded on the open disk $|z| < 1$. Let $T = (t_{jk})$, for $j, k = 1, 2, \cdots$, be defined by

$$t_{jk} = c_{k-j} \text{ if } k - j \geqq 1 \text{ and } t_{jk} = 0 \text{ otherwise.} \tag{14.2}$$

Then T is known to be bounded and its spectrum is the closure of the set of values $f(z)$ when $|z| < 1$; Wintner [13, p. 279]. Furthermore, T is semi-normal; in fact

$$TT^* - T^*T = D, \text{ where } D = B^*B \text{ and } B = (c_{j+k-1}); \tag{14.3}$$

[7, p. 517], [8, p. 838].

It will be supposed that T is not normal, so that not all c_k are zero. Then $\mathrm{sp}(T)$ is the closure of a connected open set and hence its projections on the real and imaginary axes are closed intervals. According to Theorem I, the spectra of the real and imaginary parts of T are then closed intervals. This last result was first proved by Hartman and Wintner [3, p. 868].

According to Corollary 2 of Theorem I, $W(T)$ is the closed convex hull of $\mathrm{sp}(T)$. This result is also known; see Wintner [13, p. 278].

It is noteworthy that all examples of non-normal, semi-normal operators furnished by (14.2) do have spectra with positive two dimensional Lebesgue measures. Whether also relation (3.6) holds for these operators, in general (that is without any restriction on $\mathrm{sp}(T)$ such as, for instance, that made in Corollary 2 of Theorem II) will remain open, although (3.5) and even (3.2) always hold.

If T is defined by (14.2) then $\mu_2(\mathrm{sp}(T))$ is not greater than the double integral of $|f'(z)|^2$ taken over the disk $|z| \leqq 1$, hence,

$$\mu_2(\mathrm{sp}(T)) \leqq \pi \sum_{k=1}^{\infty} k|c_k|^2, \tag{14.4}$$

with the equality surely holding if the mapping $z \to f(z)$ ($|z| < 1$) is one to one. It follows from (14.3) however (cf. [7, p. 517]) that for $x = (x_1, x_2, \cdots) \in \mathfrak{H}$, $(Dx, x) = \sum_{j=1}^{\infty} |\sum_{k=1}^{\infty} c_{j+k-1} x_k|^2$, and hence, using the Schwarz inequality,

$$\|D\| \leqq \sum_{k=1}^{\infty} k\,|\,c_k\,|^2. \tag{14.5}$$

Consequently, relation (3.6) certainly holds whenever equality holds in (14.4).

15. **Singular integral operators.** Another class of semi-normal operators is given by the operators $T = H + iJ$, where

$$Hf = h(x)f(x) + (1/i\pi)\phi(x) \int_a^b \phi^*(t)(t-x)^{-1} f(t)dt, \tag{15.1}$$

and

$$Jf = xf(x), \tag{15.2}$$

where $f \in \mathfrak{H} = L^2(a,b)$, $-\infty < a < b < \infty$, and $h(x)$ is real-valued, $\phi(x)$ is complex-valued, and both h and ϕ are bounded and measurable on (a,b). See [9] and the references cited there.

If D is defined by (1.1) and (1.3), then $\|D\| = 2\pi^{-1}\|\phi\|^2$ (cf. [9]) and (3.5) becomes

$$2\int_a^b |\phi(t)|^2 dt \leqq \mu_2(W(T)); \tag{15.3}$$

in case $\phi(x) \equiv 1$, this becomes

$$2(b-a) \leqq \mu_2(W(T)). \tag{15.4}$$

In case also $h(x) \equiv 0$, that is, if $T_0 = H_0 + iJ$ is defined by (15.2) and

$$H_0 f = (1/i\pi) \int_a^b (t-x)^{-1} f(t)\,dt, \tag{15.5}$$

then $\mathrm{sp}(H_0) = [-1,1]$ (Koppelman and Pincus [4], see also [9]) while $\mathrm{sp}(J) = [a,b]$. Hence, if R denotes the closed rectangle $R = \{(x,y)\colon -1 \leqq x \leqq 1,\ a \leqq x \leqq b\}$, then, by (15.4), $R = W(T_0)$.

Addendum (May 25, 1965). The assertions of the two corollaries of Theorem I have also been proved by Stampfli (*Hyponormal operators and spectral density*, Trans. Amer. Math. Soc. **117** (1965), 469–476).

Concerning the sets $\mathrm{sp}(T)$ and $W(T)$, see the remarks on p. 482 of Berberian (*The numerical range of a numerical operator*, Duke Math. J. **31** (1964), 479–484) and the reference given there to Schreiber; also, as was called to the author's attention by the referee, the paper of Orland (*On a class of operators*, Proc. Amer. Math. Soc. **15** 1964), 75–79).

Relative to some of the results of §4, it can be noted that in the above mentioned paper of Stampfli, he has proved that if T is semi-normal and if $\mathrm{sp}(T)$ is a subset of a smooth simple closed curve, then T is normal. Thus the possibility suggested at the end of §4 that an isolated part of $\mathrm{sp}(T)$ (when $\mathfrak{H} = \Omega$) might be such a curve can now be ruled out.

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