

ON THE BEHAVIOR OF SOLUTIONS OF QUASI-LINEAR ELLIPTIC EQUATIONS⁽¹⁾

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Recently it has been noted that the minimal surface equation, and other related quasi-linear elliptic equations have the property that the boundary values of a solution on only part of its domain, may impose bounds upon the solution of all points of its domain. A result of this type first appeared in [6] as a consequence of the fact that the minimal surface equation possesses a solution; e.g., the minimal surface of Scherk; which becomes positively infinite on a straight boundary segment. Finn [4] has obtained stronger results of this type, by showing that if D is a domain, bounded in part by an arc Γ ; and if ϕ is a solution of an elliptic equation in D whose gradient becomes infinite as Γ is approached; then ϕ majorizes in D , any solution which it majorizes on $\partial D - \Gamma$. In particular, the catenoid $\phi_0(r) = -a \cos h^{-1} r/a$ is a solution of the minimal surface equation in $r > a$; and $\lim_{r \rightarrow a} \partial \phi_0 / \partial r = -\infty$, while $\lim_{r \rightarrow a} \phi_0(r)$ is finite. It follows therefore that $\phi_0(r)$ majorizes in $a < r < b$, any solution which is $\leq \phi_0(b)$ on $r = b$, and hence any such solution is uniformly bounded in $a < r < b$.

By applying the above argument to a solution defined in the punctured disc, and letting $a \rightarrow 0$, Finn obtained an elegant new proof of the removability of isolated singularities of solutions of the minimal surface equation. These results extend immediately to the class of radially symmetric variational problems in n -variables, whose radially symmetric solutions have the essential properties of the catenoid. This class was characterized by Finn in [4]. Extensions to a wide class of quasi-linear elliptic equations in 2 variables was given by the author, in reference [1], by constructing "catenoid-like" super-solutions.

Using the catenoid, or "catenoid-like" solutions or super-solutions as comparison functions the argument of Finn leads to the following theorem, which is valid for the class of equations possessing such solutions or super-solutions: *Let D be a domain lying exterior to a circle, and bounded in part, by an arc Γ of the circle. Then there is a uniform bound on ϕ in D_0 which depends only upon the supremum of ϕ on $\partial D - \Gamma$.*

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We remark that the uniformity of the bound results from the fact that the comparison function remains finite on the inner radius. Solutions, or super-solutions, which become infinite on the inner radius imply, by a classical argument, the existence of a pointwise bound which depends on the distance to the boundary, and the supremum of ϕ on $\partial D - \Gamma$. Results of this nature have been given by J. C. C. and J. A. Nitsche for a class of radially symmetric variational problems in two variables [7].

The theorem stated above includes the results which can be obtained by using comparison functions which become infinite on straight boundary segments, and is therefore stronger and more widely applicable than Theorem 1 of reference [1]. It is contained implicitly in the theorems of Finn in [4] and is used there in applications. Through an apparent oversight however, the general statement appearing there (Theorem 6) is somewhat weaker.

It is our purpose in this paper, to generalize some of the results of [1] and [4] to a class of quasi-linear equations in n -variables. This is accomplished by extending the method of constructing radially symmetric super-solutions which was developed in [1] for equations in 2-variables, to an analogous class in n -variables. It turns out, however, that the radially symmetric super-solutions so constructed, remain super-solutions when all but $j \geq 2$ of the variables are suppressed. The strongest results are then obtained by using the super-solutions arising by taking $j = 2$.

Let $x = (x_1, \dots, x_n)$; $p = (p_1, \dots, p_n)$; and let $A(x, u, p)$ denote a symmetric, positive definite $n \times n$ matrix function. Assume that the elements $A_{ij}(x, u, p)$ of A are defined and continuous for all values of the variables x, u, p . We shall consider the quasi-linear elliptic equation

$$(1) \quad L[\phi] = \sum A_{ij}(x, \phi, \nabla \phi) \phi_{x_i x_j} = 0.$$

We define super-solutions and sub-solutions as in [1]. Let $v(x)$ be a continuous function in a domain D . $v(x)$ is a super-solution for equation (1), in D , if for every closed bounded sub-domain $S \subset D$, and every solution ϕ , continuous in S ; $v \geq \phi$ on ∂S implies $v \geq \phi$ in the interior of S . If $v \leq \phi$ on ∂S implies $v \leq \phi$ on the interior, then v is a sub-solution.

Let $P(p)$ denote the matrix with elements $P_{ij} = \delta_{ij} + p_i p_j$, and note that if $v(x)$ is a C^1 function, $\sum_{i,j} P_{ij}(\nabla v) dx_i dx_j$ is the Euclidean metric on the surface $z = v(x)$. It follows from Lemma 3 that the eigenvalues of the product matrix PA are all real and positive. Let $d_i(x, u, P)$ denote the eigenvalues of PA indexed in order of decreasing magnitude, and put $\Delta(x, u, p) = d_1 / d_n$.

The minimal surface equation in n -variables is an equation of the form of equation (1), for which $A = P^{-1}$. Hence $\Delta = 1$ for the minimal surface equation. If Δ is uniformly bounded, then we shall say that equation (1) is of "minimal surface type." This definition generalizes the concept of minimal surface type;

which was originally introduced by Finn in reference [2], for equations in two independent variables. Equations of minimal surface type arise in a natural way, in the study of regular parametric variational problems (see [3, Theorem 2]).

We shall assume that Δ is a bounded function of x, u for each fixed p , and we introduce the notation

$$\Delta'(x, p) = \sup_u \Delta(x, u, p),$$

$$\bar{\Delta}(p) = \sup_{x, u} \Delta(x, u, p),$$

$$\Delta^*(\rho) = \max_{|p|=\rho} \Delta(p).$$

If F is a function of x, u, p and f is a C^1 function of x , we shall abbreviate $F(x, f, \nabla f)$ by $F[f]$; i.e., $\Delta(x, f, \nabla f) = \Delta[f]$ etc. Also, we shall abbreviate $\Delta^*(|\nabla f|)$ by $\Delta^*[f]$.

We shall not require that Δ is bounded, but we shall impose certain growth conditions upon Δ^* . Thus, our results will apply to a class of equations which includes the class of equations of minimal surface type.

1. Statement of results. In Theorems 1 and 2, we shall assume that D is a bounded open domain, whose boundary is the union of two disjoint sets Γ_1, Γ_2 . We shall assume that ϕ is a solution of equation (1) in D , and that

$$\liminf_{x \rightarrow \Gamma_1} \phi(x) = m > -\infty, \quad \limsup_{x \rightarrow \Gamma_1} \phi(x) = M < \infty.$$

We introduce the following notation. Let T_{n-k} denote an arbitrary $n-k$ plane, $2 \leq k \leq n$, and let $r_k(x)$ denote the distance from the point x to T_{n-k} . Thus, in an appropriate coordinate system $\bar{x}_1, \dots, \bar{x}_n$, T_{n-k} is the locus $\bar{x}_1 = \bar{x}_2 = \dots = \bar{x}_k = 0$, and $r_k(x) = \{\bar{x}_1^2 + \dots + \bar{x}_k^2\}^{1/2}$.

THEOREM 1. Assume that D lies exterior to a cylinder C of the form $r_2(x) = a$, and assume that Γ_2 lies on C . Then, if $\int^\infty \Delta^*(\rho)/\rho^3 d\rho$ is convergent, and $x \in D$, there is a bound on $|\phi(x)|$ which depends only upon $m, M, \Delta^*(\rho), a$; and the distance from x to C . If $\int^\infty \Delta^*(\rho)/\rho^2 d\rho$ is convergent, then $|\phi(x)|$ is uniformly bounded in D .

COROLLARY 1. Assume that $\int^\infty \Delta^*/\rho^3 d\rho$ is convergent, and that D lies in a half space defined by an $n-1$ plane T , and Γ_2 lies on T . Then there is a bound on $|\phi(x)|$ which depends only upon $m, M, \Delta^*(\rho)$ and the distance from x to T .

It should be noted that for a very general class of nonconvex domains, Theorem 1 imposes a limitation upon the boundary values for which the boundary value problem is solvable. In order to make a precise statement, it is convenient to introduce the concept of "inner boundary point." Let D be a domain, and let x_0 be a point of D . Suppose that there is a cylinder C of the form $r_2(x) = a$, with

the property that x_0 lies on C ; and such that there is a deleted neighborhood of x_0 whose intersection with the exterior of C (i.e., with the set $r_2(x) > a$) lies in D . Under these conditions, x_0 is said to be an inner boundary point of D . This definition generalizes the definition originally given by Finn [4] for domains in 2 dimensions.

COROLLARY 2. *Assume that $\int^\infty \Delta^*(\rho)/\rho^2 d\rho$ is convergent, and assume that D is a domain which possesses at least one inner boundary point. Then it is possible to prescribe boundary data on D in such a way that the boundary value problem is not solvable.*

THEOREM 2. *Assume that $\int^\infty \Delta^*(\rho)/\rho^3 d\rho$ is convergent. Assume also that Γ_2 consists of a finite number of disjoint point sets $\Gamma_2^1, \dots, \Gamma_2^m$ lying respectively in distinct $n-2$ planes $T_{n-2}^1, \dots, T_{n-2}^m$, and such that $\Gamma_2^i \cap T_{n-2}^j = \emptyset$ if $i \neq j$. If $n = 2$, T_0^i , and hence Γ_2^i are to be interpreted as points. The conclusion is that throughout D , $m \leq \phi \leq M$.*

We remark that Theorem 2 implies that a solution of the minimal surface equation in n -variables, is bounded in the neighborhood of a singularity lying on an $n-2$ plane.

It will become evident, in proving Theorems 1 and 2, that they remain true if $r_2(x)$ is replaced by $r_k(x)$. However, a moments reflection will indicate that the most general situation is obtained by putting $k = 2$.

THEOREM 3. *Let $\phi(x)$ be a solution of equation (1). Let $k_i[\phi]$, $i = 1, \dots, n$, denote the principal curvatures of the surface $z = \phi(x)$, indexed in order of decreasing numerical value. Then at points where the $k_i[\phi]$ do not vanish simultaneously, $k_1[\phi] > 0$, $k_n[\phi] < 0$, and*

$$(3) \quad \frac{1}{(n-1)\Delta[\phi]} \leq \frac{|k_n[\phi]|}{k_1[\phi]} \leq (n-1)\Delta[\phi].$$

Note that if equation (1) is of minimal surface type, then the ratio $|k_n|/k_1$ is uniformly bounded from zero and infinity. In the two variable case, this implies the known result that the spherical image mapping is quasi-conformal on solution surfaces of equations of minimal surface type.

2. Basic lemmas.

LEMMA 1. *Let $v(x)$ be a C^2 function in a domain D . Let $k_i[v]$, $i = 1, \dots, n$, denote the principal curvatures of the surface $z = v(x)$, indexed in order of decreasing numerical value. Assume that $k_1[v] > 0$, and that $k_n[v] < 0$. Let c be an arbitrary constant, h an arbitrary vector, and α a positive constant. Let $v' = v + c$, $\bar{v}(\bar{x}) = c + \alpha v(x)/\alpha$, where $\bar{x} = x + h$; and let $v^*(x^*) = c + \alpha v(x^*)/\alpha$, where $x^* = Rx + h$, and R is a rotation matrix. The conclusions are:*

(a) If $|k_n[v]|/k_1[v] > (n-1)\Delta[v]$ at each point of D , then $L[v] < 0$ in D . If $|k_n[v]|/k_1[v] < 1/((n-1)\Delta[v])$, then $L[v] > 0$ in D .

(b) If $|k_n[v]|/k_1[v] > (n-1)\Delta'[v]$, then v' is a super-solution in D . If $|k_n[v]|/k_1[v] < 1/((n-1)\Delta'[v])$, then v' is a sub-solution in D .

(c) If $|k_n[v]|/k_1[v] > (n-1)\bar{\Delta}[v]$, then \bar{v} is a super-solution. If

$$|k_n[v]|/k_1[v] < 1/((n-1)\bar{\Delta}[v]),$$

then \bar{v} is a sub-solution.

(d) If $|k_n[v]|/k_1[v] > (n-1)\Delta^*[v]$, then v^* is a super-solution. If

$$|k_n[v]|/k_1[v] < 1/((n-1)\Delta^*[v]),$$

then v^* is a sub-solution.

LEMMA 2 (FINN). Let D be a bounded domain in E^n whose boundary is the union of two closed sets Γ_α , Γ_β . Assume that each interior point of Γ_α is the end point of a line segment lying in D , and let s denote arc length along these segments. Let $M[\phi] = 0$, be an elliptic partial differential equation which possesses a super-solution $\phi_1(x)$ with the following properties: (i) $\phi_1(x)$ is continuously differentiable in the interior of D , and continuous on Γ_β . (ii) $\lim_{x \rightarrow P} \partial\phi_1/\partial s = -\infty$, where P is any point interior to Γ_α , and the limit is taken along the above mentioned line segment. (iii) If c is any constant, then $\phi_1 + c$ is again a super-solution. Let $\phi(x)$ be a solution of the equation $M[\phi] = 0$, with the following properties: (i) $\phi(x)$ is continuous in D . (ii) $\nabla\phi$ is continuous on Γ_α . (iii) $\liminf_{x \rightarrow \Gamma_\beta} (\phi_1 - \phi) \leq 0$. The conclusion is that $\phi \leq \phi_1$ everywhere in D . An analogous result holds if the equation has an appropriate sub-solution.

This lemma is a somewhat abridged version of a lemma given in [4] by Finn. A special case is stated and proved in [1]. The proof will not be given here.

Proof of Lemma 1. In order to prove Lemma 1, we shall need two additional lemmas.

LEMMA 3. Let $a = [a_{ij}]$, $b = [b_{ij}]$ be symmetric matrices, and consider the quadratic forms $Q = \sum a_{ij}u_iu_j$, $P = \sum b_{ij}u_iu_j$. Assume that Q is positive definite. Then the eigenvalues of the product matrix ba^{-1} are all real, and they are the stationary values of the ratio P/Q .

Proof. Put $\alpha = P/Q$, then the stationary values of α satisfy the equation (cf. Lemma 2 of [1])

$$(4) \quad \det(b - \alpha a) = 0.$$

It follows from standard results of matrix theory, that the eigenvalues of ba^{-1} are all real (cf. the simultaneous diagonalization theorem).

LEMMA 4. Let $v(x)$ be a C^2 function in a domain D , having the property that for every constant c , $L[v + c] < 0$. Then v is a super-solution in D . If for every c , $L[v + c] > 0$, then v is a sub-solution.

Proof. The proof proceeds by showing that if ϕ is any solution in D , then $v - \phi$ cannot have a minimum at an interior point of D . The proof will not be given here, since it does not differ significantly from the proof for equations in two independent variables. The reader is referred to [1], where the two variable case is proven.

Proof of Lemma 1. Let $I[v]$, $II[v]$ denote respectively, the first and second fundamental forms of the surface $z = v(x)$, i.e.,

$$(5) \quad \begin{aligned} I[v] &= \sum P_{ij}[v] dx_i dx_j, \\ II[v] &= (1/W) \sum v_{x_i x_j} dx_i dx_j, \end{aligned}$$

where $W = (1 + |\nabla v|^2)^{1/2}$. Define

$$(6) \quad I^*[v] = \sum A_{ij}^{-1}[v] dx_i dx_j.$$

The principal curvatures $k_i[v]$ of the surface $z = v(x)$ are then the stationary values of the ratio $II[v]/I[v]$. Denote the stationary values of the ratio $II[v]/I^*[v]$, indexed in order of decreasing magnitude, by $k_i^*[v]$, $i = 1, \dots, n$. Then, according to Lemma 3,

$$(7) \quad k_1^*[v] + \dots + k_n^*[v] = \text{trace } VA = (1/W)L[v],$$

where V denotes the matrix with elements $v_{x_i x_j}$. For simplicity, during the remainder of the proof of Lemma 1, we shall omit the argument v from the quantities $I[v]$, $II[v]$, $I^*[v]$, $k_i[v]$, $k_i^*[v]$, $\Delta[v]$, $\Delta^*[v]$; and denote them simply by I , k_i , Δ , etc. Since, by assumption $k_1 > 0$, and $k_n < 0$,

$$(8) \quad k_n^* = \min \frac{I^*}{I} = \min \left(\frac{II}{I} \cdot \frac{I}{I^*} \right) \leq k_n d_n.$$

Since k_i^* is a value of $(II/I) \cdot (I/I^*)$, it is clear that

$$(9) \quad k_i^* \leq k_1 d_1, \quad i = 1, \dots, n-1.$$

Similarly,

$$(10) \quad \begin{aligned} k_1^* &= \max \left(\frac{II}{I} \cdot \frac{I}{I^*} \right) \geq k_1 d_n, \\ k_i^* &\geq k_n d_1, \quad i = 2, \dots, n. \end{aligned}$$

Hence,

$$(11) \quad k_1 d_1 \left\{ \frac{1}{\Delta} - (n-1) \frac{|k_n|}{k_1} \right\} \leq k_1^* + \dots + k_n^* \leq k_1 d_n \left\{ (n-1) \Delta - \frac{|k_n|}{k_1} \right\}.$$

Thus, using (7), if $|k_n|/k_1 > (n-1)\Delta$, then $L[v] < 0$, and if $|k_n|/k_1 < 1/(n-1)\Delta$, then $L[v] > 0$. This proves part (a). Parts (b), (c), and (d) follow from Lemma 4,

and the fact that the ratio of any two of the principal curvatures of a surface is invariant under rigid motions and similarity transformations. We shall give the proof of part (d). Note that the surface $z = v^*(x^*)$ is obtained from the surface $z = v(x)$ by a rigid motion, followed by a similarity transformation. Therefore

$$(12) \quad \frac{k_n[v^*(x^*)]}{k_1[v^*(x^*)]} = \frac{k_n[v(x)]}{k_1[v(x)]}.$$

Since $|\nabla v(x)| = |\nabla v^*(x^*)|$, it is clear that $\Delta^*[v(x)] \geq \Delta[v^*(x^*)]$. Thus, if it is assumed that $|k_n|/k_1 > (n-1)\Delta^*$, then

$$(13) \quad \frac{|k_n[v^*]|}{k_1[v^*]} > \Delta[v^*].$$

It is obvious that (13) continues to hold if v^* is replaced with $v^* + c$. Under the above assumption on v , Lemma 4 now implies that v^* is a super-solution. In like fashion, if $|k_n|/k_1 < 1/(n-1)\Delta^*$, then

$$|k_n[v^* + c]|/k_1[v^* + c] < 1/(n-1)\Delta[v^* + c],$$

and v^* is a sub-solution. This proves part (d); parts (b) and (c) may be proved in a similar manner.

We remark that since $k_1[-v] = -k_n[v]$, it follows that $k_n[-v]/k_1[-v] = k_1[v]/k_n[v]$. Hence if v is a super-solution satisfying one of the various criteria of Lemma 1, then $-v$ is a sub-solution satisfying the analogous criteria.

LEMMA 5. Assume that $\int^\infty \Delta^*(\rho)/\rho^3 d\rho$ converges, and put

$$(14) \quad f(\mu) = \int_\mu^\infty \frac{n\Delta^*(\rho)}{\rho(1+\rho^2)} d\rho.$$

Let $f^{-1}(v)$ denote the inverse of the function $f(\mu)$, and let $a < b$. Define

$$(15) \quad U(s; a, b) = \int_s^b f^{-1}(\log(t/a)) dt.$$

Then $U_k(x; a, b) = U(r_k(x); a, b)$ is a super-solution, and $-U_k(x; a, b)$ is a sub-solution for equation (1) in the domain $r_k(x) > a$, $k = 2, 3, \dots, n$.

Proof. It suffices to show that the function $U_k(x; a, b)$ satisfies the hypothesis of part (d) of Lemma 1, in the special case where $r_k(x) = (x_1^2 + \dots + x_k^2)^{1/2}$. The general case then follows, since for such functions the property of being a super-solution is invariant under rigid motions. The proof proceeds by showing that the principal curvatures of a surface of the form $z = v(r_k(x))$ are: $v'/(r_k W)$ with multiplicity $k-1$, v''/W^3 with multiplicity 1, and 0 with multiplicity $n-k$. As before, $W = (1 + |\nabla v|^2)^{1/2}$. It is easily verified that $U_s < 0$, and $U_{ss} > 0$. Hence for the surface $z = U_k(x; a, b)$,

$$(16) \quad \frac{|k_n[U_k]|}{k_1[U_k]} = \frac{|U_s(r_k)|(1 + |U_s|^2)}{r_k U_{ss}} = n\Delta^*(|\nabla U_k|) > (n-1)\Delta^*(|\nabla U_k|).$$

Therefore, by Lemma 1, U_k is a super-solution in the domain $r_k(x) > a$. An analogous procedure shows that $-U_k$ is a sub-solution. (Cf. also, the remarks following the proof of Lemma 1.) It remains to determine the principal curvatures of a surface of the form $z = v(r_k(x))$. The principal curvatures are the stationary values of the ratio $\Pi[v]/I[v]$, hence they are the eigenvalues of the matrix $(1/W)VP^{-1}[v]$, where as before V is the matrix with elements $V_{ij} = v_{x_i x_j}$, and $P[v]$ is the matrix with elements $P_{ij}[v] = \delta_{ij} + v_{x_i x_j}$. Let $\sigma(r_k) = v'/r_k$, $\tau(r_k) = (1/r_k)(v'/r_k)'$; and let Φ_k denote the matrix with elements $\Phi_{k,ij}$, where $\Phi_{k,ij} = x_i x_j$ if $i, j \leq k$, and $\Phi_{k,ij} = 0$ otherwise. Also, let I_k denote the matrix with 1's in the first k places along the main diagonal, and 0's elsewhere. Noting that $\Phi_k^2 = r_k^2 \Phi_k$, one may easily verify that $P^{-1}[v] = I - (\sigma^2/W^3)\Phi_k$. A calculation then yields

$$(17) \quad \left(\frac{1}{W}\right)VP^{-1} = \left(\frac{\sigma}{W}\right)I_k + \left(\frac{\tau - \sigma^3}{W^3}\right)\Phi_k.$$

Now, the rank of a matrix of the form $aI_k + b\Phi_k$ is at most k . Hence 0 is an eigenvalue of multiplicity at least $n - k$. The rank of Φ_k is 1, and its trace is r_k^2 . It follows that the other eigenvalues of $aI_k + b\Phi_k$ are: a , with multiplicity $k - 1$, and $a + br_k^2$ with multiplicity 1. The desired result now follows from (17).

The function $U(s; a, b)$ has the following useful properties: (i) $U(s; a, b)$ is monotonically decreasing in s , and $U(b; a, b) = 0$. (ii) $\lim_{s \rightarrow a} \partial U / \partial s = -\infty$. (iii) For any fixed s , $\lim_{a \rightarrow 0} U(s; a, b) = 0$. (iv) $\lim_{s \rightarrow a} U(s; a, b)$ is finite if and only if $\int^\infty \Delta^*(\rho) / \rho^2 d\rho$ is convergent. These properties are easy consequences of the definition. They are derived in detail in [1] for a function $U(r; r_1, r_2)$, which is essentially the same as the function $U(x; a, b)$ defined here. In case $\int^\infty \Delta^*(\rho) / \rho^2 d\rho$ is convergent, we shall put $\lim_{s \rightarrow a} U(s; a, b) = U(a, b)$.

Note that property (ii) implies that the partial derivative of $U_k(x; a, b)$ in the direction of increasing r_k , is negatively infinite on the surface $r_k(x) = a$. Thus, U_k will satisfy the hypotheses of Lemma 2 in domains for which Γ_a lies on the surface $r_k(x) = a$.

Proof of the theorems.

Proof of Theorem 1. Choose b sufficiently large so that D lies in the region defined by $a < r_2(x) < b$. Let A_ε denote the region $a + \varepsilon < r_2(x) < b$, and let C_ε denote the cylinder $r_2(x) = a + \varepsilon$. According to Lemma 5, then, $U_2(x; a + \varepsilon, b)$ is a super-solution and $-U_2(x; a + \varepsilon, b)$ is a sub-solution in $D \cap A_\varepsilon$. We now apply Lemma 2 in $D \cap A_\varepsilon$ with $\Gamma_a = D \cap C_\varepsilon$, $\Gamma_b = \partial(D \cap A_\varepsilon) - \Gamma_a$. Letting $\varepsilon \rightarrow 0$ we have in D ,

$$m - U_2(x; a, b) \leq \phi \leq M + U_2(x; a, b).$$

If $\int^\infty \Delta^*(\rho) / \rho^2 d\rho$ is convergent, then $U_2(x; a, b) \leq U(a, b)$ and

$$m - U(a, b) \leq \phi \leq M + U(a, b)$$

holds throughout D .

Proof of Corollary 1. Let x_0 be any point of D . Then for sufficiently large a , there is a cylinder C of the form $r_2(x) = a$ such that X_0 lies exterior to C and Γ_2 lies interior to C . The result now follows by applying Theorem 1 in the intersection of D with the exterior of C .

Proof of Corollary 2. If x_0 is an inner boundary point of D , then by hypothesis, there is a neighborhood $N(x_0)$ of x_0 on ∂D , and a cylinder $r_2(x) = a$, such that x_0 lies on the cylinder, and all points of $N(x_0)$ lie either on, or in the interior of the cylinder. Choose b so that $b - a$ exceeds the diameter of D . Let $f(x)$ be a function defined on ∂D with the properties: (i) $f(x)$ is continuous on ∂D , (ii) $f(x) \equiv 0$ on $\partial D - N(x_0)$, (iii) $f(x_0) > U(a, b)$.

However, the argument used in the proof of Theorem 2 implies that any solution which has zero boundary values on $\partial D - N(x_0)$, cannot exceed $U(a, b)$ at x_0 . Thus, there can be no solution which assumes the values $f(x)$ on ∂D .

Proof of Theorem 2. Let $r_2^i(x)$ denote the distance from x to the $n - 2$ plane T_{n-2}^i , $i = 1, 2, \dots, j$; and let b_i be small enough so that the cylinder C_i , defined by $r_2^i(x) = b_i$ does not contain any points of Γ_2^j , if $i \neq j$. Let G_ε^i denote the domain $\varepsilon < r_2^i(x) < b_i$, and as in the proof of Theorem 1, apply Lemma 2 in the domain $D \cap G_\varepsilon^i$. Let $M_i' = \max_{x \in C_i \cap \bar{D}} \phi(x)$, and let $M_i^* = \max\{M_i', M\}$. It follows from Lemma 2, that in $D \cap G_\varepsilon^i$, $\phi(x) \leq M_i^* + U_2(x; \varepsilon, b_i)$. Since $\lim_{\varepsilon \rightarrow 0} U_2(x; \varepsilon, b_i) = 0$, we let $\varepsilon \rightarrow 0$, and obtain the inequality $\phi(x) \leq M_i^*$ in $D \cap G^i$, where G^i denotes the domain $r_2^i(x) \leq b_i$. Now, it is well known, that a nonconstant solution of an elliptic equation of the form of equation (1), cannot attain its maximum at an interior point. It follows therefore, that $\phi(x) \leq \max_i M_i^*$. Suppose $M_j^* = \max_i M_i^* > M$. Then $M_j^* = M_j'$, and ϕ must attain the value M_j^* on $C_j \cap D$, i.e., at an interior point of D . This is impossible, hence $M_j^* \leq M$. An analogous argument shows that $\phi(x) \geq m$ in D .

Proof of Theorem 3. Since $L[\phi] = W(k_1^*[\phi] + \dots + k_n^*[\phi]) = 0$, it is clear that either $k_i^*[\phi] = 0$ for all i , or else $k_1^*[\phi] > 0$, and $k_n^*[\phi] < 0$. Since the k_i^* are the stationary values of II/I^* , it follows that either II/I^* vanishes for all directions, at a point x , or it takes both positive and negative values. Since the principal curvatures $k_i[\phi]$ are the stationary values of $\text{II}/\text{II} = (\text{II}/\text{I}^*)(\text{I}^*/\text{I})$ and $\text{I}^*/\text{I} > 0$, it follows that the same is true of the ratio II/I . Thus at a given point, either all of the principal curvatures vanish, or else $k_1[\phi] > 0$, and $k_n[\phi] < 0$. The result now follows from part (a) of Lemma 1.

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