

# HOMOTOPY PROPERTIES OF THE SPACE OF HOMEOMORPHISMS ON $P^2$ AND THE KLEIN BOTTLE<sup>(1)</sup>

BY

MARY-ELIZABETH HAMSTROM

1. **Introduction.** This is another in a series of papers dealing with the computation of the homotopy groups of the space of homeomorphisms on a 2-manifold,  $M$ , (the homeotopy groups of McCarty [12]). Denote by  $H(M, K)$  the space of homeomorphisms on  $M$  leaving  $K$  pointwise fixed and by  $H_0(M, K)$  its identity component. Let  $\dot{M}$  denote the boundary of the manifold  $M$ . Then if  $M$  is a disc with holes,  $H_0(M, \dot{M})$  is homotopically trivial ([1] and [4]) and if  $M$  is a torus the homotopy groups of  $H_0(M)$  are isomorphic to those of  $M$  [6]. Quintas [14] related  $H(M)$  to  $H(M, x)$ , where  $M$  is a 2-manifold and  $x$  is a point of  $M$  and McCarty [11] related  $H(M, x)$  to  $H(M - x)$ . He also introduced some fiber space techniques to the study of these problems. Lickorish studied some of these problems also in his recent interesting papers [9] and [10]. Local homotopy properties are considered by McCarty in [11] and Dyer and myself in [3]. In the present paper the Klein bottle, Moebius strip and  $P^2$  are considered. In a paper under preparation I prove that  $H_0(M)$  has trivial homotopy groups if  $M$  is orientable and has two or more handles, or is nonorientable and has three or more cross caps.

2. **The Moebius strip.** Denote the Moebius strip by  $M$  in the rest of this paper. The space  $H(M)$  has two components. To see that there are at least two, let  $M$  be obtained from  $I \times I$  by identifying appropriate points of  $I \times 0$  and  $I \times 1$  and let a homeomorphism  $f$  of  $M$  be obtained by reflecting  $I \times I$  on  $I \times 1/2$  before the identification is made. Then  $f$  reverses orientation on  $\dot{M}$  and is thus not isotopic to the identity. The Moebius strip may also be obtained from  $I \times S^1$  by identifying diametrically opposite points of  $0 \times S^1$  and  $f$  may be obtained by first reflecting  $I \times S^1$  on a diameter and then making the identification.

**THEOREM 2.1.** *The space  $H(M, \dot{M})$  is homotopically trivial.*

**Proof.** First,  $H(M, \dot{M})$  is arcwise connected. To see this, let  $I_x$  denote the image of  $I \times x$  under the first identification map described above and suppose

---

Presented to the Society, April 20, 1964 under the title *Homotopy in the space of homeomorphisms on a 2-manifold*; received by the editors June 10, 1964 and, in revised form, March 10, 1965.

(<sup>1</sup>) Research supported, in part, by the National Science Foundation.

that  $g$  is an element of  $H(M, \dot{M})$ . An isotopy leaving the endpoints fixed can easily be constructed that carries  $g(I_0)$  back to an arc  $\alpha$  that is either  $I_0$  or has the properties that it intersects  $I_0$  only a finite number of times, it crosses  $I_0$  at each point of intersection except the endpoints, and at the endpoints of the closure  $k$  of each component of  $\alpha - I_0$ ,  $k$  abuts on  $I_0$  from opposite sides. It is fairly easily seen that this second case cannot occur so that there is an isotopy  $h_t$  of  $I_0$  in  $M$  such that  $h_0 = g$ ,  $h_1 = i$  and  $h_t|_{\dot{I}_0} = i$ .

In  $I \times M$ , let  $G$  be a homeomorphism of  $((0 \cup 1) \times M) \cup (I \times (\dot{M} \cup I_0))$  into  $I \times M$  such that  $G(t, x) = (t, h_t(x))$  for  $x$  in  $I_0$ ,  $G(t, x) = (t, x)$  for  $x$  in  $\dot{M}$ ,  $G(1, x) = (1, x)$  for each  $x$  and  $G(0, x) = (0, g(x))$ . The space of homeomorphism of  $M$  onto itself leaving  $\dot{M} \cup I_0$  pointwise fixed is clearly homeomorphic to the space of homeomorphisms of a disc into itself leaving its boundary pointwise fixed. Therefore, it follows from [2] that  $G$  may be extended to a homeomorphism  $G^*$  of  $I \times M$  onto itself leaving  $(t, M)$  invariant for each  $t$ . Let  $g_t$  denote the homeomorphism of  $M$  onto itself taking  $x$  onto  $y$ , where  $G^*(t, x) = (t, y)$ . This yields the required arc from  $g$  to  $i$  in  $H(M, \dot{M})$ .

Now suppose  $f$  is a mapping of  $S^k$  into  $H(M, \dot{M})$ . Consider  $I \times E^1$  and the standard covering map  $\pi$  taking it onto  $M$ ,  $\pi(I \times n)$  being  $I_0$ , where  $n$  is an integer and, in general,  $\pi(I \times x')$  being  $I_x$ , where  $x' \equiv x \pmod{1}$ . Let  $P_0$  and  $P_1$  denote the endpoints of  $I_0$  and  $Q_0$  and  $Q_1$  the points of  $I \times 0$  going into  $P_0$  and  $P_1$  under  $\pi$ . Let  $F$  be the mapping of  $I_0 \times S^k$  into  $M$  defined by  $F(x, y) = f(y)(x)$ . Then  $F(P_0, y) = P_0$  and  $F(P_1, y) = P_1$  for each point  $y$  of  $S^k$ . The space  $P_0 \times S^k$  is a strong deformation retract of  $I_0 \times S^k$ . Standard lifting theorems [7, p. 63] yield a map  $G$  of  $I_0 \times S^k$  into  $I \times E^1$  such that  $G(P_0 \times S^k) = Q_0$  and  $\pi G(x, y) = F(x, y)$  for each  $(x, y)$ . Then  $G|(I_0 \times y)$  is the unique lifting of  $F|_{I_0 \times y}$  to a map of  $I_0 \times y$  into  $I \times E^1$  taking  $(P_0, y)$  onto  $Q_0$ . This is a homeomorphism. Also, since  $F|(I_0 \times y)$  is isotopic to a mapping taking  $(x, y)$  onto  $x$  under an isotopy leaving  $F((P_0 \cup P_1), y)$  unchanged, lifting theorems yield the fact that  $G(P_1 \times S^k) = Q_1$ .

A proof almost identical to that of a similar theorem for tori [6] now yields a mapping  $z$  of  $I_0 \times S^k \times I$  into  $M$  such that  $z(x, y, 0) = F(x, y)$  and  $z(x, y, 1) = x$ . Let  $\alpha$  be the homeomorphism of  $[M \times S^k \times (0 \cup 1)] \cup [(I_0 \cup \dot{M}) \times S^k \times I]$  into  $M \times S^k \times I$  such that  $\alpha(x, y, t) = (x, y, t)$  for  $x \in \dot{M}$ ,  $\alpha(x, y, t) = (z(x, y, t), y, t)$  for  $x \in I_0$ ,  $\alpha(x, y, 0) = (f(y)(x), y, 0)$ , and  $\alpha(x, y, 1) = (x, y, 1)$ . Since the space of homeomorphisms of the disc onto itself leaving its boundary pointwise fixed is contractible, it follows from [2] that  $\alpha$  can be extended to a homeomorphism  $\alpha^*$  of  $M \times S^k \times I$  onto itself that leaves each  $(M, y, t)$  invariant. Then if  $Z_t$  is the map of  $S^k$  into  $H(M, \dot{M})$  such that  $Z_t(y)(x) = x'$ , where  $\alpha^*(x, y, t) = (x', y, t)$ ,  $Z_t$  is the required homotopy taking  $f$  into a constant mapping.

**THEOREM 2.2.** For  $k > 1$ ,  $\pi_k(H_0(M)) = 0$ ,  $\pi_1(H_0(M)) = Z$ , and  $H(M)$  has exactly two components.

**Proof.** Let  $p$  denote the map of  $H(M)$  into  $H(\dot{M})$  such that  $p(f) = f|_{\dot{M}}$  for each  $f \in H(M)$ . Then  $H(M)$  is a fiber space with projection  $p$ , base space  $H(\dot{M})$  and fiber  $H(M, \dot{M})$  (in the sense of Hu [7, p. 62]). To see this, let  $g$  be a mapping of the  $k$ -cell  $R^k$ ,  $k \geq 0$ , into  $H(M)$  and  $G_t$  a homotopy in  $H(\dot{M})$  such that  $G_0(x) = pg(x)$  for each  $x$ . Let  $\alpha$  be a homeomorphism of  $(\dot{M} \times R^k \times I) \cup (M \times R^k \times 0)$  into  $(M \times R^k \times I)$  such that  $\alpha(y, x, t) = (G_t(x)(y), x, t)$  for  $y$  in  $\dot{M}$  and  $\alpha(y, x, 0) = (g(x)(y), x, 0)$ . As in the proof of Theorem 2.1  $\alpha$  may be extended to a homeomorphism  $\alpha^*$  of  $M \times R^k \times I$  onto itself and a homotopy  $g_t$  on  $H(M)$  is constructed such that for each  $t$ ,  $g_t(x)|_{\dot{M}} = G_t(x)|_{\dot{M}}$  for each  $x$  in  $R^k$ . Thus  $pg_t(x) = G_t(x)$  and  $H(M)$  is a fiber space. The homotopy sequence of this fibering now yields the first part of the theorem (Hu [7, p. 152]), since  $\pi_i H(M, \dot{M}) = 0$  for  $i \geq 1$ .

If  $f$  is a map in  $H(M)$ , then  $p(f)$  lies in one of the two homeomorphic components of  $H(\dot{M})$ . If  $p(f)$  and  $p(f')$  lie in the same component of  $H(\dot{M})$ , there is an arc from  $p(f)$  to  $p(f')$  in that component, and thus there is an arc in  $H(M)$  from  $f$  to a map  $f''$  such that  $p(f'') = p(f')$ . Since  $H_0(M, \dot{M})$  is arcwise connected, there is an arc from  $f$  to  $f'$  in  $H(M)$ . This proves that there are at most two components of  $H(M)$ . That there are at least two components was proved in the opening paragraph of this section.

### 3. The projective plane.

**THEOREM 3.1.** *The space  $H(P^2)$  has only one component,  $H(P^2, p)$  has two components,  $\pi_i H(P^2, p) = 0$  for  $i > 1$ , and  $\pi_1 H(P^2, p) = Z$ .*

**Proof.** That  $H(P^2)$  has only one component was observed by Lickorish in [10], but it follows readily from the following argument. Let  $D$  be a disc (consider it circular) in  $P^2$  with center  $p$  and  $M$  the Moebius strip in  $P^2$  with boundary  $\dot{D}$ . Let  $f$  be an element of  $H(P^2, p)$  that takes  $\dot{D}$  onto itself, but reverses orientation. If  $f$  were isotopic to  $i$  under an isotopy leaving  $p$  fixed,  $f|_{\dot{D}}$  would be homotopic to the identity in  $P^2 - p$ , which is impossible. Thus  $H(P^2, p)$  has at least two components.

Let  $f$  be an element of  $H(P^2, p)$ ,  $E$  a disc with center  $p$  lying in  $(\text{int } D) \cap (\text{int } f(D))$  and  $g$  a homeomorphism in  $H(P^2, p)$  such that  $g|_{\dot{D}} = f|_{\dot{D}}$  and  $g|_E$  is either the identity or a reflection on a diameter. The map  $g^{-1}f = i$  on  $\dot{D}$ . Thus there is an isotopy  $F_t$  such that  $F_t|_{\dot{D} \cup p} = i$ ,  $F_1 = g^{-1}f$  and  $F_0 = i$ . The map  $g|_E$  is either the identity or a reflection. There is thus an isotopy  $G_t$  such that  $G_t|_E = g|_E$ ,  $G_1 = g$  and  $G_0$  is either the identity or is, on  $P^2 - E$ , the reflection described in §2. Then  $Z_t = G_t F_t$  is an isotopy such that  $Z_1 = f$  and  $Z_0$  is the identity or the "reflection" described in the foregoing paragraph. Thus  $H(P^2, p)$  has two components. This reflection is isotopic to the identity in  $H(P^2)$ , as can be seen by pushing  $p$  around an orientation reversing simple closed curve so that the reflection is isotopic to a map  $f$  that leaves  $p$  fixed, but gives rise to



opposite points, as is  $f'\phi^{-1}(z)$ . Note that  $\phi^*$  takes fibers into fibers and  $p\phi^* = \phi p'$ . For each  $i$ ,  $\phi^*$  and  $\phi$  induce homomorphisms of  $\pi_i H^0(S^2)$ ,  $\pi_i H^0(S^2, x')$ ,  $\pi_i(S^2)$  onto  $\pi_i H(P^2)$ ,  $\pi_i H(P^2, x)$ ,  $\pi_i(P^2)$ , and in particular, the following diagram

$$\begin{array}{ccccccccccc} \cdots & \rightarrow & \pi_2 H^0(S^2) & \xrightarrow{p'_2} & \pi_2(S^2, x') & \xrightarrow{d'_2} & \pi_1 H^0(S^2, x') & \xrightarrow{i'_1} & \pi_1 H^0(S^2) & \xrightarrow{p'_1} & \pi_1(S^2, x') & \rightarrow \cdots \\ & & \downarrow \phi_2^* & & \downarrow \phi_2 & & \downarrow \phi_1^{**} & & \downarrow \phi_1^* & & & \\ \cdots & \rightarrow & \pi_2 H(P^2, x) & \xrightarrow{i_2} & \pi_2 H(P^2) & \xrightarrow{p_2} & \pi_2(P^2, x) & \xrightarrow{d_2} & \pi_1 H(P^2, x) & \xrightarrow{i_1} & \pi_1 H(P^2) & \rightarrow \cdots \end{array}$$

where  $\phi_2^*, \phi_2$ , etc., are the induced homomorphisms, is commutative. It is observed in [3] and [12] that  $\pi_1 H^0(S^2) = Z_2$  and  $\pi_2 H^0(S^2) = 0$ . Also,  $\phi_2$  is an isomorphism,  $\pi_1 H^0(S^2, x') = Z$ , and the diagram looks as follows

$$\begin{array}{ccccccccccc} \cdots & \rightarrow & 0 & \xrightarrow{p'_2} & Z & \xrightarrow{d'_2} & Z & \xrightarrow{i'_1} & Z_2 & \xrightarrow{p'_1} & 0 & \rightarrow \cdots \\ & & \downarrow \phi_2^* & & \downarrow \phi_2 & & \downarrow \phi_1^{**} & & \downarrow \phi_1^* & & & \\ \cdots & \rightarrow & 0 & \xrightarrow{i_2} & \pi_2 H(P^2) & \xrightarrow{p_2} & Z & \xrightarrow{d_2} & Z & \xrightarrow{i_1} & \pi_1 H(P^2) & \rightarrow \cdots \end{array}$$

The image of  $Z$  under  $i'_1$  is  $Z_2$ . Hence the kernel of  $i'_1$  is  $2Z$ . Then  $d'_2$  is a 1-1 map of  $Z$  onto  $2Z$ . Since  $\phi_2$  is an isomorphism and  $d_2 \phi_2 = \phi_1^{**} d'_2$ ,  $d_2(Z) = 2Z$ . (It is an easy exercise to prove that  $\phi_1^{**}$  is also an isomorphism.) Therefore the kernel of  $i_1$  is  $2Z$  and, since  $i_1$  is onto,  $\pi_1 H(P^2) = Z_2$ . Since  $d_2$  is 1-1,  $i_2$  is onto. Hence  $\pi_2 H(P^2) = 0$ .

REMARK. The facts that  $\phi_i^{**}$  is an isomorphism for  $i \geq 1$  and  $\phi_i$  is an isomorphism for  $i \geq 2$  yield, by the five lemma,  $\phi_i^*$  is an isomorphism of  $\pi_i H^0(S^2)$  onto  $\pi_i H(P^2)$  for  $i \geq 1$ . But  $H^0(S^2)$  is homeomorphic to  $P^3$ . Therefore  $\pi_i H(P^2) = \pi_i(P^3)$  for all  $i$ . Also  $\pi_i H(P^2) = \pi_i(S^3)$  for  $i > 1$ , and for  $i > 2$ ,  $\pi_i H(P^2) = \pi_i(S^2) = \pi_i(P^2)$ . (See the opening remarks of [3].)

4. **The Klein bottle.** The Klein bottle has two representations,  $A$  and  $B$ . Let  $A$  be the representation obtained from the annulus  $S^1 \times I$  by identifying diametrically opposite points of  $S^1 \times 0$  and  $S^1 \times 1$ . Let  $C$  be the curve  $S^1 \times 1/2$ . Then  $K$  is the union of two Moebius strips  $M_0$  and  $M_1$ , sewn together along their common boundary,  $C$ . Their center curves,  $C_0$  and  $C_1$  are obtained from the identification of points of  $S^1 \times 0$  and  $S^1 \times 1$ . Coordinatize  $S^1$  by the reals mod 1 and let  $C'$  be the closed curve in  $K$  obtained from  $(0 \times I) \cup (1/2 \times I)$  by means of the identification. Lickorish [10] has shown that  $\pm C, C'$ , and a curve bounding a disc represent the four isotopy classes of orientation preserving simple closed curves on  $K$ . The four classes of orientation reversing curves are represented by  $\pm C_0$  and  $\pm C_1$ .

Let  $B$  be the representation obtained from  $S^1 \times I$  by identifying  $(x, 0)$  and

$(-x, 1)$ . In this representation,  $C'$  is represented by  $S^1 \times 1/2$ ,  $C$  by  $(1/4 \times I) \cup (3/4 \times I)$ ,  $C_0$  by  $0 \times I$  and  $C_1$  by  $1/2 \times I$ .

The fundamental group,  $\pi_1(K)$  is generated by the classes of the homeomorphisms  $\alpha_0$  and  $\alpha_1$  of  $S^1$  into  $C_0$  and  $C_1$  (assume base point to be on  $C_0$ ) and  $\pi_1(K)$  is isomorphic to the free group generated by  $[\alpha_0]$  and  $[\alpha_1]$  with the sole relation  $[\alpha_0]^2[\alpha_1]^2 = 1$ . The group  $\Lambda(K)$ , the quotient group of  $H(K) \text{ mod } H_0(K)$  is, as Lickorish points out, isomorphic to  $Z_2 \times Z_2$ . Thus  $H(K)$  has four components. These (see [10]) are generated by the identity homeomorphism,  $i$ , the homeomorphism  $f$  obtained from representation  $B$  by reflecting  $S^1 \times I$  on  $S^1 \times 1/2$  and then identifying, the homeomorphism  $h$  obtained by cutting  $K$  along  $C'$ , twisting through  $360^\circ$  and sewing back together, and  $hf$ .

**THEOREM 4.1.** *The space  $H(K, x)$  ( $x \in C'$ ) has infinitely many components,  $H(K)$  has four components,  $\pi_i H(K, p) = 0$  for each  $i \geq 1$ ,  $\pi_i H(K) = 0$  for  $i \geq 2$  and  $\pi_1 H(K) = Z$ .*

**Proof.** A covering space for  $K$  is  $S^1 \times E^1$ . The covering map  $\pi$  is such that  $\pi(S^1, t) = \pi(S^1, t')$  if  $t \equiv t' \pmod{1}$ . The proof that  $H(K, x)$  has trivial homotopy groups is the same as the proof in [6] that  $H(T^2, p)$  has trivial homotopy groups. Let  $H^0(K, x)$  denote the subset of  $H(K, x)$  consisting of the elements that are in  $H_0(K)$ . Then  $g$  is in  $H^0(K, x)$ , where  $g$  is the homeomorphism obtained either from representation  $B$  by first reflecting on the diameter containing  $(0 \cup 1/2) \times I$  and then identifying or from representation  $A$  by first rotating through  $\pi$  radians and then identifying. From the last form of  $g$ , it can be easily seen that  $g$  is in  $H_0(K)$ . Let  $Z(t)$  be the homeomorphism of  $K$  obtained by rotating  $S^1 \times I$  through  $\pi t$  radians and then identifying. Then  $Z(1) = g$  and  $Z(0) = i$ . Also,  $k$  is in  $H^0(K, x)$ , where  $k$  is obtained from representation  $B$  by pushing  $x$  clockwise around  $C'$  through  $360^\circ$  while leaving  $(S^1 \times 0) \cup (S^1 \times 1)$  pointwise fixed and then identifying. The group structure of  $H^0(K, x)$  appears below.

It again follows from [12] that  $H(K)$  is a fiber space with fiber  $H(K, x)$  and base space  $K$ . The lower part of the exact homotopy sequence for this fiber space is

$$\dots \rightarrow \pi_1 H(K, x) \xrightarrow{i_1} \pi_1 H(K) \xrightarrow{p_1} \pi_1(K) \xrightarrow{d_1} \pi_0 H(K, x) \xrightarrow{i_0} \pi_0 H(K).$$

Since  $\pi_i(K) = 0$  for  $i \geq 2$ , it follows that  $\pi_i H(K) = \pi_i H(K, x) = 0$  for  $i \geq 2$ . Since  $\pi_1 H(K, x) = 0$ ,  $p_1$  is 1-1. The kernel of  $i_0$  is  $H^0(K, x)$ , so this is the image of  $d_1$ . Since  $\pi_1 H(K)$  is isomorphic to the kernel of  $d_1$ , knowledge of this kernel yields the structure of  $\pi_1 H(K)$ . A study of the meaning of  $d_1$  shows that, with the base point for  $\pi_1(K)$  on  $C_0$ ,  $d_1[\alpha_0] = [g]$ , the isotopy class in  $H(K, x)$  of  $g$ ,  $d_1[\alpha_1] = [kg]$ , but  $d_1[\alpha_0]^2 = d_1[\alpha_1]^2 = [i]$ . Since  $[gkg] = [k^{-1}]$ , it follows readily that the infinite cyclic subgroup of  $\pi_1(K)$  generated by  $[\alpha_0]^2 (= [-\alpha_1]^2)$  is the kernel of  $d_1$ . Thus  $\pi_1 H(K) = Z$  and it follows that  $\pi_0 H^0(K, x)$  is isomorphic to the free group on two generators  $a, b$  with the relation  $a^2 = b^2 = 1$ .

5. **Appendix on generators and Lie groups.** If the Moebius strip  $M$  is considered as being obtained from the annulus  $S^1 \times I$  by the identification of diametrically opposite points of  $S^1 \times 0$ , the "rotations" of  $M$  obtained by first rotating  $S^1 \times I$  and then identifying form a subgroup  $L_1(M)$  of  $H_0(M)$  whose fundamental group is  $Z$ . The arguments in §2 indicate that the injection of  $L_1(M)$  into  $H_0(M)$  induces an isomorphism of the fundamental groups.

In [8], Kneser proved that the space of rotations of  $S^2$ ,  $SO(3)$  (the identity component of my  $H^0(S^2)$ ), is a deformation retract of  $H_0(S^2)$ . This implies the results mentioned in the remark at the end of §3. The mapping  $\phi^*$  is a homeomorphism of the identity component of the space of antipodal homeomorphisms of  $S^2$  onto  $H(P^2)$  and  $\phi^*|SO(3)$  takes  $SO(3)$  onto a group,  $L(P^2)$ . Since  $\phi_n^*$  is an isomorphism of  $\pi_n H^0(S^2)$  onto  $\pi_n H(P^2)$ , it follows that the injection map of  $L(P^2)$  into  $H(P^2)$  induces isomorphisms of the homotopy groups.

It is also true that  $H^0(S^2, x')$  is a deformation retract of  $H(S^2, x')$ . The identity component,  $H_0^0(S^2, x')$  is simply the group of rotations about the diameter through  $x'$ . Let  $L(P^2, x) = \phi^* H_0^0(S^2, x')$ . Clearly the elements of  $L(P^2, x)$  are obtained by rotating  $D$  and  $M$  (see first paragraph of §3). The injection of  $L(P^2, x)$  into  $H_0(P^2, x)$  clearly induces isomorphisms of the homotopy groups.

Finally, let  $L_1(K)$  be the subgroup of  $H(K)$  consisting of the "rotations" obtained by first rotating  $S^2 \times I$  in representation  $B$  and then identifying. The fundamental group of  $L_1(K)$  is  $Z$  and it should be clear from §4 that the injection into  $H(K)$  induces isomorphisms of the homotopy groups.

The group  $SO(3)$  is a transitive Lie group operating on  $S^2$ . Also,  $L(P^2)$  is a transitive Lie group operating on  $P^2$ , so it is of interest that the injections induce isomorphisms of the homotopy groups. It is seen in [6] that if  $T = S^1 \times S^1$ , then the subgroup  $L(T)$  generated by the rotations of the two copies of  $S^1$  has the same homotopy groups as  $H(T)$ . It is a transitive Lie group acting on the torus and the injection of  $L(T)$  into  $H(T)$  induces isomorphisms of the homotopy groups.

Mostow proves in [13] that the only other compact 2-manifold on which a Lie group acts transitively is the Klein bottle,  $K$ . This Lie group,  $L(K)$  has  $Z$  as its fundamental group and the higher homotopy groups are trivial. Also,  $L(K)$  has  $L_1(K)$  as a subgroup. The injection of  $L_1(K)$  in  $L(K)$  induces an isomorphism of the fundamental groups. It would seem, considering the fact that  $SO(3)$  is a deformation retract of  $H_0(S^2)$ , that  $L(P^2)$ ,  $L(T)$  and  $L(K)$  are deformation retracts of  $H_0(P^2)$ ,  $H_0(T)$  and  $H_0(K)$ . I leave this as a conjecture.

McCarty [11, Theorem 4.4] proves that if  $x$  is a point of a compact manifold with boundary,  $M$ , then  $\pi_i H(M, x)$  is isomorphic to  $\pi_i H(M - x)$ , the latter group having the compact open topology. The plane,  $E^2$ , the open annulus  $A$  and the open Moebius strip  $M^\circ$  [13] are the only noncompact 2-manifolds on which transitive Lie groups act. The group  $L(M^\circ)$  is isomorphic to  $L(K)$  and has the "rotation" group,  $L_1(M^\circ)$  as a deformation retract. It follows from

McCarty's result and Theorem 3.1, that  $H(M^\circ)$  is isomorphic to  $Z$  and the comments of the earlier part of this section imply that the injection of  $L_1(M^\circ)$  into  $H(M^\circ)$  (thus that of  $L(M^\circ)$  into  $H(M^\circ)$ ) induces an isomorphism of the fundamental group.

Kneser also proved that  $H_0(E^2)$  has as a deformation retract the space of rotations about the origin. The simplest Lie group acting transitively on  $E^2$  is the translation group, which is clearly homotopically trivial. However, the group of rigid motions,  $L(E^2)$  also acts transitively. This has the group of rotations about the origin as a deformation retract. Thus the injection of  $L(E^2)$  into  $H_0(E^2)$  induces isomorphisms of the homotopy groups.

Finally, McCarty's work implies that for each  $i$ ,  $\pi_i H_0(A)$  is isomorphic to  $\pi_i H_0(E^2, 0)$ . It follows that if  $L_1(A)$  denotes the group of rotations of  $A$ , the injection into  $H_0(A)$  induces isomorphisms of the homotopy groups. The Lie group  $L(A)$  operating transitively on  $A$  has  $L_1(A)$  as a deformation retract. Thus the injection of  $L(A)$  into  $H_0(A)$  induces isomorphisms of the homotopy groups.

NOTE<sup>(2)</sup>. The group  $L(K)$  may be roughly described as follows. Consider  $E^2$  as a covering space of  $K$ . Identify  $(x, y)$  with  $(x, y + 1)$  for each  $(x, y)$  and  $(x, y)$  with  $(x + \pi, -y)$ . Then the mapping of  $K$  obtained by taking each point of  $(x, y)$  into  $(x + a, u, \sin(x + a) + v \cos(x + a) + y)$  is a homeomorphism. The collection of such homeomorphisms for real  $a, w, v$  is a Lie group acting transitively on  $K$ . This is  $L(K)$ .

If the points  $(x, y)$  and  $(x + \pi, -y)$  only are identified,  $M^\circ$  is the identification space and the same homeomorphisms of  $E^2$  yield  $L(M^\circ)$ . If  $(x, y)$  is identified with  $(x + 2\pi, y)$ , the identification space is  $A$  and the same homeomorphisms of  $E^2$  yield  $L(A)$ . In each of these cases, the rotation group is a deformation retract of the transitively acting Lie group.

#### REFERENCES

1. J. W. Alexander, *On the deformation of an  $n$ -cell*, Proc. Nat. Acad. Sci. U.S.A. **9** (1923), 406-407.
2. Eldon Dyer and Mary-Elizabeth Hamstrom, *Completely regular mappings*, Fund. Math. **45** (1958), 103-118.
3. Mary-Elizabeth Hamstrom and Eldon Dyer, *Regular mappings and the space of homeomorphisms on a 2-manifold*, Duke Math. J. **25** (1958), 521-531.
4. Mary-Elizabeth Hamstrom, *Some global properties of the space of homeomorphisms on a disc with holes*, Duke Math. J. **29** (1962), 657-662.
5. ———, *Regular mappings and decompositions of product spaces*, Notices Amer. Math. Soc. **10** (1963), 492.
6. ———, *The space of homeomorphisms on a torus*, Notices Amer. Math. Soc. **10** (1963), 275.
7. S. T. Hu, *Homotopy theory*, Academic Press, New York, 1959.
8. H. Kneser, *Die Deformationsätze der einfach zusammenhängenden Flächen*, Math. Z. **25** (1926), 362-372.

(2) I wish to thank Professor R. L. Bishop of the University of Illinois for his enlightening comments at this point.

9. W. B. R. Lickorish, *A representation of orientable, combinatorial 3-manifolds*, Ann. of Math. **76** (1962), 531–540.
10. ———, *Homeomorphisms of nonorientable 2-manifolds*, Proc. Cambridge Philos. Soc. **59** (1963), 307–317.
11. G. S. McCarty, Jr., *Local connectivity in homeomorphism groups*, Bull. Amer. Math. Soc. **67** (1961), 420–421.
12. ———, *Homeotopy groups*, Trans. Amer. Math. Soc. **106** (1963), 293–304.
13. G. D. Mostow, *The extensibility of local Lie groups of transformations and groups on surfaces*, Ann. of Math. **52** (1950), 606–636.
14. Louis V. Quintas, *On the space of homeomorphisms of a multiply punctured surface of genus  $g$*  (to appear).
15. J. H. Roberts, *Local arcwise connectivity in the space  $H^n$  of homeomorphisms of  $S^n$  onto itself*, Summary of Lectures, Summer Institute on Set Theoretic Topology, Madison, Wisconsin, 1955; p. 100.

UNIVERSITY OF ILLINOIS,  
URBANA, ILLINOIS