

REPRESENTATIONS OF GRADED LIE ALGEBRAS

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Introduction. Let L be a graded Lie algebra (see §2) over a commutative ring R with unit. We shall deal with graded representations of L .

In §2 we define the universal enveloping algebra U of L and prove a Poincaré-Birkhoff-Witt theorem for U .

In §3 we prove that if L is a finite dimensional graded Lie algebra over a field of characteristic $\neq 2$, then L has a faithful finite dimensional representation. A key lemma for this proof is a result on (nongraded) Lie algebras which states that if L is a finite dimensional Lie algebra over a field of characteristic 0 then L has a faithful, finite dimensional representation f such that $f(x)$ is nilpotent whenever $\text{ad } x$ is nilpotent.

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1. The symmetric algebra built on a graded module. Let Γ be an additive group, always assumed to have a subgroup Γ_0 of index 2 whose elements are called even. The elements of the complementary coset Γ_1 are called odd. If Γ is the group of the integers, Γ_0 is necessarily the subgroup of the even integers.

For fixed Γ, Γ_0 , define, for all $\gamma \in \Gamma$,

$$1^\gamma = 1, (-1)^\gamma = \begin{cases} -1, & \gamma \in \Gamma_1 \\ 1, & \gamma \in \Gamma_0 \end{cases}, \quad (-1)^{\gamma\delta} = ((-1)^\gamma)^\delta = ((-1)^\delta)^\gamma.$$

Henceforth, “ R ” will denote a commutative ring with identity in which multiplication by 2 is an additive automorphism.

Let L be a Γ -graded R -module. That is $L = \sum_{\gamma \in \Gamma} L_\gamma$, the L_γ 's being R -submodules and the sum being direct. We say the elements of L_γ are homogeneous of degree γ . The degree of a homogeneous element x_j will sometimes be written as $d(x_j)$ or d_j . Written as $x_\gamma, \gamma \in \Gamma$, the element will always denote a homogeneous element of L .

If $\Gamma = (0, 1)$ the homogeneous submodules will be written L_0 and L_1 and L will be said to be semi-graded (SG). Generally, setting $L_0 = \sum_{\gamma \text{ even}} L_\gamma, L_1 = \sum_{\gamma \text{ odd}} L_\gamma$

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we see that L has a SG-structure called the Γ_0 -associated semi-graded structure.

Let $T(L)$ be the tensor algebra over R built on the module L . $T(L) = \sum_{i=0}^{\infty} T^i(L) = R + L + L \otimes_R L + \dots$. Let $S(L) = T(L)/J$, where J is the two-sided ideal in $T(L)$ generated by the elements $x_\lambda \otimes x_\mu - (-1)^{\lambda\mu} x_\mu \otimes x_\lambda$. $S(L)$ is called the symmetric algebra in the graded sense based on the graded R -module L . If L is R -free, $S(L)$ may be identified with the algebra of polynomials in the basis elements (x_λ) , with coefficients in R , and with relations $x_\lambda x_\mu = (-1)^{\lambda\mu} x_\mu x_\lambda$.

2. Graded Lie algebras and their enveloping algebras. Let L be a graded Lie algebra (GLA) over the ring R . That is, L is Γ -graded R -module, together with an R -bilinear multiplication $[\]: L_\lambda \times L_\mu \rightarrow L_{\lambda+\mu}$ such that, for homogeneous x_λ, x_μ, x_ν :

1. $[x_\lambda, x_\mu] = -(-1)^{\lambda\mu} [x_\mu, x_\lambda]$,
- 2a. $(-1)^{\lambda\nu} [x_\lambda, [x_\mu, x_\nu]] + (-1)^{\lambda\mu} [x_\mu, [x_\nu, x_\lambda]] + (-1)^{\nu\mu} [x_\nu, [x_\lambda, x_\mu]] = 0$,
- 2b. $[x_\lambda, [x_\lambda, x_\lambda]] = 0$.

If $\Gamma = (0, 1)$, the R -submodules will be written as L_0 and L_1 . We have $[L_0, L_0] \subset L_0$, $[L_0, L_1] \subset L_1$, $[L_1, L_1] \subset L_0$. Such an L will be called a semi-graded Lie algebra (SGLA). Generally, setting $L_0 = \sum_{\gamma \in \Gamma_0} L_\gamma$, $L_1 = \sum_{\gamma \in \Gamma_1} L_\gamma$, we see that L has a SGLA structure, called the Γ_0 -associated SGLA structure.

Let $U(L) = T(L)/I$, where I is the two-sided ideal in $T(L)$ generated by the elements $x_\lambda \otimes x_\mu - (-1)^{\lambda\mu} x_\mu \otimes x_\lambda - [x_\lambda, x_\mu]$. $U(L)$ is called the universal enveloping algebra of L .

Let M be a graded, unitary R -module and let $E^*(M)$ be the graded R -algebra of all graded R -endomorphisms of M . A graded L -module structure on M is a linear map $\rho: L \rightarrow E^*(M)$, homogeneous of degree 0, such that, for homogeneous x_λ, x_μ , $\rho([x_\lambda, x_\mu]) = \rho(x_\lambda)\rho(x_\mu) - (-1)^{\lambda\mu}\rho(x_\mu)\rho(x_\lambda)$. Extending ρ to a unitary R -algebra homomorphism of $T(L)$ into $E^*(M)$, we have $\rho(I) = 0$. Hence ρ induces a unitary homomorphism of U into $E^*(M)$, i.e., M has the structure of a graded U -module over the graded algebra U , in which the grading is as follows. Let u_λ be the image of x_λ in $U(L)$. The grading on elements $u_{\lambda_1}^{e_1} \dots u_{\lambda_n}^{e_n}$ is $\sum e_j \lambda_j$. We obtain in this fashion a 1-1 correspondence between graded L -module structures and unitary, graded U -module structures.

THEOREM 2.1. *Suppose L is homogeneously free over R and let (x_j) be a simply ordered, free, homogeneous R -basis of L . Let u_j be the image of x_j in U . Then 1 and the ordered monomials*

- 2.1. $u_{j_1} \dots u_{j_n}$, with $j_1 \leq \dots \leq j_n$, and $j_k < j_{k+1}$ whenever d_{j_k} is odd,
for $k = 1, \dots, n - 1$,

form a free R -basis for U .

Proof. As in the nongraded case, the fact that these ordered monomials span U is obtained from the "straightening" procedure.

Let U_n be the canonical image of T_n , where $T_n = \sum_{i \leq n} T^i(L)$. The U_n constitute an increasing filtration of the R -module U . U_0 is spanned by 1. Assume U_{n-1} is spanned by 1 and the ordered monomials $u_{j_1} \cdots u_{j_{n-1}}$. For the monomials $u_{j_1} \cdots u_{j_n}$ there are two cases to consider.

1. If $j_r > j_{r+1}$, for some r , then $u_{j_1} \cdots u_{j_r} u_{j_{r+1}} \cdots u_{j_n} = \varepsilon u_{j_1} \cdots u_{j_{r+1}} u_{j_r} \cdots u_{j_n} + u_{j_1} \cdots [u_{j_r}, u_{j_{r+1}}] \cdots u_{j_n}$, with $\varepsilon = \pm 1$ and $u_{j_1} \cdots [u_{j_r}, u_{j_{r+1}}] \cdots u_{j_n} \in U_{n-1}$.

2. If $j_r = j_{r+1}$ with d_{j_r} odd, then $u_{j_1} \cdots u_{j_r} u_{j_{r+1}} \cdots u_{j_n} = (1/2) u_{j_1} \cdots u_{j_{r-1}} [u_{j_r}, u_{j_r}] \cdots u_{j_n} \in U_{n-1}$.

Hence, by induction on the number of inversions of order in (j_1, \dots, j_n) we show that the ordered monomials in 2.1 span U_n . First, if there are repetitions of indices i_r in (i_1, \dots, i_n) with d_{i_r} odd, then, by using 1 and 2 above, we see that $u_{i_1} \cdots u_{i_n} \in U_{n-1}$, whence by the induction hypothesis the monomial $u_{i_1} \cdots u_{i_n}$ is a sum of the ordered monomials in U_{n-1} . Assuming then no repetitions of indices corresponding to elements of odd degree we see that the theorem holds trivially when the number of inversions is zero. Assume the theorem is true for p inversions. For $p + 1$ inversions by using 1 on the first contiguous inversion that occurs reading from left to right, we have, by the induction hypothesis, that $u_{i_1} \cdots u_{i_n}$ is the sum of ordered monomials in U_n .

It remains to be shown that the ordered monomials are R -linearly independent. To prove this we define a graded L -module structure on $S(L)$, denoted by $(x, z) \rightarrow x \cdot z$, hence that of a graded $U(L)$ -module structure on $S(L)$. With some modifications necessary to take the grading into account, the proof proceeds as in [1, Lemma 3.5, p.271].

The modifications are as follows. For $P = z_{i_1}^{e_1} \cdots z_{i_p}^{e_p} \in S(L)$, the degree of P is $\sum e_k d_{i_k}$, and the height of P is $\sum e_k$. Let S^γ denote the subspace of $S(L)$ consisting of elements of degree γ , and let S_q denote the subspace consisting of polynomials of height $\leq q$. Then for each integer p there is a unique homomorphism $L \otimes S_p \rightarrow S(L)$ such that

- (0) $L_\lambda \cdot S^\gamma \subset S^{\lambda+\gamma}$,
- (1'a) $x_j \cdot z_I = z_j z_I, d_j$ even (odd), $j \leq I$ ($j \leq I$ with $j \notin I$),
- (1'b) $x_j \cdot z_I = (1/2)[x_j, x_j] \cdot z_j, d_j$ odd, $j \leq I$ with $I = (j, J)$,
- (3) $x_j \cdot (x_k \cdot z_J) = (-1)^{\lambda\gamma} x_k \cdot (x_j \cdot z_J) + [x_j, x_k] \cdot z_J, z_J \in S_{p-1}, \lambda = d(x_j), \gamma = d(x_k)$.

Equations (2) and (4) of the above reference remain unchanged. At the inductive step it is easily seen that $x_j \cdot (x_k \cdot z_I) - (-1)^{\lambda\mu} x_k \cdot (x_j \cdot z_I) = [x_j, x_k] \cdot z_I$, where $\lambda = d(x_j), \mu = d(x_k)$, whenever $j, k \leq I$ and $z_I \in S_p$. For if $d(x_j)$ is odd and $j = k$ with $j \notin I$ we have $2x_j \cdot (x_j \cdot z_I) = 2x_j \cdot (z_j z_I) = [x_j, x_j] \cdot z_I$; whereas if $j \in I, I = (j, J)$, we have $2x_j \cdot (x_j \cdot z_I) = 2x_j \cdot (x_j \cdot (z_j z_J)) = 2x_j \cdot ((1/2)[x_j, x_j] \cdot z_J) = [x_j, [x_j, x_j]] \cdot z_J + [x_j, x_j] \cdot (x_j \cdot z_J) = [x_j, x_j] \cdot z_I$.

The other cases that arise under the condition that $j, k \leq I$ and the remainder of the proof of the existence of the L -module structure are as in the reference with the necessary sign changes arising from (3). This proves Theorem 2.1.

Now let G^n be the R -module U_n/U_{n-1} , and let G be the direct R -module sum of the G^n . G has a multiplication induced by that of U such that $G^n G^m \subset G^{n+m}$. G is called the graded algebra associated to the filtered algebra U . Let ψ_n be the canonical map of T^n into U_n , Θ_n the canonical map of U_n onto G^n , and let $\phi_n = \Theta_n \circ \psi_n$. Since T^n is supplementary to T_{n-1} in T_n , ϕ_n is surjective. The ϕ_n 's define an R -linear map ϕ of T onto G .

PROPOSITION 2.2. *The map ϕ of T onto G is an epimorphism of algebras sending 1 into 1, and ϕ annihilates the two-sided ideal generated by the elements $x_\lambda \otimes x_\mu - (-1)^{\lambda\mu} x_\mu \otimes x_\lambda$.*

Proof. For $t \in T^n$ and $t' \in T^m$, by definition of the multiplication in G , we have $\phi(t)\phi(t') = \phi(tt')$. Hence, ϕ is an algebra homomorphism, and it is clear that $\phi(1) = 1$. Now for $x_\lambda \in L_\lambda$, $x_\mu \in L_\mu$, we have that $x_\lambda \otimes x_\mu - (-1)^{\lambda\mu} x_\mu \otimes x_\lambda \in T^2$ and the canonical image of this in U_2 is equal to that of $[x_\lambda, x_\mu]$, hence belongs to U_1 . Hence $\phi(x_\lambda \otimes x_\mu - (-1)^{\lambda\mu} x_\mu \otimes x_\lambda) = 0$. The above proposition shows that there is a unique epimorphism ω of the algebra $S(L)$ onto the algebra G such that the following diagram is commutative:

$$\begin{array}{ccc} T^n & \longrightarrow & U_n \\ \downarrow & & \downarrow \\ S^n & \longrightarrow & G^n \end{array}$$

We have the following result.

COROLLARY. *Under the hypothesis of Theorem 2.1 the above epimorphism ω is an isomorphism of $S(L)$ onto G .*

Proof. It suffices to show that $\omega_n: S^n(L) \rightarrow G^n$ is injective. Since the monomials $x_{i_1}^{e_1} \cdots x_{i_p}^{e_p}$ with $i_1 < \cdots < i_p$, $\sum e_i = n$, and $e_k = 1$ if d_{i_k} is odd, form a basis for $S(L)$, it will suffice to show that ω_n sends these elements onto R -linearly independent elements of G^n . Now $\omega_n(x_{i_1}^{e_1} \cdots x_{i_p}^{e_p})$ is the coset mod U_{n-1} of $u_{i_1}^{e_1} \cdots u_{i_p}^{e_p}$, and it is clear from Theorem 2.1 that these cosets are R -linearly independent elements of G^n .

3. Representations of graded Lie algebras.

DEFINITION. Let L be a GLA over a field K , and let V be a vector space over K . A pre-representation of L is a linear map $\rho: L \rightarrow E_K(V)$, such that $\rho[x_\lambda, x_\mu] = \rho(x_\lambda)\rho(x_\mu) - (-1)^{\lambda\mu}\rho(x_\mu)\rho(x_\lambda)$. The term "representation" will always refer to a graded representation, that is, V will be a graded L -module and L_λ will send V_μ into $V_{\lambda+\mu}$.

LEMMA 3.1. *Let A be a ring and let B be a subring of A such that A is generated as a right B -module, with generators (u_i) . Then if k is a left ideal in B , $k' = \sum u_i k$ is a left ideal in A .*

Proof. Let $a \in A$, $h \in k'$. Then $a = \sum u_i b_i$ with b_i in B and $h = \sum u_i k_i$, with k_i in k . Hence $ah = (\sum_j u_j b_j)(\sum_i u_i k_i) = \sum_{i,j} u_j u_i b_j k_i$. But $u_j b_j u_i = \sum_e u_e b_{eij}$, with b_{eij} in B . Hence $ah = \sum_e u_e b_{eij} k_i \in k'$.

PROPOSITION 3.2. *Let $L = L' \oplus L''$ be a finite dimensional GLA over a field K of characteristic $\neq 2$. Assume that L'' is a subalgebra of L and that each homogeneous component of L' is of odd degree. Then every faithful, finite dimensional pre-representation (resp. representation) of L'' can be extended to a faithful, finite dimensional pre-representation (resp. representation) of L .*

Proof. Let (x_i) be a simply ordered, homogeneous K -basis of L such that x_i , for $i = 1, \dots, m$ (resp. $i = m + 1, \dots, n$) constitutes a K -basis for L' (resp. L''). Let u_i be the image of x_i in $U(L)$ and let V be the space spanned by $(u_{i_1} \cdots u_{i_r} \mid 1 \leq i_1 < \cdots < i_r \leq m)$. Then, as a vector space, $U(L) \simeq U(L'') \oplus VU(L')$. The pre-representation ρ of L'' , when extended to $U(L')$, has a $U(L')$ -kernel k , such that $L \cap k = (0)$ in $U(L)$. If ρ is a representation of L'' , then k is a homogeneous ideal in $U(L')$.

Let $k' = k + Vk$. By Lemma 3.1, k' is a left ideal in $U(L)$ and is homogeneous whenever k is homogeneous. Since k is of finite co-dimension in $U(L')$ and since V is finite-dimensional and $U(L) = U(L'') + VU(L')$, k' is of finite co-dimension in $U(L)$. Finally, $L \cap k' = (0)$; for let $y_1 \in L'$, $y_2 \in L''$, $v \in V$ and $k_1, k_2 \in k$ with $y_1 + y_2 = k_2 + vk_1$, i.e. $(k_2 - y_2) + (vk_1 - y_1) = 0$. Then the Birkhoff-Witt theorem (2.1) implies that $y_2 = k_2$ and $y_1 = vk_1$. Since ρ is faithful, the first of these relations gives $y_2 = 0$. If $y_1 \neq 0$ then the second relation gives $k_1 \in K$ contrary to the definition of k . Hence also $y_1 = 0$, so that we have indeed $L \cap k' = (0)$. Hence, by left multiplication in $U(L)/k'$, L has a faithful finite dimensional pre-representation. If k' is homogeneous $U(L)/k'$ has a gradation with which the pre-representation is a representation.

COROLLARY. *If L is a finite dimensional SGLA over a field of characteristic $\neq 2$, L has a faithful, finite dimensional representation.*

Proof. $L = L_0 + L_1$, where L_0 is a Lie algebra. By Ado's theorem L_0 has a faithful, finite dimensional representation (of degree 0). Hence L has a faithful finite dimensional representation.

LEMMA 3.3. *Let L be a finite dimensional Lie algebra over a field F of characteristic 0. Let N be the maximal nilpotent ideal of L and let ρ be a finite dimensional representation of L that is nilpotent on N . Let $x \in L$ and suppose $\text{ad } x$ is nilpotent. Then $\rho(x)$ is nilpotent.*

Proof. Let S be a maximal semi-simple subalgebra of L and let R be the radical of L . Then we have the semidirect sum decomposition $L = S + R$. Accordingly, if $x \in L$, write $x = s + r$, with s in S and r in R . The endomorphism of L/R induced

by $\text{ad } x$ coincides with that induced by $\text{ad } s$. Thus if $\text{ad}_L x$ is nilpotent so is $\text{ad}_S s$. It follows that every representation of S maps s onto a nilpotent linear endomorphism. In fact by [Theorem 17, p. 100 of [2]], there exists an $h \in S$ such that $[h, s] = s$. Then $Fh + Fs$ is a solvable subalgebra of S whose derived algebra is Fs . Now the assertion follows from Lie's theorem, which states that if L is a solvable Lie algebra over a field of characteristic 0, then every finite dimensional semi-simple L -module is annihilated by $[L, L]$. In particular, $\text{ad}_L s$ is nilpotent. It follows that $Fs + N$ is nilpotent on L . Now $[\text{ad}(s + r), \text{ad } s] = \text{ad}[r, s]$, and $[r, s] \in N$. Hence $F(s + r) + Fs + N$ is nilpotent on L . Therefore $\text{ad } r$ is nilpotent on L . Hence $r \in N$. Now $\rho(s)$ is nilpotent, $\rho(N)$ is nilpotent and $[\rho(s), \rho(N)] \subset \rho(N)$, so that $F\rho(s) + \rho(N)$ is nilpotent. Hence $\rho(x)$ is nilpotent.

PROPOSITION 3.4. *Let L be a finite dimensional, evenly graded Lie algebra over a field of characteristic 0. Then L has a faithful, finite dimensional representation.*

Proof. By Lemma 3.3 and Ado's theorem, L has a faithful, finite dimensional pre-representation ρ^* , such that $\rho^*(x)$ is nilpotent whenever $\text{ad } x$ is nilpotent. Let V be the representation space of ρ^* . For each $\gamma \in \Gamma$, let V_γ be a copy of V and let π_γ be an isomorphism of V_γ onto V . Let $V_0 = V$ and let π_0 be the identity map on V . For $x \in L_\lambda$ and $v \in V_\gamma$ define

$$\rho(x)(v) = \pi_{\lambda+\gamma}^{-1}(\rho^*(x)(\pi_\gamma(v))).$$

One verifies directly that ρ is a representation of L on $\sum_{\gamma \in \Gamma} V_\gamma$. Moreover, if V' is the submodule of $\sum_{\gamma \in \Gamma} V_\gamma$ that is generated by V , then ρ induces a faithful representation of L on V' . Hence it suffices to show that V' is finite dimensional. Note that if $x \in L_\gamma$ and $\text{ad } x$ is not nilpotent then γ must be of finite order, because L is finite dimensional. Also the set of γ 's for which such an x exists must be finite. Hence the group generated by these γ 's is a finite subgroup Δ of Γ .

Now we choose an ordered, homogeneous basis $x_1, \dots, x_r, y_1, \dots, y_s$ of L such that each $\text{ad } x_i$ is nilpotent and each y_j lies in some L_γ with $\gamma \in \Delta$. Each $\rho^*(x_i)$ is nilpotent, so that there is a non-negative integer q such that $\rho^*(x_i)^q = 0$, for all i . If $v \in V_\gamma$ and $x_i \in L_\lambda$ we have $\rho(x_i)^k(v) = \pi_{\lambda+\gamma}^{-1}(\rho^*(x_i)^k(\pi_\gamma(v)))$, whence $\rho(x_i)^q = 0$.

Now V' consists of all linear combinations $\rho(x_1)^{e_1} \dots \rho(x_r)^{e_r} \rho(y_1)^{f_1} \dots \rho(y_s)^{f_s}(v)$, with $v \in V$. These are 0 if one of the e_i 's is $\geq q$. Hence it suffices to take $0 \leq e_i < q$. The elements $\rho(y_1)^{f_1} \dots \rho(y_s)^{f_s}(v)$ lie in the finite dimensional space $\sum_{\gamma \in \Delta} V_\gamma$. Hence it is clear that V' is finite dimensional.

COROLLARY. *If L is a finite dimensional GLA over a field of characteristic 0, then L has a faithful, finite dimensional representation.*

Proof. $L = \sum_{\gamma \text{ even}} L_\gamma + \sum_{\gamma \text{ odd}} L_\gamma$. By Proposition 3.4, $\sum_{\gamma \text{ even}} L_\gamma$ has a faithful, finite dimensional representation which, by Proposition 3.2 may be extended to a faithful, finite dimensional representation.

PROPOSITION 3.5. *Let L be a finite dimensional GLA over a field of characteristic $\neq 2$. Then L has a faithful finite dimensional representation.*

Proof. $L = \sum_{\gamma \text{ even}} L_{\gamma} + \sum_{\gamma \text{ odd}} L_{\gamma}$. By Proposition 3.2 it suffices to show that $L' = \sum_{\gamma \text{ even}} L_{\gamma}$ has a faithful, finite dimensional representation.

Let (x_1, \dots, x_n) be an ordered, homogeneous basis for L' such that $\text{ad}_{L'} x_i$ is nilpotent, for $i = 1, \dots, r$, and x_i , for $i = r + 1, \dots, n$, lies in some L_{γ} with γ of finite order. For $i = 1, \dots, r$ there is a p -polynomial $m_i(\lambda) = \lambda^{p^{m_i}}$ such that $z_i = x_i^{p^{m_i}}$ is in the center of $U(L')$. For $i = r + 1, \dots, n$ there is a p -polynomial $m_i(\lambda) = \lambda^{p^{m_i}} + a_1 \lambda^{p^{m_i}-1} + \dots + a_{m_i} \lambda$ such that $z_i = x_i^{p^{m_i}} + a_1 x_i^{p^{m_i}-1} + \dots + a_{m_i} x_i$ is in the center of $U(L')$.

Now $U(L')$ is graded, and the center of $U(L')$ is homogeneous. Hence, by dropping some terms of exponent $< p^{m_i}$ if necessary, we may assume z_i is homogeneous. Hence, by Lemma 5.4 of [2], the elements $z_1^{h_1} \dots z_n^{h_n} u_1^{\lambda_1} \dots u_n^{\lambda_n}$, $h_i \geq 0$, $0 \leq \lambda_i < p^{m_i}$, form a basis for $U(L')$. Let B be the ideal in U generated by the z_i . Then B is homogeneous since each generator is homogeneous. As in the proof of Theorem 5.11 of [2] the cosets of the elements $u_1^{\lambda_1} \dots u_n^{\lambda_n}$, $0 \leq \lambda_i < p^{m_i}$ form a basis for U/B . Hence this algebra is a finite dimensional graded algebra and the canonical mapping $a \rightarrow \bar{a}$, $a \in L'$, is a monomorphism of L' into $\bar{U} = U/B$. It follows that L' has a faithful, finite dimensional representation.

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