NORMAL OPERATORS AND UNIFORMLY ELLIPTIC SELF-ADJOINT PARTIAL DIFFERENTIAL EQUATIONS(1)

BY

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1. Let $C_H^1(V)$ be the class of functions with Hölder-continuous first derivatives in a region V of Euclidean n-space R^n. Let $a_{ik} \in C_H^1(V)$, $i, k = 1, \cdots, n$, with $a_{ik} = a_{ki}$, such that

(1) $$\frac{1}{\lambda} \sum_i \xi_i^2 < a_{ik}(x) \xi_i \xi_k < \lambda \sum_i \xi_i^2$$

in V for some constant $\lambda > 0$ and all reals ξ_i; here Einstein's summation convention is adopted. Consider the self-adjoint uniformly elliptic partial differential equation

(2) $$Eu \equiv \frac{\partial}{\partial x_k}\left(a_{ik}(x) \frac{\partial u}{\partial x_i}\right) = 0.$$

We shall present a method of finding solutions with given singularities and given behavior near the boundary of V. Extremal properties of such solutions, to be called *principal solutions*, will be established.

A more detailed description of the problem and its significance will be given in No. 6 below, after the necessary concepts have been introduced.

Our approach is based on the linear operator method previously used for the Laplacian [6], [7]. The generalization, while parallel, reveals delicate differences. The existence of the L_0-operator, one of our basic tools, for every regular subregion is trivial in the case of Riemann surfaces; in the present general context it is a deep result established by G. Fichera on pp. 195–202 of [4]. The convergence of the principal functions of exhausting subregions offered difficulties not overcome in the present paper, and extremal properties are given for regular subregions only.

For further extensions and applications of our method we refer to [8] and [9]. In a new direction (not containing earlier work) the problem was given an elegant formulation in [1].

§1. Preliminaries.

2. We shall call a subregion Ω of $V \subset R^n$ *Hölder-bordered* if

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(a) $\partial\Omega$ is compact in V,

(b) every $x \in \partial\Omega$ has a neighborhood N_x and a homeomorphism h of N_x with the unit ball $B \subset R^n$ such that $h(N_x \cap \partial\Omega)$ is the intersection of B with a coordinate plane P, $h(N_x \cap \Omega)$ is one of the half-balls constituting $B \setminus P$, and h^{-1} is in $C_H^1(B)$.

A Hölder-bordered region shall be called Hölder-regular, or simply *regular*, if

(c) $\bar{\Omega}$ is compact in V,

(d) each component of $V - \Omega$ is noncompact in V.

3. At a point x of the boundary $\partial\Omega$ of a Hölder-bordered region we denote by $\partial / \partial n$ the normal derivative in the Euclidean metric, directed toward the exterior of Ω unless specified otherwise. We set

$$a = \left[\sum_i \left(\sum_k a_{ik} \cos(x_k, n) \right)^2 \right]^{1/2} \tag{3}$$

and define the *conormal derivative* $\partial / \partial \nu$ by

$$a \frac{\partial}{\partial \nu} = a_{ik} \cos(x_k, n) \frac{\partial}{\partial x_i}. \tag{4}$$

We shall call $\int_{\partial\Omega} a(\partial u / \partial \nu)\, dS$ the *flux of* u across the border of a Hölder-bordered Ω, u being in C_H^1 in an open set containing $\partial\Omega$.

Solutions of (2) in C_H^1 will be referred to simply as "solutions". By a "solution in a set E" will be meant the restriction to E of a solution in an open set containing E. For a regular Ω and solutions u, v in $\bar{\Omega}$ the Green's formula reads [5]

$$\int_{\partial\Omega} a\left(v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) dS = 0, \tag{5}$$

where dS is the surface element of $\partial\Omega$. As a consequence, the mean value property, the Harnack inequality, and the maximum principle are valid [3].

The norm over V is defined as $M(u) = M(u,u)$, where

$$M(u,v) = \int_V a_{ik} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_k} \, dV. \tag{6}$$

The triangle inequality

$$M^{1/2}(u+v) \leqq M^{1/2}(u) + M^{1/2}(v) \tag{7}$$

is proved in the usual manner.

4. Let V_0 be a compact set in $V \subset R^n$. We consider a family $\{u\}$ of solutions in V with $\operatorname{sgn} u \,|\, V_0 \neq \text{const}$.

LEMMA. *There exists a positive constant $q < 1$, independent of u, such that*

$$\max_{V_0} |u| \leqq q \sup_V |u|. \tag{8}$$

Proof. The lemma is trivial if $\sup_V |u| = 0$ or ∞. In other cases we normalize by $\sup_V |u| = 1$. Suppose there did not exist any constant $q < 1$ with $\max_{V_0} |u| \leqq q$. Then there would exist a sequence $\{u_n\}$ with $\max_{V_0} |u_n| \to 1$. The points $\{z_n\}$ in V_0 where the maxima are taken have a subsequence, again denoted by $\{z_n\}$, tending to $z_0 \in V_0$, say. The coefficients a_{ik} being continuous in the compact set V_0, the set $\{u_n\}$ of uniformly bounded solutions is compact [2, p. 344]. A subsequence tends to a solution u_0 with $|u_0(z_0)| = 1$, sgn $u_0 | V_0 \neq$ const. This violates the maximum principle.

§2. The problem.

5. Let V_1 be the complement in V of the closure of a regular region, and let F be the family of solutions f on $\alpha_1 = \partial V_1$.

DEFINITION. *L is a normal operator if for $f \in F$ the function Lf is a solution in $\bar{V}_1$ with the following properties:*

$$Lf|\alpha_1 = f, \tag{9}$$

$$L(c_1 f_1 + c_2 f_2) = c_1 Lf_1 + c_2 Lf_2, \tag{10}$$

$$\min f \leqq Lf \leqq \max f, \tag{11}$$

$$\int_{\alpha_1} a \frac{\partial Lf}{\partial \nu} dS = 0. \tag{12}$$

The existence of operators L will be established in §§4–5.

6. Let s be a solution in $\bar{V}_1$. We shall construct a solution p in V that imitates the behavior of s in $\bar{V}_1$. More precisely, for a given s and L we require that $p | \bar{V}_1 = s + L(p - s)$. This means that $p | \bar{V}_1$ deviates from s by a bounded solution, and we shall give explicit bounds for the deviation. Moreover, $p - s$ enjoys boundary properties determined by the choice of L.

V_1 need not be connected. For instance, given a region $V^* \subset R^n$ and points $z_i \in V^*$, $i = 1, \cdots, m$, we may choose V as $V^* \setminus \bigcup \{z_i\}$. Given disjoint neighborhoods N_i of the z_i, V_1 can be taken as the union of the punctured neighborhoods $V_{1i} = N_i \setminus z_i$ and of a neighborhood $V_{1\beta} = V_1^*$ of the ideal boundary β of V^* with $\bar{V}_{1\beta} \cap \{\bigcup \bar{V}_{1i}\} = \varnothing$.

In V_{1i}, s may have a singularity at z_i, and L may be the operator giving the solution of the Dirichlet problem in N_i. In $V_{1\beta}$, s can have an arbitrary growth when approaching β, and L can be given separately in each component of $V_{1\beta}$. Thus we are dealing with the problem of constructing solutions with given singularities and prescribed boundary behavior.

The problem is given further interest by the fact that certain operators L give significant extremum properties to the corresponding p. We shall call p the *principal solution* for given s, L.

7. Once s and L have been prescribed, the principal solution is unique up to an additive constant. Indeed, for two such functions p', p'' we have

$$\max_{V-V_1} (p' - p'') = \max_{\alpha_1} (p' - p'')$$

by the maximum principle, and

$$\max_{\bar{V}_1} (p' - p'') = \max_{\alpha_1} (p' - p'')$$

by (11). Consequently $\max_V(p' - p'')$ is taken on α_1, and $p' - p''$ must reduce to a constant.

Without loss of generality we assume that $s|\alpha_1 = 0$, for otherwise we can replace s by $s - Ls$.

The function p is constant if and only if $s \equiv 0$. Sufficiency is seen from $p = Lp$ and the maximum principle. Necessity is obvious.

For the existence of p it is necessary that the flux of s vanish,

$$\int_{\alpha_1} a \frac{\partial s}{\partial \nu} \, dS = 0. \tag{13}$$

This follows directly from $\int_{\alpha_1} a(\partial p / \partial \nu) \, dS = 0$ and (12). The essence of the following main existence theorem is that (13) is also sufficient.

§3. Main existence theorem.

8. Let V be a region in R^n, and V_1 the complement in V of a regular subregion of V, with border α_1. Let L be a normal operator (9)–(12) for V_1, and s with $s|\alpha_1 = 0$ a solution in $\bar{V}_1$ of the elliptic equation (2) whose coefficients satisfy the conditions stated in No. 1. We shall write Lp for $L(p|\alpha_1)$.

THEOREM. *The vanishing of the flux* (13) *of s is necessary and sufficient for the existence of a principal solution p on V of* (2), *characterized by*

$$p|\bar{V}_1 = s + Lp. \tag{14}$$

The function is unique up to an additive constant and it reduces to a constant if and only if $s \equiv 0$.

Proof. Let V_0 be a regular subregion of V with border $\alpha_0 = \partial V_0 \subset V_1$, $\alpha_1 \subset V_0$. Let L' be the operator that for functions on α_0 gives solutions of the Dirichlet problem in $\bar{V}_0$. It suffices to find $p|\alpha_0$, for then

$$p|\bar{V}_0 = L'p, \qquad p|\bar{V}_1 = s + Lp, \tag{15}$$

where again $L'p$ stands or $L'(p|\alpha_0)$. We set $K = LL'$ and seek $p|\alpha_0$ with

$$p = s + Kp \tag{16}$$

on α_0.

9. We shall show that

$$p = \sum_0^\infty K^n s \tag{17}$$

converges uniformly on α_0. Then K can be applied term by term, for

$$\left| K \sum_0^\infty K^n s - \sum_1^{m+1} K^n s \right| = \left| K \sum_{m+1}^\infty K^n s \right| \leqq \left| \sum_{m+1}^\infty K^n s \right|$$

which tends to 0 as $m \to \infty$. Consequently $Kp = \sum_1^\infty K^n s = p - s$ as desired.

Let $\omega \in C^1$ in $\bar{V}_0 \cap \bar{V}_1$ satisfying (2) in $V_0 \cap V_1$ with $\omega \,|\, \alpha_0 = 0$, $\omega \,|\, \alpha_1 = 1$. For a solution σ in $\bar{V}_0 \cap \bar{V}_1$ with $\int_{\alpha_0} a(\partial\sigma / \partial\nu)\, dS = 0$ we have by (5)

$$\int_{\alpha_0} \sigma a \frac{\partial \omega}{\partial \nu} dS = \int_{\alpha_1} \sigma a \frac{\partial \omega}{\partial \nu} dS. \tag{18}$$

This holds, in particular, for $\sigma = s$, $L'\phi$, $L\psi$, $K\phi$, where ϕ, ψ are restrictions to α_0, α_1 of solutions in open sets containing α_0, α_1, respectively.

We claim that

$$\int_{\alpha_1} K^n s\, a \frac{\partial \omega}{\partial \nu}\, dS = 0 \tag{19}$$

for $n \geqq 0$. By assumption this is true for $n = 0$. Suppose it holds for $n = i$. Then by (18) the integral vanishes over α_0 as well:

$$\int_{\alpha_0} L' K^i s\, a \frac{\partial \omega}{\partial \nu}\, dS = 0.$$

Again α_0 can be replaced by α_1, and subsequently L' by $LL' = K$. The statement follows.

From $\cos(n,\nu) > 0$ [5] and the maximum principle we have $\partial\omega / \partial\nu > 0$. This together with $a > 0$ gives sgn $K^n s \,|\, \alpha_1 \neq$ const. Lemma 4 applies:

$$|Ks| \,|\, \alpha_0 \leqq q \max_{\alpha_0} |s|, \qquad |K^n s| \,|\, \alpha_0 \leqq q^n \max_{\alpha_0} |s|,$$

whence the uniform convergence of (17). On setting

$$m = \min_{\alpha_0} s, \qquad M = \max_{\alpha_0} s, \tag{20}$$

one can also easily see (cf. [7]) that

$$q^n m \leqq K^n s \,|\, \alpha_0 \leqq q^n M. \tag{21}$$

10. We have actually proved more than Theorem 8 states:

THEOREM. *The principal solution is given by the Neumann series* $p|\alpha_0 = \sum_0^\infty K^n s$ *and satisfies the inequalities*

(22) $$\frac{m}{1-q} \leqq p|\bar{V}_0 \leqq \frac{M}{1-q},$$

(23) $$\frac{m}{1-q} \leqq p-s|\bar{V}_1 \leqq \frac{M}{1-q}.$$

Indeed, (21) gives these bounds for $p|\alpha_0$, and by the maximum principle they hold for $p|\bar{V}_0$, hence for $p|\alpha_1$, $p-s|\alpha_1$, and $p-s|\bar{V}_1$.

§4. Normal operators for regular regions.

11. In this section V_1 stands for a regular region in R^n with disconnected border partitioned into disjoint sets α_1 and β of components. Let f be the restriction to α_1 of a solution in an open set containing α_1. Denote by $E(\bar{V}_1)$ the family of solutions in $\bar{V}_1$ and set

(24) $$U = \{u \mid u \in E(\bar{V}_1),\ u|\alpha_1 = f\}.$$

In this class we single out the functions u_0 and u_1 determined by the conditions

(25) $$\frac{\partial u_0}{\partial \nu}\Big|\beta = 0,$$

(26) $$u_1|\beta = c(\text{const.}), \qquad \int_{\alpha_1} a\frac{\partial u_1}{\partial \nu}\, dS = 0.$$

The existence of u_0 has been demonstrated by G. Fichera [4, pp. 195–202]. The flux of u_1 can be taken across β, where it is negative for $c = \min f$, positive for $c = \max f$, and increases monotonically with c. This ensures the existence of the constant c with property (26).

The functions u_0 and u_1 are uniquely determined and are related to f by linear operators:

(27) $$u_0 = L_0 f, \qquad u_1 = L_1 f.$$

It is readily seen that L_0, L_1 are normal in the sense of No. 5.

12. For $\lambda \in R$ set

(28) $$u_\lambda = (1-\lambda)u_0 + \lambda u_1 = L_\lambda f,$$

where the operator L_λ is again normal. We shall derive an extremal property of u_λ in the class

(29) $$U_0 = \left\{u \,\middle|\, u \in U,\ \int_{\alpha_1} a\frac{\partial u}{\partial \nu}\, dS = 0\right\}.$$

This property will serve to establish the existence of normal operators for arbitrary Hölder-bordered regions V_1.

For $u, v \in U_0$ set

$$A(u,v) = \int_{\alpha_1} ua \frac{\partial v}{\partial \nu} dS, \qquad B(u,v) = \int_{\beta} ua \frac{\partial v}{\partial \nu} dS, \tag{30}$$

where $\partial / \partial \nu$ is taken toward the interior of V_1 on α_1, exterior to it on β. Write

$$A(u) = A(u,u), \qquad B(u) = B(u,u), \tag{31}$$

and let $M(u) = M(u,u)$ stand for the norm (6) over V_1.

LEMMA. *On a regular $\bar{V}_1 \subset R^n$, the function u_0 minimizes $M(u)$, and u_1 minimizes $A(u) + B(u)$, in U_0.*

We shall prove, more generally, that

$$B(u) + (2\lambda - 1)A(u) = \lambda^2 A(u_1) - (1 - \lambda)^2 A(u_0) + M(u - u_\lambda). \tag{32}$$

Thus u_λ minimizes the functional on the left and the minimum is given by the first two terms on the right. We also have an explicit expression for the deviation from this minimum: $M(u - u_\lambda)$.

Proof. Clearly $A(u - u_\lambda) = B(u, u_0) = B(u_1, u) = 0$ for $u \in U_0$. In the decomposition

$$M(u - u_\lambda) = B(u) + B(u_\lambda) - B(u, u_\lambda) - B(u_\lambda, u)$$

we rewrite the last three terms:

$$\begin{aligned}
B(u_\lambda) &= \lambda(1 - \lambda)\, B(u_0, u_1) \\
&= \lambda(1 - \lambda)\, (B(u_0, u_1) - B(u_1, u_0)) \\
&= \lambda(1 - \lambda)\, (A(u_1) - A(u_0)), \\
B(u, u_\lambda) &= \lambda(B(u, u_1) - B(u_1, u)) \\
&= \lambda(A(u_1) - A(u)), \\
B(u_\lambda, u) &= (1 - \lambda)\, (B(u_0, u) - B(u, u_0)) \\
&= (1 - \lambda)\, (A(u) - A(u_0)).
\end{aligned}$$

Here the transfers from β to α_1 are justified by Green's formula. The lemma follows.

§5. Operators for noncompact regions.

13. Now let V_1 be an arbitrary Hölder-bordered region in R^n with border $\alpha_1 = \partial V_1$ and let $\Omega \subset V_1$ be a regular subregion with border $\partial\Omega = \alpha_1 \cup \beta_\Omega$, $\beta_\Omega \subset V_1$. Denote by $u_{\Omega\lambda}$ the function u_λ of No. 12 constructed for Ω. The family $\{u_{\Omega\lambda}\}$ of uniformly bounded functions is normal (cf. No. 4): for every nested exhausting

sequence of the Ω's there is a subsequence $\{\Omega_n\}$ such that the corresponding functions $u_n = u_{\Omega_n\lambda}$ converge to a limiting solution u_λ in V_1. The convergence is uniform on compact subsets of V_1, and hence on those of $\bar{V}_1$, and u_λ belongs to the class U_0 defined by (24), (29) for the noncompact $\bar{V}_1$.

We set

$$B_\Omega(u,v) = \int_{\beta\Omega} ua \frac{\partial v}{\partial \nu}\, dS, \tag{33}$$

$$B(u) = \lim_{\Omega \to V_1} B_\Omega(u), \tag{34}$$

where $B_\Omega(u)$ stands for $B_\Omega(u,u)$; we consider functions u with a finite $B(u)$. For $B_{\Omega_n}(u)$ we write $B_n(u)$ and set

$$F_n(u) = B_n(u) + (2\lambda - 1)A(u). \tag{35}$$

Lemma. *Every limiting function $u_\lambda = \lim_{n\to\infty} u_n$ minimizes the functional*

$$F(u) = B(u) + (2\lambda - 1)\,A(u) \tag{36}$$

in U_0 and the value of the minimum is

$$F(u_\lambda) = \lim_{n\to\infty} F_n(u_n). \tag{37}$$

Proof. We have

$$F(u_\lambda) = \lim_{m\to\infty} F_m(u_\lambda) = \lim_{m\to\infty} \lim_{n\to\infty} F_m(u_n).$$

Moreover, $F_m(u_n) \leqq F_n(u_n)$ for $m \leqq n$, and consequently

$$F(u_\lambda) \leqq \liminf_{n\to\infty} F_n(u_n).$$

On the other hand

$$F_n(u_n) \leqq F_n(u) \leqq F(u)$$

and

$$\limsup_{n\to\infty} F_n(u_n) \leqq \inf_{U_0} F(u) \leqq F(u_\lambda).$$

We conclude that

$$\min_{U_0} F(u) = F(u_\lambda) = \lim_{n\to\infty} F_n(u_n). \tag{38}$$

14. The norm of $u \in U_0$ over $\bar{V}_1$ is

$$M(u) = B(u) - A(u).$$

Theorem. *On an arbitrary Hölder-bordered $\bar{V}_1 \subset R^n$ there are unique functions u_0 and u_1 that minimize in U_0 the functionals $M(u)$ and $A(u) + B(u)$, respectively.*

We shall again derive the more general result

$$B(u) + (2\lambda - 1)\, A(u) = \lambda^2 A(u_1) - (1 - \lambda)^2 A(u_0) + M(u - u_\lambda). \tag{39}$$

Proof. For any limiting function $u_\lambda = \lim u_n$ and for $u \in U_0$ set $h = u - u_\lambda$. Then for $\varepsilon \in R$,

$$F_n(u_\lambda + \varepsilon h) = F_n(u_\lambda) + \varepsilon^2 M_n(h) + \varepsilon I_n, \tag{40}$$

where

$$I_n = B_n(u_\lambda, h) + B_n(h, u_\lambda) + (2\lambda - 1)\, A(u_\lambda, h).$$

If $M(h) < \infty$, the first three terms in (40) have limits as $n \to \infty$ and a fortiori I_n tends to a limit I:

$$F(u_\lambda + \varepsilon h) = F(u_\lambda) + \varepsilon^2 M(h) + \varepsilon I.$$

By the minimum property of u_λ, $dF/d\varepsilon = 0$ for $\varepsilon = 0$ and we infer that $I = 0$. For $\varepsilon = 1$ this gives

$$F(u) = F(u_\lambda) + M(u - u_\lambda),$$

which by (37) is the desired deviation formula (39).

That two minimizing functions are identical follows from

$$M(u' - u'') = F(u') - F(u'') = 0.$$

The uniqueness enables us to introduce the operators L_0, L_1 for the arbitrary Hölder-bordered V_1: $u_\lambda = L_\lambda f$. Due to the uniform convergence of u_n, the operators are normal.

§6. Extremal properties of principal solutions.

15. In this section $V \subset R^n$ is a regular region with border β, (A_{ik}) signifies the adjoint matrix of (a_{ik}), and A is its determinant. Designate by

$$d(x, y) = [A_{ik}(y)(x_i - y_i)(x_k - y_k)]^{1/2} \tag{41}$$

the "elliptic distance" between $x, y \in V$ and set

$$\sigma(x, y) = \frac{(d(x, y))^{2-n}}{(n - 2)\omega_n \sqrt{A(y)}}, \tag{42}$$

where ω_n is the area of the unit n-sphere. (In the case $n = 2$, (42) is replaced by the corresponding logarithmic singularity.)

Let C_a, $C_b \subset V$ be balls with disjoint closures, centered at a, b. The Green's functions of (2) in $\bar{C}_a$, $\bar{C}_b$ with singularities at a, b can be written (up to sign)

$$s(x, a) = \sigma(x, a) + h(x), \tag{43}$$

$$s(x, b) = -\sigma(x, b) + k(x), \tag{44}$$

where for some $\kappa > 0$ and all $i,j = 1,\cdots,n$ the function $h(x)$ has the properties

$$h(x) = O(r^{\kappa+2-n}), \quad \frac{\partial h(x)}{\partial x_i} = O(r^{\kappa+1-n}), \quad \frac{\partial^2 h(x)}{\partial x_i \partial x_k} = O(r^{\kappa-n}) \tag{45}$$

in terms of the Euclidean distance $r = |x - a|$, and $k(x)$ satisfies analogous conditions near b. By definition, $s(x,a) \mid \partial C_a = s(x,b) \mid \partial C_b = 0$. The flux of $s(x,a)$ across ∂C_a is -1 and that of $s(x,b)$ across ∂C_b is 1.

16. For given λ, $\mu \in R$ let $P_{\mu+\lambda}$ be the class of solutions in $\bar{V} - a - b$ with singularities

$$p \mid \bar{C}_a = (\lambda + \mu)s(x,a) + e(x), \tag{46}$$

$$p \mid \bar{C}_b = -(\lambda + \mu)s(x,b) + f(x), \tag{47}$$

where e and f are solutions of (2) in $\bar{C}_a$ and $\bar{C}_b$ respectively, $f(b) = 0$. The flux of p across ∂C_a is $-(\lambda + \mu)$ and that across ∂C_b is $\lambda + \mu$. Let $p_i \in P_1$ be the principal solution determined by L_i $(i = 0, 1)$ and set

$$p_{\mu\lambda} = \mu p_0 + \lambda p_1 \in P_{\mu+\lambda}, \tag{48}$$

$$B(p) = \int_\beta p\, a \frac{\partial p}{\partial \nu}\, dS. \tag{49}$$

Denote by h_i the function h corresponding to p_i.

THEOREM. *The functions p_0 and p_1 minimize the functionals $B(p) - e(a)$ and $B(p) + e(a)$, respectively, in P_1.*

More generally, $p_{\mu\lambda}$ has in $P_{\mu+\lambda}$ the minimum property

$$B(p) + (\lambda - \mu)e(a) = \lambda^2 h_1(a) - \mu^2 h_0(a) + M(p - p_{\mu\lambda}). \tag{50}$$

Proof. We again start with $M(p - p_{\mu\lambda}) = B(p) + B(p_{\mu\lambda}) - B(p, p_{\mu\lambda}) - B(p_{\mu\lambda}, p)$. In analogy with (30) let A_a, A_b stand for integrals taken over ∂C_a and ∂C_b. Then $B(p_{\mu\lambda})$ is the sum of

$$\mu\lambda[A_a(s + h_0, s + h_1) - A_a(s + h_1, s + h_0)]$$

and a similar expression in terms of A_b. Since $s \mid \partial C_a = 0$, and by Green's formula $A_a(h_0, h_1) - A_a(h_1, h_0) = 0$, the only nonvanishing terms are the mean values $A_a(h_i, s) = -h_i(a)$ (e.g. Miranda [5, formulas (6.5), (7.5), and (9.3)]). By virtue of the normalization $k(b) = 0$ the contribution of the A_b-terms vanishes and we obtain

$$B(p_{\mu\lambda}) = \mu\lambda(h_1(a) - h_0(a)).$$

Similarly

$$B(p, p_{\mu\lambda}) = \lambda[(\mu + \lambda)h_1(a) - e(a)],$$

$$B(p_{\mu\lambda}, p) = \mu[e(a) - (\mu + \lambda)h_0(a)],$$

and the theorem follows.

17. If $\mu + \lambda = 0$ the competing class P_0 consists of regular solutions u in V with $u(b) = 0$.

COROLLARY. *The difference* $p_0 - p_1$ *minimizes in* P_0 *the functional* $M(u) - 2u(a)$. *The value of the minimum is* $h_1(a) - h_0(a)$ *and the deviation from the minimum is* $M(u - p_0 + p_1)$.

Using the special case $u = 0$ one obtains an explicit expression for the norm of $p_0 - p_1$:

$$M(p_0 - p_1) = h_0(a) - h_1(a). \tag{51}$$

18. For solutions with finite norm we obtain a bound for $u(a)$:

COROLLARY. *The inequality*

$$(u(a))^2 \leqq M(u)(h_0(a) - h_1(a)) \tag{52}$$

holds for all regular solutions $u \in P_0$.

Proof. Replace u by cu, $c = \text{const.}$, to obtain

$$c^2 M(u) - 2cu(a) = h_1(a) - h_0(a) + M(cu - p_0 + p_1).$$

Since the last term is nonnegative the desired result follows on substituting $c = u(a) / M(u)$.

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