ON THE WEDDERBURN PRINCIPAL THEOREM FOR COMMUTATIVE POWER-ASSOCIATIVE ALGEBRAS

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Introduction. Let A be a strictly power-associative algebra of characteristic not two with radical N and such that the difference algebra $A - N$ is separable. Then we say that A has a Wedderburn decomposition if A has a subalgebra $S \cong A - N$ with $A = S + N$ (vector space direct sum).

Since the characterization of the simple, and hence semisimple, commutative strictly power-associative algebras is nearly complete (see [10]) it is desirable to see if a Wedderburn decomposition can be given for them. In §2 the problem is reduced to the case where A has a unity element and $A - N$ is simple. In §3 we show that if $A - N$ is simple and does not have two as the maximum number of pairwise orthogonal primitive idempotents, then A has a Wedderburn decomposition (Corollary 3.3). The counterexample to a general decomposition theorem (§5) shows that this is the best possible result of that type. Our other major result (Theorem 4.1) is that stable algebras have Wedderburn decompositions.

It is known that associative [1, Theorem 23, p. 47], alternative [12], and Jordan [2], [11] algebras have Wedderburn decompositions. In each of these cases the proof is essentially effected in two stages, namely, $N^2 = 0$ and $N^2 \neq 0$. In our results we do not have anything corresponding to the case $N^2 = 0$ but our basic tools (Lemma 3.2 and Theorem 4.1) are conceptually based on the same idea as the usual induction proof employed in the case $N^2 \neq 0$.

As a matter of terminology, by an algebra we will always mean a finite dimensional vector space with a multiplication defined which satisfies both distributive laws. An algebra A is called power-associative if $x^\alpha x^\beta = x^{\alpha+\beta}$ for all positive integers α and β and every $x \in A$. A is called strictly power-associative if A_K is power-associative for every scalar extension K of the base field. In [7] Kokoris shows that, for commutative algebras of characteristic not 2, 3, or 5, power-associativity and strict power-associativity are equivalent concepts. The radical of a strictly power-associative algebra is the unique maximal nil ideal of A and a non-nil algebra with zero radical is called semisimple. A is called separable if A_K is semisimple for every scalar extension K of the base field.

The basic structure theory of commutative strictly power-associative algebras of characteristic not 2, 3, or 5 was given by Albert in [4] and extended to charac-

Presented to the Society, November 1, 1963; received by the editors July 10, 1963.

teristic not 2 by Kokoris in [7]. Any reference to [4] will thus be understood to imply a reference to the corresponding results in [7]. In particular $A - N$ is semisimple and every semisimple commutative strictly power-associative algebra of characteristic not two has a unity element and is a direct sum of simple ideals.

These results depend on the following well-known decomposition of A. For an idempotent $e \in A$, we have $A = A_e(1) + A_e(1/2) + A_e(0)$ where $x \in A_e(\lambda)$ if and only if $ex = \lambda x$ for $\lambda = 0,\ 1/2, 1$. Moreover $A_e(1)$ and $A_e(0)$ are orthogonal subalgebras of A, $A_e(1/2)A_e(1/2) \subseteq A_e(1) + A_e(0)$, and for $\lambda = 0, 1$ we have $A_e(\lambda)A_e(1/2) \subseteq A_e(1/2) + A_e(1-\lambda)$.

Unless otherwise specified we will understand that the generic symbol A represents a commutative strictly power-associative algebra of characteristic not two such that $A - N$ is separable. We will always let N represent the radical of A and we assume $N \neq 0, A$ since otherwise A has a trivial Wedderburn decomposition.

I would like to thank Professor R. H. Oehmke, who proposed this problem as a doctoral research project at Michigan State University. This paper was written while I held a fellowship from the Institute of Science and Technology at Ann Arbor, Michigan.

I am also indebted to D. J. Rodabaugh who observed that if A_K has a Wedderburn decomposition for some finite scalar extension K of F, then A has a Wedderburn decomposition. For if B_0 is a subalgebra of A_K such that $B_0 \cong A_K - N_K = (A-N)_K$ then the remainder of the proof is just as in [1, p. 48] since the associativity of A was not used there and the use of $NN = 0$ was not necessary if we use the fact that $\xi_i \xi_j = \sum_{k=1}^{r} \alpha_{ijk}\xi_k$ with $\alpha_{ijk} \in F$.

This observation applies to most of my results but it should specially be noted relative to those that have a restriction concerning nodal subalgebras.

1. **Pairwise orthogonal idempotents.** Based upon and related to the decomposition of A by a single idempotent, Albert has given in [4, §5] a decomposition of A relative to a set of pairwise orthogonal idempotents $e_1, e_2, \cdots, e_t$ for which $e_1 + e_2 + \cdots + e_t$ is a unity element of A. It is shown that we can write A in a vector space direct sum $A = \sum_{i \leq j} A_{ij}$ for $i, j = 1, 2, \cdots, t$ where $A_{ii} = A_{e_i}(1)$, and $A_{ij} = A_{ji} = A_{e_i}(1/2) \cap A_{e_j}(1/2)$ when $i \neq j$. Moreover, if $g = e_i + e_j$ for $i \neq j$, then g is an idempotent with $A_g(1) = A_{ii} + A_{ij} + A_{jj}$, $A_g(1/2) = \sum_{k \neq i,j}(A_{ik} + A_{jk})$, and $A_g(0) = \sum_{k,l \neq i,j} A_{kl}$. For i, j, k, l distinct we have

$$\begin{aligned} A_{ii}^2 &\subseteq A_{ii}, \\ A_{ii}A_{ij} &\subseteq A_{ij} + A_{jj}, \\ A_{ii}A_{jj} &= A_{ij}A_{kl} = A_{ii}A_{kl} = 0, \\ A_{ij}A_{jk} &\subseteq A_{ik}, \\ A_{ij}^2 &\subseteq A_{ii} + A_{jj}. \end{aligned}$$

Since these relations are basic to much of our work we will generally use them without specific reference.

Also related to pairwise orthogonal idempotents we have the following lemma.

(1.1) LEMMA. *Let $[u_1], [u_2], \cdots, [u_t]$ be pairwise orthogonal idempotents in $A - M$, M a nil ideal of A, and let $u = u_1 + u_2 + \cdots + u_t$. Then there exists an idempotent e and pairwise orthogonal idempotents $e_1, e_2, \cdots, e_t$ such that $e = e_1 + e_2 + \cdots + e_t$, $[e] = [u]$, and $[e_i] = [u_i]$* for $i = 1, 2, \cdots, t$. *Moreover, if A has* 1 *as a unity element and $[1] = [u]$, then $e = 1$.*

Proof. The proof of the first part of the lemma is by induction and the proof of the case $t = 1$ is that of Lemma 1 of [2, p. 1].

Let $w = u_1 + u_2$ for $t \geqq 2$. Then $u = w + u_3 + \cdots + u_t$ for pairwise orthogonal idempotents $[w], [u_3], \cdots, [u_t]$. By the induction hypothesis there exists an idempotent e and pairwise orthogonal idempotents $f, e_3, \cdots, e_t$ such that $e = f + e_3 + \cdots + e_t$, $[e] = [u]$, $[f] = [w]$, and $[e_i] = [u_i]$ for $i = 3, \cdots, t$. If $[f][x] = [x]$ for $x \in A$ we can write $x = x_1 + x_{1/2} + x_0$ with $x_\lambda \in A_f(\lambda)$ and have $[x_1] + [x_{1/2}] + [x_0] = [x] = [f][x] = [x_1] + 1/2[x_{1/2}]$. Hence $[x] = [x_1]$ if $x \in A$ such that $[f][x] = [x]$. Now $[f][u_1] = ([u_1] + [u_2])[u_1] = [u_1]$ so there exists an element $x_1 \in A_f(1)$ such that $[x_1] = [u_1]$. Moreover x_1 is not nilpotent since $[u_1]$ is not. Hence the associative algebra $F[x_1] \subseteq A_f(1)$ is not nilpotent and so contains an idempotent $e_1 = g(x_1)$ for $g \in F[x_1]$. Thus $[e_1] = [g(x_1)] = \alpha[x_1]$ for $\alpha = g(1) \in F$ and $\alpha[x_1] = [e_1] = [e_1]^2 = \alpha^2[x_1]^2 = \alpha^2[x_1]$. But $e_1 \notin M$ so $\alpha[x_1] \neq 0$. Thus $\alpha = 1$ and $[e_1] = [x_1] = [u_1]$. Now $e_2 = f - e_1$ is an idempotent in $A_f(1)$; $e_2 e_1 = (f - e_1)e_1 = e_1 - e_1 = 0$, $[e_2] = [f - e_1] = [f] - [e_1] = [w] - [u_1] = [u_2]$, and since e_1 and $e_2 \in A_f(1)$ they are orthogonal to e_i for $i = 3, \cdots, t$. Thus $e = f + e_3 + \cdots + e_t = e_1 + e_2 + e_3 + \cdots + e_t$ where the e_t are pairwise orthogonal idempotents with $[e] = [u]$ and $[e_i] = [u_i]$ for $i = 1, 2, \cdots, t$.

For $1 \in A$, $1 - (e_1 + e_2 + \cdots + e_t)$ is either zero or an idempotent of A. But $[1] = [u_1] + \cdots + [u_t] = [e_1] + \cdots + [e_t]$ means that $1 - (e_1 + \cdots + e_t) \in M$, so it is nilpotent. Hence it is zero and $1 = e_1 + e_2 + \cdots + e_t$ as desired.

As a consequence of Lemma 1.1 we immediately have Corollary 1.2.

(1.2) COROLLARY. *If M is a nil ideal of A, then A has t pairwise orthogonal idempotents if and only if $A - M$ has t pairwise orthogonal idempotents.*

2. **Reduction to A with unity and $A - N$ simple.** Let $\mathscr{A}$ be the class of all commutative strictly power-associative algebras A that have a Wedderburn decomposition and for which $A - N$ is simple.

(2.1) THEOREM. *Let A be a commutative strictly power-associative algebra of characteristic not two so that $A - N = B_1 \oplus \cdots \oplus B_t$ where B_i is simple and has a unity element $[u_i]$. Let e_i be as in Lemma* 1.1. *Then A has a Wedderburn decomposition if and only if $A_{e_i}(1)$ is in $\mathscr{A}$ for $i = 1, 2, \cdots, t$.*

Proof. Let $e = e_1 + e_2 + \cdots + e_t$ as in Lemma 1.1 and let $A_1 = A_{e_1}(1)$, $A_{12} = A_{e_1}(1/2)$, and $A_2 = A_{e_1}(0)$. Also let R_i be the radical of A_i and $N_i = N \cap A_i$ for $i = 1, 2$.

(2.2) Remarks. When B is a subspace of A, then $B - N$ is the subspace of $A - N$ consisting of all classes $[b]$ for $b \in B$. When B is a subalgebra of A, then $B - N$ is a subalgebra of $A - N$ and is isomorphic to $B - N_b$ where $N_b = N \cap B$. We also remark that for a nil ideal M of A, $A - M$ is semisimple if and only if M is the radical of A.

Thus we have $A_1 - N_1 \cong A_1 - N \cong B_1$, $A_{12} - N = 0$, and

$$A_2 - N_2 \cong A_2 - N \cong B_2 \oplus \cdots \oplus B_t.$$

So by (2.2) $N_i = R_i$ for $i = 1, 2$. Also $A_{12} \subseteq N$.

First assume $A = S + N$ is a Wedderburn decomposition of A and let $S_1 = S \cap A_1$. One then easily sees that $A_1 = S_1 + N_1$ which is a Wedderburn decomposition for $A_1 = A_{e_1}(1)$. The same argument holds for $A_{e_i}(1)$ for $i = 2, \cdots, t$.

The sufficiency of the condition is proved by induction on t. For $t = 1$, $A_2 - N = 0$ so $A_{12} + A_2 \subseteq N$ and since A_1 is in $\mathscr{A}$ it has a Wedderburn decomposition, say $A_1 = S_1 + N_1$. Then $A = S_1 + N$ is a Wedderburn decomposition for A.

If $t > 1$, then $A_2 - N_2 \cong B_2 \oplus \cdots \oplus B_t$ where $[u_i] = [e_i]$ is the unity element of B_i for $i = 2, \cdots, t$. Moreover $(A_2)_{e_i}(1) = A_{e_i}(1)$ is in $\mathscr{A}$ so by the induction hypothesis A_2 has a Wedderburn decomposition, say $A_2 = S_2 + N_2$. Then $A = (S_1 \oplus S_2) + N$ is a Wedderburn decomposition of A.

3. $A - N$ simple and of degree other than two.

(3.1) Theorem. *Let A be a commutative strictly power-associative algebra with a unity element and of characteristic not two such that A has three pairwise orthogonal idempotents and $A - N$ is simple. Then A has a Wedderburn decomposition.*

Proof. The proof is by induction on n, the dimension of A, so $n \geqq 3$ since A has three pairwise orthogonal idempotents. The theorem is trivial if $n = 3$. We now give a lemma to accomplish the induction step.

(3.2) Lemma. *If A, of dimension n, is as in the theorem, if A has a proper ideal $M \neq N$, and if every algebra as in the theorem and of dimension less than n has a Wedderburn decomposition, then A has a Wedderburn decomposition.*

Proof. For convenience we will write $d(B)$ for the dimension of a subspace B.

Assume first that M is nil. Now $A - N \cong (A - M) - (N - M)$ so $N - M$ is the radical of $A - M$. Also $(A - M) - (N - M)$ is simple since $A - N$ is simple and $A - M$ has a unity element since A has one. And by Corollary 1.2, $A - M$ has three pairwise orthogonal idempotents. But M is a proper ideal of A and we have $d(A-M)<n$ so by hypothesis $A-M$ has a subalgebra C_0 such that $C_0 \cong (A-M)-(N-M) \cong A-N$. Therefore A has a subalgebra $C_1 \neq 0$, A such that $M \subset C_1$ (that is $M \subseteq C_1$ and $M \neq C_1$) and $C_0 \cong C_1 - M$. Thus we have a proper subalgebra C_1 of A such that $C_1 - M \cong A - N$. Similar to the considerations for $A-M$ above we see that M is the radical of C_1, $C_1 - M$ is simple, C_1 has three pairwise orthogonal idempotents, and $d(C_1)<n$. So by hypothesis C_1 has a subalgebra $C \cong C_1 - M$. Thus C is a subalgebra of A such that $C \cong A - N$. But $C \cap N$ is a nil ideal of C so $C \cap N = 0$ since $C \cong A - N$ which is simple. Therefore $C+N$ is a subspace of A with $d(C+N) = d(C) + d(N)$ $= d(A-N) + d(N) = d(A)$. So $A = C + N$ and this is a Wedderburn decomposition for A.

If M is not nil, then $M \nsubseteq N$. Also $N \nsubseteq M$ since $A-N$ is simple. If $M \cap N \neq 0$, then it is a proper nil ideal of A different from N so by the last paragraph A has a Wedderburn decomposition. If $M \cap N = 0$, then $M+N$ is an ideal of A and $(M+N)-N$ is a nonzero ideal of $A-N$. But $A-N$ is simple so $A = M+N$ with $M \cap N = 0$. This is a Wedderburn decomposition of A and completes the proof of Lemma 3.2.

The remainder of the proof of the theorem simply amounts to repeated applications of the lemma to various ideals of A until we have reduced A to an algebra for which we can give a Wedderburn decomposition. Because this process is long and needs some preliminaries we have postponed it until §6.

Theorem 3.1 concerns the case where A has three pairwise orthogonal idempotents. If, on the other hand, the unity element 1 of A is a primitive idempotent (i.e. $1 \neq e_1 + e_2$ for orthogonal idempotents e_1 and e_2) then as in [4, pp. 526–527] $A = 1 \cdot F + M$ where M is nil. If the characteristic of A is zero, then it was shown that M is a subalgebra of A and hence it is the radical so we have a Wedderburn decomposition of A. More generally if A has no nodal subalgebras (i.e. a subalgebra $B = e \cdot F + R$ where e is the unity of B and R is nil but *not* a subalgebra of B), then $1 \cdot F + M$ is a Wedderburn decomposition of A.

Combining this observation with Theorems 2.1 and 3.1 we have an immediate corollary.

(3.3) COROLLARY. *Let A be a commutative strictly power-associative algebra of characteristic not two and having no nodal subalgebras. Suppose further that any B_i, having two pairwise orthogonal idempotents, has three where $A - N = B_1 \oplus \cdots \oplus B_t$ with B_i simple. Then A has a Wedderburn decomposition.*

4. **Stable algebras.** An algebra A is stable with respect to an idempotent e if $A_e(\lambda)A_e(1/2) \subseteq A_e(1/2)$ for $\lambda = 0, 1$ and it is stable if it is stable with respect to each of its idempotents.

Since part of the proof of Theorem 4.2 can be generalized without any extra work we give that part separately.

Let P be a property of algebras such that if A has property P then each of its subalgebras has property P. Let $\mathscr{P}$ be the class of all commutative strictly power-associative algebras of characteristic not two having property P with $A - N$ separable for A in $\mathscr{P}$.

(4.1) THEOREM. *Every algebra in $\mathscr{P}$ has a Wedderburn decomposition if and only if every algebra in $\mathscr{P}$ that has at most two pairwise orthogonal idempotents has a Wedderburn decomposition.*

Proof. The necessity of the condition is obvious so we assume that every algebra in $\mathscr{P}$ that has at most two pairwise orthogonal idempotents has a Wedderburn decomposition. Thus we take $n = d(A) \geqq 3$ and assume that every algebra of $\mathscr{P}$ with dimension less than n has a Wedderburn decomposition. Evidently we can assume A has three pairwise orthogonal idempotents. If $A - N$ is simple, then A has a Wedderburn decomposition by Theorem 3.1 (we can assume A has a unity as shown in the proof of Theorem 4.2).

Thus we can assume the existence of a non-nil proper ideal D of A. So D has an idempotent and hence a principle idempotent, say e (e is principle if $A_e(0)$ is nil). Write $D = D_e(1) + D_e(1/2) + D_e(0)$ and let M be the radical of D. According to Albert [4, Theorem 7, p. 524] $D_e(1/2) + D_e(0) \subseteq M$ since e is principle. We write $M = J + D_e(1/2) + D_e(0)$ where $J = M \cap D_e(1)$ and $A = A_e(1) + A_e(1/2) + A_e(0)$. We can now proceed as in [4, p. 525] to show that M is an ideal of A. Thus $M \subseteq N$.

Now $D \neq 0, A$ so $0 < d(D) < n$ and by the induction hypothesis $D = T + M$ where T is a semisimple subalgebra of D (and hence of A) and $T \cap M = 0$. Thus $T \subseteq D_e(1)$ and $D_e(1) = T + J$ is a Wedderburn decomposition of $D_e(1)$. Likewise $D \neq 0, A$ means that $0 < d(A_e(0)) < n$ so $A_e(0) = S + N_0$ where S is a semisimple subalgebra of $A_e(0)$ (and hence of A), N_0 is the radical of $A_e(0)$, and $S \cap N_0 = 0$. Note that $D_e(0) \subseteq N_0$ since $D_e(0)$ is a nil ideal of $A_e(0)$. Let $N_a = J + A_e(1/2) + N_0$. Then $N \subseteq N_a$ and just as in the proof of Theorem 2.1 $N_0 = N \cap A_e(0)$. But $S \subseteq A_e(0)$ and $T \subseteq A_e(1)$ are semisimple subalgebras of A so $S \oplus T$ is a semisimple subalgebra of A. Moreover $(S \oplus T) \cap N = 0$ since $S \cap N = S \cap N_0 = 0$ and $T \cap N = T \cap J = 0$. Hence $A = A_e(1) + A_e(1/2) + A_e(0) = T + (J + A_e(1/2) + N_0) + S = (S \oplus T) + N$ is our desired Wedderburn decomposition of A.

(4.2) THEOREM. *If A is a stable commutative strictly power-associative algebra with no nodal subalgebras and with characteristic not two, then A has a Wedderburn decomposition.*

Proof. Let P be the property of being stable, having no nodal subalgebras, and having characteristic not two. By Theorem 4.1 we can then assume A has at most two pairwise orthogonal idempotents.

Since A is non-nil it has a principle idempotent, say e. Then by [4, Theorem 7, p. 524] $A_e(1/2) + A_e(0) \subseteq N$. Let R_1 be the radical of $A_e(1)$, $N_1 = N \cap A_e(1)$, and $M = R_1 + A_e(1/2) + A_e(0)$. Clearly $N \subseteq M$ and since R_1 is an ideal of $A_e(1)$ it is easily seen that M is an ideal of A. If $x \in M$, then $x = a + n$ for some $a \in R_1$ and $n \in N$. Thus $x^2 \in a^2 + N$ and by induction $x^k \in a^k + N$ for every positive integer k. But a is nilpotent so for some k, $x^k \in N$, x^k is nilpotent, and M is a nil ideal of A. Thus $M = N$ and $R_1 = N_1$. So if $A_e(1)$ has a Wedderburn decomposition, say $A_e(1) = S + N_1$, then $A = S + N$ is a Wedderburn decomposition for A. So without loss of generality we can assume A has a unity element 1 to begin with.

Suppose that A does not have two orthogonal idempotents. Then 1 is a primitive idempotent. With this and the assumption of no nodal subalgebras we can use the first part of the proof of Theorem 9 [4, pp. 526–527] to conclude that

$$A = 1 \cdot F + N$$

is a Wedderburn decomposition of A.

Thus we can assume $1 = u + v$ for primitive orthogonal idempotents u and v. Then $A = A_1 + A_{12} + A_2$ as in §1, where we are letting $A_{11} = A_1$ and $A_{22} = A_2$, and as above $A_1 = uF + R_1$ and $A_2 = vF + R_2$ where R_i is the radical of A_i. Let $N_i = N \cap A_i$ and $N_{12} = N \cap A_{12}$ as usual.

Let $x \in A_{12}$. If $x^2 \notin R_1 + R_2$, then x is said to be nonsingular and it is known [4, Lemma 10, p. 517] that $x^2 = \alpha + g$ for $g \in R_1 + R_2$ and α a nonzero element of F. If $x^2 \in R_1 + R_2$, then x is said to be singular.

Suppose every element in A_{12} is singular. If $x, y \in A_{12}$, then

$$2xy = x^2 + y^2 - (x-y)^2 \in R_1 + R_2$$

so $A_{12}^2 \subseteq R_1 + R_2$. Let $M = R_1 + A_{12} + R_2$. Then by stability

$$AM \subseteq A_1R_1 + A_{12} + A_{12}^2 + A_2R_2 \subseteq M.$$

Moreover M is nil, for if not, then M has an idempotent $f = f_1 + f_{12} + f_2$ with $f_i \in R_i$ and $f_{12} \in A_{12}, f_{12} \neq 0$. Computing $f^2 = f$ and equating the components in A_{12} we get $(f_1 + f_2)f_{12} = f_{12}$. Let T be the linear transformation given by $T(x) = xf_{12}$ for all $x \in A_1 + A_2$. Then it is known [4, p. 517] that T is nilpotent. But $(f_1 + f_2)f_{12} = f_{12}$ means that $T^k(f_1 + f_2) = f_{12}$ for every positive integer k. Thus $f_{12} = 0$ which is a contradiction. Therefore M is a nil ideal, $M \subseteq N$, $M = N$, $R_i = N_i$, $A_{12} \subseteq N$, and $A = (uF + vF) + N$ is a Wedderburn decomposition of A.

Thus we can assume there is a nonsingular element $x \in A_{12}$. Let

$$M = R_1 + R_1A_{12} + R_2A_{12} + R_2.$$

Then the proof by Albert in [5, ending on p. 331] that M is an ideal is valid here since the only use of simplicity there was to obtain a nonsingular element in A_{12}. As in the last paragraph M is nil so $M \subseteq N$ and $R_i = N_i$.

REMARK. The assumptions on A in [5] are more restrictive than ours but one sees that the loss of algebraic closure there is repaired by our assumption that A has no nodal subalgebras. Because of [7] the only difficulty in assuming A has characteristic not two occurs in proving Lemma 5 [5, p. 326] when A has characteristic three. But then taking $x = y = w$ in formula (5) of [7, p. 364] we get $(wz)w = (((wz)w)w)w$ which enables us to prove the lemma as before.

For our uses we state the pertinent parts of Lemmas 3 and 7 of [5].

(4.3) LEMMA. *If x is a nonsingular element of A_{12}, then there exists a quantity $c \in F[x^2] \subseteq A_1 + A_2$ such that $w^2 = 1$ for $w = cx \in A_{12}$. Moreover $A_{12} = wB \dotplus G$ where $B = \{b \in A_1 + A_2 : w(wb) = b\}$ and $G = \{g \in A_{12} : gw = 0\}$.*

REMARKS. There are some comments that need to be made regarding this lemma.

The first comment deals with notation. In the rest of this section and in §6 we will use $B + C$ to indicate the sum of the subspaces B and C, whereas before it indicated the *direct* sum. If, as in the lemma, we wish to emphasize that the sum is direct we will use the dot over the plus sign.

Next we would like to indicate briefly how we intend to use Lemma 4.3 to construct a Wedderburn decomposition for A. Let $w = w_1$. Then we will show that we can keep "breaking elements w_i out of G" where $w_i w_j = \delta_{ij}$ (the Kronecker delta) until what remains of G is a set of singular elements $G_{(m)} \subseteq N_{12}$. From this we see that $A = (uF + w_1F + \cdots + w_mF + vF) + N$ is a Wedderburn decomposition of A.

Finally one easily sees that $B = \{\alpha + b : \alpha \in F$ and $b \in N_1 + N_2$ such that $w(wb) = b\}$. In particular this means that $wB = \{\alpha w + wb : \alpha \in F$ and $b \in N_1 + N_2$ such that $w(wb) = b\}$. The importance in this for us is that $wB \subseteq wF \dotplus N_{12}$.

Let $e = 1/2(1 + w)$. Then e is an idempotent and for $x \in A$, $ex = 1/2(1 + w)x = 1/2x$ if and only if $wx = 0$. Therefore w is in the annihilator of $A_e(1/2)$. More importantly we see that $G = A_{12} \cap A_e(1/2)$. And since A is stable, it is evident that $[A_f(1/2)]^{2m-1} \subseteq A_f(1/2)$ for any idempotent f and every positive integer m. Thus $G^{2m-1} \subseteq G$ for every positive integer m.

If z is a nonsingular element in G, then, according to Lemma 4.3, there is a quantity $c \in F[z^2]$ such that $y^2 = 1$ for $y = cz$. But then

$$y = \alpha_1 z + \alpha_2 z^3 + \cdots + \alpha_k z^{2k-1}$$

and by the last paragraph $z^{2m-1} \in G$ for every positive integer m so $y \in G$ and $wy = 0$. Applying Lemma 4.3 with respect to u and then with respect to e, we can write $A_{12} = yB_y \dotplus G_y$ and $A_e(1/2) = yB_{y1} \dotplus G_{y1}$ where

$$B_y = \{b \in A_1 + A_2 : y(yb) = b\}, \quad G_y = \{g \in A_{12} : gy = 0\},$$

$$B_{y1} = \{b \in A_e(1) + A_e(0): y(yb) = b\},$$

and

$$G_{y1} = \{g \in A_e(1/2): gy = 0\}.$$

(4.4) LEMMA. *For $y \in G$ with $y^2 = 1$ we know that every element $h \in A_e(1/2)$ has a unique representation in the form $h = yb + g$ for $yb \in yB_{y1}$ and $g \in G_{y1}$. But for $h \in G$ we also have $yb \in yB_y$ and $g \in G \cap G_y$.*

Proof. $G_{y1} \subseteq A_e(1/2)$ so $gw = 0$ as noted above. But we have $h \in A_{12}$ so $(yb)_1 + (yb)_2 + g_1 + g_2 = 0$ where the subscripts refer to the subspaces A_1, A_2, and A_{12}. Examining the $A_1 + A_2$ component of the equation $0 = wg = w(g_1 + g_2) + wg_{12}$ we have $wg_{12} = 0$ since A is stable. Similarly $yg_{12} = 0$. Thus $g_{12} \in G \cap G_y$.

Since A is stable $(yb)_{12} = [y(b_1 + b_{12} + b_2)]_{12} = y(b_1 + b_2)$. Thus

$$b_1 + b_2 \in (A_1 + A_2) \cap (A_e(1) + A_e(0))$$

such that $y[y(b_1 + b_2)] = b_1 + b_2$, so $y(b_1 + b_2) \in yB_y \cap yB_{y1}$. Therefore $h = (yb)_{12} + g_{12}$ where $(yb)_{12} \in yB_{y1}$ and $g_{12} \in G_{y1}$. But h has a unique representation in that form; namely, $h = yb + g$ so we must have $yb = (yb)_{12} \in yB_y$ and $g = g_{12} \in G \cap G_y$ which proves the lemma.

Previous to Lemma 4.3 we had reached the point where A_{12} had a nonsingular element and $N_i = R_i$. We can now put the intermediate pieces together by induction to give a Wedderburn decomposition for A.

By Lemma 4.3 A_{12} contains an element w_1 such that $w_1^2 = 1$ and

$$A_{12} = w_1 B_1 \dotplus G_1$$

where $B_1 = \{\alpha + b: \alpha \in F$ and $b \in N_1 + N_2$ such that $w_1(w_1 b) = b\}$ and $G_1 = \{g \in A_{12}: gw_1 = 0\}$.

If every element of G_1 is singular then let $M_1 = N + G_1$. For $x = n + g \in M_1$, $x^2 = n^2 + 2ng + g^2 \in N$ so x^2 is nilpotent, x is nilpotent, and M_1 is nil. In particular for $x, y \in G_1$ we have $2xy = x^2 + y^2 - (x - y)^2 \in N$ so $G_1^2 \subseteq N$. Thus $A_{12}M_1 \subseteq N + A_{12}G_1 \subseteq N + (w_1 F + N + G_1)G_1 \subseteq N + G_1^2 \subseteq N \subseteq M_1$ and $A_1 M_1 \subseteq u(N + G_1) + N_1(N + G_1) \subseteq N + G_1 = M_1$. Likewise $A_2 M_1 \subseteq M_1$ so M_1 is a nil ideal of A. Hence $G_1 \subseteq N$ and $A = (uF + w_1 F + vF) + N$ is a Wedderburn decomposition of A. Thus we can continue by assuming G_1 has a nonsingular element.

For notation in the general case we will have $w_i \in A_{12}$ with $w_i^2 = 1$ and will write $A_{12} = w_i B_i \mathring{+} G_i$ by Lemma 4.3 where B_i and G_i are defined in terms of w_i as in the case $i = 1$ above. Let $e_i = 1/2(1 + w_i)$.

Assume that $A_{12} = w_1 F + \cdots + w_{m-1} F + N_{12} + G_{(m-1)}$ where

$$G_{(m-1)} = \bigcap_{i=1}^{m-1} G_i,$$

$G_{(m-1)}$ has a nonsingular element x, and $w_i w_j = \delta_{ij}$ for $i, j = 1, 2, \cdots, m - 1$.

From Lemma 4.3, as before, there is an element w_m in $G_{(m-1)}$ such that $w_m^2 = 1$ and $w_m w_i = 0$ for $i = 1, 2, \cdots, m-1$. Let $G_{(m)} = G_m \cap G_{(m-1)}$. Then we wish to show that we can write $A_{12} = w_1 F + \cdots + w_m F + N_{12} + G_{(m)}$.

Let h be in $G_{(m-1)}$. Then $h \in G_i$ for each $i = 1, 2, \cdots, m-1$ so taking $G = G$ and $y = w_m$ in Lemma 4.4 the element h has a unique representation in the form $h = w_m b_i + g_i$, $i = 1, 2, \cdots, m-1$, where $w_m b_i \in w_m B_m$ and $g_i \in G_i \cap G_m$. But by Lemma 4.3, h also has the unique representation $h = w_m b + g$ for $w_m b \in w_m B_m$ and $g \in G_m$. Thus $g_i = g$ for $i = 1, 2, \cdots, m-1$ so $g \in G_{(m)}$ as desired. For if $a \in A_{12}$, we have $a = \alpha_1 w_1 + \cdots + \alpha_{m-1} w_{m-1} + n_{m-1} + h$ where n_{m-1} is in N_{12} and h is in $G_{(m-1)}$. But by our last result we can write this as

$$\begin{aligned} a &= \alpha_1 w_1 + \cdots + \alpha_{m-1} w_{m-1} + n_{m-1} + (\alpha_m w_m + n_m + g) \\ &= \alpha_1 w_1 + \cdots + \alpha_{m-1} w_{m-1} + \alpha_m w_m + n + g, \end{aligned}$$

with $n \in N$ and $g \in G_{(m)}$ as desired.

This inductive process cannot continue indefinitely since $d(G_{(m)}) < d(G_{(m-1)})$ so for some m, $G_{(m)}$ must consist of singular elements. Then as before $G_{(m)} \subseteq N$ and $A = (uF + w_1 F + \cdots + w_m F + vF) + N$ is a Wedderburn decomposition of A.

5. **Example.** Let A be the 6-dimensional commutative algebra with basis $e_{11}, e_{12}, e_{21}, e_{22}, m, n$ and multiplication table $e_{11}^2 = e_{11}$, $e_{22}^2 = e_{22}$, $e_{11}e_{12} = e_{22}e_{12} = 1/2e_{12}$, $e_{11}e_{21} = e_{22}e_{21} = 1/2e_{21}$, $e_{11}n = e_{12}m = n$, $e_{22}m = e_{21}n = m$, $e_{12}e_{21} = 1/2(e_{11} + e_{22} + m + n)$, and all other products zero.

If we restrict A to have a base field F of characteristic not 2, 3, or 5 and carry out the computation of Lemma 4 in [3, p. 554], we find that A is power-associative since A is commutative by definition.

The radical N of A is spanned by m and n, $N^2 = 0$, and $A - N \cong F_2^+$ with basis $[e_{11}], [e_{12}], [e_{21}], [e_{22}]$ where F_2 is the algebra of all 2 by 2 matrices over F. Suppose A had a subalgebra $S \cong A - N$. Then S would have the usual matrix basis $g_{11}, g_{12}, g_{21}, g_{22}$ for F_2^+ and there would be an automorphism σ of $A - N$ such that $\sigma([e_{ij}]) = [g_{ij}]$. But this is a change of basis for the 2 by 2 matrices so there is a nonsingular element $[y] = \alpha[e_{11}] + \beta[e_{12}] + \gamma[e_{21}] + \delta[e_{22}]$ in $A - N$, with $\Delta = \alpha\delta - \beta\gamma \neq 0$, such that $[g_{ij}] = [y] \circ [e_{ij}] \circ [y]^{-1}$ (note that this multiplication takes place in F_2). But

$$[y]^{-1} = \Delta^{-1}(\delta[e_{11}] - \beta[e_{12}] - \gamma[e_{21}] + \alpha[e_{22}])$$

so computing $[g_{ij}] = [y] \circ [e_{ij}] \circ [y]^{-1}$ we have

$$\begin{aligned} g_{11} &= \Delta^{-1}(\alpha\delta e_{11} - \alpha\beta e_{12} + \gamma\delta e_{21} - \beta\gamma e_{22} + \varepsilon_1 m + \varepsilon_2 n), \\ g_{12} &= \Delta^{-1}(-\alpha\gamma e_{11} + \alpha^2 e_{12} - \gamma^2 e_{21} + \alpha\gamma e_{22} + \theta_1 m + \theta_2 n), \\ g_{21} &= \Delta^{-1}(\beta\delta e_{11} - \beta^2 e_{12} + \delta^2 e_{21} - \beta\delta e_{22} + \lambda_1 m + \lambda_2 n), \\ g_{22} &= \Delta^{-1}(-\beta\gamma e_{11} + \alpha\beta e_{12} - \gamma\delta e_{21} + \alpha\delta e_{22} + \pi_1 m + \pi_2 n). \end{aligned}$$

Equating coefficients of m and n in the products $g_{ij}g_{kl}$ (for example the coefficients of m and n in $g_{11}g_{12}$ and $1/2g_{12}$ are equal since $g_{11}g_{12} = 1/2g_{12}$) yields equations in $\alpha, \beta, \gamma, \delta, \varepsilon_1, \varepsilon_2, \cdots, \pi_1, \pi_2$ which force $\Delta = 0$. But this is a contradiction so A has no subalgebra $S \cong A - N$ and hence A has no Wedderburn decomposition.

This example of course shows we can not prove the Wedderburn Principal Theorem for the class of all commutative strictly power-associative algebras. Moreover it shows that one needs more than a restriction on the base field, for in our example the base field is arbitrary other than the restriction that the characteristic not be 2, 3, or 5.

In connection with Theorem 4.2 we note that by using Theorem 2 of [9, p. 698] we were able to show that the above example is not stable with respect to any idempotent.

6. **Completion of the proof of Theorem 3.1.** Before continuing the proof we need a few preliminaries.

The linearization of $x^2x^2 = (x^2x)x$ gives

$$\begin{aligned} &4[(xy)(zw) + (xz)(yw) + (xw)(yz)] \\ &\quad = x[y(zw) + z(wy) + w(yz)] + y[x(zw) + z(wx) + w(xz)] \\ &\quad\quad + z[x(yw) + y(wx) + w(xy)] + w[x(yz) + y(zx) + z(xy)]. \end{aligned} \tag{6.1}$$

We will also make use of (5) and (8) of [4, pp. 505–506]. We state them as

$$[w_{1/2}(x_1y_1)]_{1/2} = [(w_{1/2}x_1)y_1 + (w_{1/2}y_1)x_1]_{1/2}, \tag{6.2}$$

$$[w_{1/2}(x_1y_1)]_0 = 2[(w_{1/2}x_1)y_1 + (w_{1/2}y_1)x_1]_0, \tag{6.3}$$

$$[(w_{1/2}y_1)x_0]_1 = 1/2[(w_{1/2}x_0)y_1]_1, \tag{6.4}$$

where z_λ, $\lambda = 0, 1/2, 1$, is the $A_e(\lambda)$ component of z; e an idempotent.

Before continuing, we need to explain some new notation we will use. Recall that in this section $B + C$ will only indicate the sum (not necessarily the direct sum) of the vector spaces B and C.

We have previously used the product BC but it is too restrictive for our purposes now so we introduce a new product, $B \circ C$, of the subspaces B and C. Since A has a unity element, denoted by 1, and three pairwise orthogonal idempotents we can write $1 = e_1 + e_2 + e_3$ where the e_i are pairwise orthogonal idempotents. Then as in §1 A has a corresponding decomposition as $A = \sum_{i \leq j} A_{ij}$, $i, j = 1, 2, 3$. We define $B \circ C = \sum_{i \leq j}(BC)_{ij}$ where $x \in (BC)_{ij}$ if and only if there exists an element $y \in BC$, $y = \sum_{i \leq j} y_{ij}$, such that $x = y_{ij}$. We write $B \circ B = B^{(2)}$. Evidently $BC \subseteq B \circ C$ but it may happen that $B \circ C \nsubseteq BC$. But if BC is an ideal of A, then $BC = B \circ C$ (this can easily be seen by making appropriate linear combinations and multiplications by the e_i; for example, $e_1(2e_1y - y) = y_{11}$). Since we are only interested in using the product of subspaces to construct ideals, we will use the product $B \circ C$ since it is easier to work with and may in fact be an ideal even though BC is not.

(6.5) LEMMA. *For i, j, k distinct we have*
(a) $A_{ii}(A_{ij} \circ A_{jk}) \subseteq (A_{ii}A_{ij}) \circ A_{jk}$,
(b) $A_{ik}A_{ij}^{(2)} \subseteq (A_{ik}A_{ij}) \circ A_{ij}$,
(c) $A_{ij}(A_{ik} \circ A_{jk}) \subseteq A_{ik}^{(2)} + A_{jk}^{(2)}$,
(d) $A_{ii}A_{ij}^{(2)} \subseteq (A_{ii} \circ A_{ij}) \circ A_{ij}$.

Proof. Let $g = e_i + e_j$. Then $A_g(1) = A_{ii} + A_{ij} + A_{jj}$, $A_g(1/2) = A_{ik} + A_{jk}$, and $A_g(0) = A_{kk}$ as in §1. By (6.2) we have $[w_{jk}(x_{ii}y_{ij})]_{1/2} = [(x_{ii}w_{jk})y_{ij} + x_{ii}(y_{ij}w_{jk})]_{1/2} = [x_{ii}(y_{ij}w_{jk})]_{1/2}$ since $x_{ii}w_{jk} = 0$. From (6.3) we get $[w_{jk}(x_{ii}y_{ij})]_0 = 2[x_{ii}(y_{ij}w_{jk})]_0$. So $x_{ii}(y_{ij}w_{jk}) = [w_{jk}(x_{ii}y_{ij})]_{ik} + [w_{jk}(x_{ii}y_{ij})]_{jk} + 1/2[w_{jk}(x_{ii}y_{ij})]_{kk} \in A_{jk} \circ (A_{ii}A_{ij})$. But $A_{ij}A_{jk} \subseteq A_{ik}$ so $A_{ij}A_{jk} = A_{ij} \circ A_{jk}$ which proves (a).

We note that $(A_{ik}A_{ij})A_{ij} \subseteq A_{jk}A_{ij} \subseteq A_{ik}$ so using (6.2) and (6.3) as before we have $w_{ik}(x_{ij}y_{ij}) = [(w_{ik}x_{ij})y_{ij} + (w_{ik}y_{ij})x_{ij}]_{ik} \in (A_{ik}A_{ij}) \circ A_{ij}$. Moreover $A_{ik}A_{ij}^{(2)} = A_{ik}A_{ij}^2$. That proves (b).

To prove (c) take x_{ij}, y_{ik}, w_{jk}, and e_j in (6.1) to obtain $x_{ij}(y_{ik}w_{jk}) + w_{jk}(x_{ij}y_{ik}) = y_{ik}(x_{ij}w_{jk}) + e_j[x_{ij}(y_{ik}w_{jk}) + w_{jk}(x_{ij}y_{ik})]$ as a result of simplifying and noting that $e_j[y_{ik}(x_{ij}w_{jk})] = 0$. Multiplying this by e_i gives $e_i[x_{ij}(y_{ik}w_{jk})] = e_i[y_{ik}(x_{ij}w_{jk})]$. Interchanging the roles of i and j and of y and w in this gives $e_j[x_{ij}(y_{ik}w_{jk})] = e_j[w_{jk}(x_{ij}y_{ik})]$. Adding the last two equations, we have

$$x_{ij}(y_{ik}w_{jk}) = e_i[y_{ik}(x_{ij}w_{jk})] + e_j[w_{jk}(x_{ij}y_{ik})] \in e_iA_{ik}^2 + e_jA_{jk}^2 \subseteq A_{ik}^{(2)} + A_{jk}^{(2)}.$$

Now $A_{ik}A_{jk} \subseteq A_{ij}$ so $A_{ik}A_{jk} = A_{ik} \circ A_{jk}$ and we have (c).

If we substitute x_{ii}, y_{ij}, w_{ij}, and e_i in (6.1), we get $x_{ii}(y_{ij}w_{ij}) = -1/2[y_{ij}(x_{ii}w_{ij}) + w_{ij}(x_{ii}y_{ij})] + e_i[y_{ij}(x_{ii}w_{ij}) + w_{ij}(x_{ii}y_{ij})] + y_{ij}[e_i(x_{ii}w_{ij})] + w_{ij}[e_i(x_{ii}y_{ij})]$ which is in $A_{ij} \circ (A_{ii} \circ A_{ij})$. But $A_{ii}A_{ij}^2 = A_{ii}(A_{ij}^2)_{ii} = A_{ii}A_{ij}^{(2)}$ so we have proved (d).

Let e be an idempotent of A and define $B_e = \{x \in A_e(1) : xA_e(1/2) \subseteq A_e(0)\}$ and $C_e = \{x \in A_e(1) : xA_e(1/2) = 0\}$. Obviously $C_e \subseteq B_e \subseteq A_e(1)$. Moreover by [4, Lemma 1, p. 506] C_e is an ideal of A, B_e is an ideal of $A_e(1)$, $B_e^2 \subseteq C_e$, and $A_e(1) - B_e$ is a Jordan algebra.

Let $f = e_1 + e_2$, $h = e_1 + e_3$, and $m = e_2 + e_3$. Then if $g = e_i + e_j$ is one of these idempotents we will use the notation $B_{gi} = B_g \cap A_{ii}$, $B_{gij} = B_g \cap A_{ij}$, $B_i = B_{e_i}$, $C_i = C_{e_i}$, $C_{gi} = C_g \cap A_{ii}$, and $C_{gij} = C_g \cap A_{ij}$. Note that if $b_{ij} \in B_{gij} \subseteq A_{ij}$, then $b_{ij}A_g(1/2) = b_{ij}(A_{ik} + A_{jk}) = b_{ij}A_{ik} + b_{ij}A_{jk} \subseteq A_g(1/2)$ so $b_{ij}A_g(1/2) = 0$ and $B_{gij} = C_{gij}$.

Moreover, if $B_{gij} = 0$, then $B_g = B_i + B_j$. For one easily sees that $B_i \subseteq B_g$ and if $x \in B_g = B_{gi} + B_{gj}$, say $x = x_i + x_j$, then $x_i \in A_{ii}$ such that $x_i(A_{ik} + A_{jk}) \subseteq A_{kk}$. But B_g is an ideal of $A_g(1)$ so $x_iA_{ij} \subseteq B_{gj} \subseteq A_{jj}$. Therefore $x_i(A_{ij} + A_{ik}) \subseteq A_{jj} + A_{kk}$ and $x_i \in B_i$. Likewise $x_j \in B_j$ so $B_g = B_i + B_j$.

We are now ready to continue the proof of Theorem 3.1.

Let $B = B_f + B_h + B_m$. Then as in [4, p. 510] B is an ideal of A. By Lemma 3.2 we can assume $B = 0$, N, or A.

For $B = 0$ Albert proved in [4, Theorem 1, pp. 512–514] that A is a Jordan algebra. So by the results of Penico in [11] A has a Wedderburn decomposition. (At the time [11] was published the simple Jordan algebras of degree one and dimension greater than one were unknown. In [6] Jacobson shows they are isomorphic to the base field. This completed the classification of the simple Jordan algebras, and since no new type appeared, the proof in [11] is valid for all Jordan algebras of characteristic not two.)

Let $B = A$ and suppose the ideals C_f, C_h, and C_m are all nil. Then $A_{11} = B_{f1} + B_{h1}$ since $B_m \cap A_{11} = 0$. But we know that B_{f1} is an ideal of A_{11} since B_f is an ideal of $A_f(1)$ and $A_{11} \subseteq A_f(1)$. Moreover, $B_{f1}^2 \subseteq B_f^2 \subseteq C_f$ so B_{f1} is a nil ideal of A_{11}. Likewise B_{h1} is a nil ideal of A_{11}. But then $A_{11} = B_{f1} + B_{h1}$ is nil which is a contradiction since $e_1 \in A_{11}$. Thus one of the ideals C_f, C_h, or C_m is a proper non-nil ideal of A and by Lemma 3.2 A has a Wedderburn decomposition.

Thus we can assume $B = N$.

The above indicates our use of Lemma 3.2. Since we will make a few more such reductions, we label some cases to facilitate following the argument.

The following outline covers the remaining possibilities.

(A) $N = B = B_f = C_f$. This comes from assuming $C_g \neq 0$ where g is one of f, h, or m and without loss of generality we assume $g = f$. Clearly $C_f \neq A$ so by Lemma 3.2 we can assume $C_f = N$. So $N = C_f \subseteq B_f \subseteq B = N$.

(B) $C_f = C_h = C_m = 0$, $B = N$.

Case (A) has two subcases:

(A.1) $I_f = 0$ where

$$\begin{aligned} I_f &= \{\textstyle\sum (y_0 w_{1/2})_1 : y_0 \in A_f(0) \text{ and } w_{1/2} \in A_f(1/2)\} \\ &= (A_{13}A_{33})_{11} + (A_{23}A_{33})_{22}. \end{aligned}$$

(A.2) $I_f = N$. This comes from $I_f \neq 0$. For clearly $I_f \neq A$ so by Lemma 3.2 we can assume $I_f = N$.

(A) $N = B = B_f = C_f$. Let $I = I_f + N$ where I_f is defined in (A.1) above. If $x_1 \in A_f(1)$, then by (6.4) we have

$$x_1(y_0 w_{1/2})_1 = [x_1(y_0 w_{1/2})]_1 = 2[(x_1 w_{1/2})y_0]_1 = 2[(x_1 w_{1/2})_{1/2} y_0]_1 \in I_f$$

so I_f is an ideal of $A_f(1)$ and I is an ideal of $A_f(1)$. Since $N = C_f$, we have $NA_{13} = NA_{23} = 0$. Combining these results we get $AI \subseteq I + A_{13}I + A_{23}I = I + A_{13}I_f + A_{23}I_f$. Now $A_{13}I_f = A_{13}I_{f1} + A_{13}I_{f2} = A_{13}I_{f1} = A_{13}(A_{13}A_{33})_{11}$ where $I_{fi} = I_f \cap A_{ii}$. By Corollary 1.2, $A - N$ has three pairwise orthogonal idempotents since A has, so by [4, Theorem 1, p. 512] $A - N$ is a Jordan algebra since $A - N$ is simple. Moreover, $N \subseteq A_f(1)$ so $A_{13}A_{33} \subseteq A_{13} + N_1$. Therefore

$A_{13}(A_{13}A_{33})_{11} \subseteq A_{13}N_1 \subseteq N \subseteq I$. In the same manner we have $A_{23}I_f \subseteq I$ and I is an ideal of A.

But $I \neq 0$ since $N \neq 0$ and $I \neq A$ since $e_3 \notin I$; so by Lemma 3.2 we can assume that $I = N = C_f$. Thus $I_f \subseteq N$ and $AI_f \subseteq I_f + A_f(1/2)I_f = I_f + A_f(1/2)C_f = I_f$ so I_f is a nil ideal of A. This brings us to cases (A.1) and (A.2).

(A.1) $N = B = B_f = C_f$ and $I_f = 0$. Hence $A_{33}A_{13} \subseteq A_{13}$ and $A_{33}A_{23} \subseteq A_{23}$. As noted in case (A), $A - N$ is a Jordan algebra and hence is stable. But $N = C_f \subseteq A_f(1)$ so we have $A_{11}A_{13} \subseteq A_{13}$ and $A_{22}A_{23} \subseteq A_{23}$. Since $A_{11}A_{23} = A_{22}A_{13} = 0$ we can combine these results into

$$A_{ii}A_{j3} \subseteq A_{j3} \text{ for } i = 1,2,3;\ j = 1,2. \tag{6.6}$$

Let $H_f = A_f(1/2) + [A_f(1/2)]^{(2)} = A_{13} + A_{23} + A_{13} \circ A_{23} + A_{13}^{(2)} + A_{23}^{(2)}$. Now $A_{11}(A_{13} + A_{23}) \subseteq A_{13} + A_{23} \subseteq H_f$ by (6.6), $A_{11}(A_{13} \circ A_{23}) \subseteq (A_{11}A_{13}) \circ A_{23} \subseteq A_{13} \circ A_{23} \subseteq H_f$ by (6.6) and (a) of Lemma 6.5, and $A_{11}A_{13}^{(2)} \subseteq (A_{11} \circ A_{13}) \circ A_{13} \subseteq (A_{11}A_{13}) \circ A_{13} \subseteq A_{13} \circ A_{13} \subseteq H_f$ by (6.6) and (d) of Lemma 6.5. Similarly $A_{11}A_{23}^{(2)} \subseteq H_f$ so $A_{11}H_f \subseteq H_f$. By similar uses of (6.6) and Lemma 6.5 we can show in general that $A_{ij}H_f \subseteq H_f$; so H_f is an ideal of A and we can assume $H_f = 0$, N, or A by Lemma 3.2.

If $H_f = 0$ or N, then $A_f(1/2) = 0$ since $N \subseteq A_f(1)$. Thus $A = A_f(1) \oplus A_f(0)$, $A_f(0)$ is an ideal of A with $A_f(0) \neq 0$, N, or A, and A has a Wedderburn decomposition.

If $H_f = A$, then $A_{11} = (A_{13}^2)_{11}$ and $A_{22} = (A_{23}^2)_{22}$; so $N_1A_{11} = N_1(A_{13}^{(2)})_{11} \subseteq N_1A_{13}^{(2)} \subseteq (N_1 \circ A_{13}) \circ A_{13}$ by (d) of Lemma 6.5 where $N_i = N \cap A_{ii}$ and $N_{ij} = N \cap A_{ij}$. But $N_1 = C_{f1}$ so by (6.6) $N_1 \circ A_{13} = N_1A_{13} = 0$. Thus $N_1A_{11} = 0$. But $e_1 \in A_{11}$ so $N_1 = e_1N_1 = 0$. In the same manner we obtain $N_2 = 0$ so $N = N_{12} \subseteq A_{12}$. Then by (b) of Lemma 6.5, $N_{12}A_{11} \subseteq N_{12}A_{13}^{(2)} \subseteq (N_{12}A_{13}) \circ A_{13} = 0$ again because $N = C_f$. But this gives $N = N_{12} = e_1N_{12} = 0$ which is a contradiction. That completes the proof in case (A.1).

Before taking up case (A.2) we prove two lemmas.

(6.7) Lemma. *If $N = B_1 + B_2 + B_3$ and $H_g = A_g(1/2) + [A_g(1/2)]^{(2)}$ where $g = f, h$, or m, then $H_g + N$ is a nonzero ideal of A.*

Proof. As noted in case (A), $A - N$ is a Jordan algebra and hence it is stable. This and having $N \subseteq A_{11} + A_{22} + A_{33}$ gives

$$A_{ii}A_{ij} \subseteq A_{ij} + B_j \text{ for } i \neq j;\ i,j = 1,2,3. \tag{6.8}$$

Without loss of generality we can assume $g = f$. Then the proof that $H_f + N$ is an ideal of A is essentially the same as the proof in case (A.1) that H_f was an ideal of A. We only indicate this by considering two of the relations that need to be checked.

By (6.8) and (a) of Lemma (6.5) we have $A_{11}(A_{13} \circ A_{23}) \subseteq (A_{11}A_{13}) \circ A_{23} \subseteq (A_{13} + B_3) \circ A_{23} \subseteq A_{13} \circ A_{23} + N \subseteq H_f + N$. By (6.8) and (d) of Lemma (6.5) we have $A_{11}A_{13}^{(2)} \subseteq (A_{11} \circ A_{13}) \circ A_{13} \subseteq (A_{13} + B_3) \circ A_{13} \subseteq A_{13}^{(2} + N \subseteq H_f + N$.

(6.9) LEMMA. *If* $A = H_f + N = H_h + N = H_m + N$, *then* H_f *is a subalgebra of* A. *If we also have* $C_1 = C_2 = C_3 = 0$, *then* $A = H_f + N$ *is a Wedderburn decomposition for* A.

Proof. By (c) of Lemma 6.5, $(A_{13}^2)_{33} = [A_{13}(A_{12}A_{23})]_{33} \subseteq (A_{12}^{(2)} + A_{23}^{(2)})_{33} = (A_{23}^{(2)})_{33} = (A_{23}^2)_{33}$. Similarly $(A_{23}^2)_{33} \subseteq (A_{13}^2)_{33}$ so $(A_{23}^2)_{33} = (A_{13}^2)_{33}$. Denote these as S_3. By a similar argument we set $S_2 = (A_{23}^2)_{22} = (A_{12}^2)_{22}$ and $S_1 = (A_{12}^2)_{11} = (A_{13}^2)_{11}$.

The proof that $A_{12}H_f \subseteq H_f$ given in case (A.1) is valid here since (6.6) was not used. Therefore $(A_{13} \circ A_{23})H_f = A_{12}H_f \subseteq H_f$.

Clearly $A_{13}(A_{13} + A_{23}) \subseteq H_f$. Also $A_{13}A_{23}^{(2)} \subseteq (A_{13}A_{23}) \circ A_{23} = A_{12} \circ A_{23} = A_{13} \subseteq H_f$ by (b) of Lemma 6.5. Similarly $A_{13}A_{12}^{(2)} \subseteq A_{13} \subseteq H_f$. From these and the relations for S_1 and S_3 we have $A_{13}A_{13}^{(2)} = A_{13}[(A_{13}^2)_{11} + (A_{13}^2)_{33}] = A_{13}[(A_{12}^2)_{11} + (A_{23}^2)_{33}] \subseteq A_{13}(A_{12}^{(2)} + A_{23}^{(2)}) \subseteq H_f$. Thus $A_{13}H_f \subseteq H_f$ and by symmetry $A_{23}H_f \subseteq H_f$.

By symmetry and what we have just checked of H_fH_f it remains to show that $A_{13}^{(2)}A_{13}^{(2)}$ and $A_{13}^{(2)}A_{23}^{(2)}$ are subsets of H_f. But $A_{13}^{(2)}A_{13}^{(2)} = A_{13}^{(2)}[(A_{13}^2)_{11} + (A_{13}^2)_{33}] = A_{13}^{(2)}[(A_{12}^2)_{11} + (A_{23}^2)_{33}] \subseteq A_{13}^{(2)}A_{12}^{(2)} + A_{13}^{(2)}A_{23}^{(2)}$. These summands are handled in the same manner so we will only consider the latter. We note that $A_{13}^{(2)}A_{23}^{(2}$ $= A_{13}^2A_{23}^2$ since $A_{ii}A_{jj} = 0$ for $i \neq j$. Taking $x_{13}, y_{13}, z_{23}, w_{23}$ in (6.1) gives $4(x_{13}y_{13})(z_{23}w_{23}) + 4(x_{13}z_{23})(y_{13}w_{23}) + 4(x_{13}w_{23})(y_{13}z_{23}) = x_{13}[y_{13}(z_{23}w_{23}) + z_{23}(y_{13}w_{23}) + w_{23}(y_{13}z_{23})] + y_{13}[x_{13}(z_{23}w_{23}) + z_{23}(x_{13}w_{23}) + w_{23}(x_{13}z_{23})] + z_{23}[x_{13}(y_{13}w_{23}) + y_{13}(x_{13}w_{23}) + w_{23}(x_{13}y_{13})] + w_{23}[x_{13}(y_{13}z_{23}) + y_{13}(x_{13}z_{23}) + z_{23}(x_{13}y_{13})]$ which is in $A_{13}[A_{13}A_{23}^2 + A_{23}(A_{13}A_{23})] + A_{23}[A_{13}(A_{13}A_{23}) + A_{23}A_{13}^2] \subseteq A_{13}H_f + A_{23}H_f \subseteq H_f$ by our previous results. Also

$$4(x_{13}z_{23})(y_{13}w_{23}) + 4(x_{13}w_{23})(y_{13}z_{23}) \in (A_{13}A_{23})^2 \subseteq A_{12}^2 \subseteq H_f.$$

So $A_{13}^{(2)}A_{23}^{(2)} = A_{13}^2A_{23}^2 \subseteq H_f$ and H_f is a subalgebra of A.

Now assume we also have $C_1 = C_2 = C_3 = 0$. Evidently $S_1A_{12} = (A_{13}^2)_{11}A_{12} \subseteq A_{12}A_{13}^{(2)} \subseteq (A_{12}A_{13}) \circ A_{13} \subseteq A_{23} \circ A_{13} \subseteq A_{12}$ by (b) of Lemma 6.5. Likewise $S_1A_{13} \subseteq A_{13}$. So for $x \in S_1$ we get $x(A_{12} + A_{13}) \subseteq A_{12} + A_{13}$ while $x \in B_1$ implies that $x(A_{12} + A_{13}) \subseteq B_2 + B_3 \subseteq A_{22} + A_{33}$. Therefore $S_1 \cap N = S_1 \cap B_1 \subseteq C_1 = 0$. Similarly $S_2 \cap N = S_3 \cap N = 0$ so $A = H_f + N$ is a Wedderburn decomposition of A.

(A.2) $N = B = B_f = C_f = I_f \subseteq A_{11} + A_{22}$. Thus $B_{f12} = B_3 = 0$ and $N = B_1 + B_2$. So by Lemma 6.7 $H_f + N$, $H_h + N$, and $H_m + N$ are nonzero ideals of A. If one of them, say $H_f + N$, is N then $H_f \subseteq N$, $A_f(1/2) = 0$, $A = A_f(1) \oplus A_f(0)$, $A_f(0)$ is a proper non-nil ideal of A and A has a Wedderburn decomposition by Lemma 3.2. Thus we can assume $A = H_f + N = H_h + N = H_m + N$.

If $C_1 = C_2 = 0$ (we already have $C_3 \subseteq B_3 = 0$), then by Lemma 6.9 A has a Wedderburn decomposition. Therefore we assume, without loss of generality, that $C_1 \neq 0$. Clearly $C_1 \neq A$ so by Lemma 3.2 we can further assume that $C_1 = N$.

From $N = I_f = B_f$ we notice that $N^2 = I_f B_f = 0$. For if $b_1 \in B_f \subseteq A_f(1)$ and $(y_0 w_{1/2})_1 \in I_f \subseteq A_f(1)$, then $b_1 w_{1/2} \in A_f(0)$; so by (6.4), $b_1(y_0 w_{1/2})_1 = [b_1(y_0 w_{1/2})]_1 = 2[(b_1 w_{1/2}) y_0]_1 = 0$.

But we also have $N = C_1 \subseteq A_{11} = (A_{13}^2)_{11} + N$. From this and (d) of Lemma 6.5 we get $NA_{11} \subseteq NA_{13}^{(2)} + N^2 = NA_{13}^{(2)} \subseteq (N \circ A_{13}) \circ A_{13} = 0$. But $e_1 \in A_{11}$ so $N = e_1 N = 0$ which is a contradiction.

There remains case (B).

(B) $C_f = C_h = C_m = 0$, $B = N$. Recalling the preliminaries we see that $B_{gij} = 0$ for $g = f, h, m$; $i \neq j$; $i, j = 1, 2, 3$, and so $N = B_1 + B_2 + B_3$. In addition $C_i \subseteq C_g$, for if $x \in C_i$, then $x(A_{ij} + A_{ik}) = 0$. But $x \in A_{ii}$ so $xA_{jk} = 0$. Therefore $xA_g(1/2) = x(A_{ik} + A_{jk}) = 0$ and $x \in C_g$. Thus $C_1 = C_2 = C_3 = 0$. We then proceed as in the first part of case (A.2) using Lemmas 3.2, 6.7, and 6.9 to show that A has a Wedderburn decomposition. That completes the proof of case (B) and consequently of Theorem 3.1.

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