GLOBAL STRUCTURE IN VON NEUMANN ALGEBRAS

BY

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1. **Introduction.** A von Neumann algebra on a separable Hilbert space has an essentially unique representation as a direct integral of factors. In this paper we shall investigate the extent to which such decompositions reduce the classification problem for von Neumann algebras to that for factors. G. W. Mackey [18] has studied the corresponding questions that arise in the direct integral theory for representations of a separable locally compact group G. The essential concept that he introduced was the *dual* $\hat{G}$ of G, the set of unitary equivalence classes of separable, irreducible, unitary representations of G. $\hat{G}$, together with a natural σ-algebra of sets (a "Borel structure"), may be used as an index space for direct integrals of irreducible representations. We follow Mackey's program by introducing *canonical index spaces* $\hat{\mathscr{F}}$ and $\tilde{\mathscr{F}}$ for direct integrals of factors.

Let $\mathfrak{H}_n$ be a fixed Hilbert space of dimension n, $1 \leqq n \leqq \infty = \aleph_0$ and $\mathscr{A}_n$ be the set of all von Neumann algebras on $\mathfrak{H}_n$. In [5] we introduced a standard Borel structure on $\mathscr{A}_n$. Let $\mathscr{A} = \bigcup_{n=1}^{\infty} \mathscr{A}_n$ have the Borel structure generated by the structures on the $\mathscr{A}_n$. The relative structure on the set of factors $\mathscr{F}$ is standard [5, Corollary 3 of Theorem 3]. We define $\hat{\mathscr{F}}$ and $\tilde{\mathscr{F}}$ to be the spatial and algebraic isomorphism classes in $\mathscr{F}$, together with the quotient structures.

We have been unable to determine if $\hat{\mathscr{F}}$ and $\tilde{\mathscr{F}}$ are smooth, i.e., if they have countably separated structures. The problem is of considerable interest, as when a group G has a nonsmooth dual, direct integrals provide one with representations of G having unusual (i.e., nontype I) global structure (see [4, §4]). In §2 and §3 we show that points in $\hat{\mathscr{F}}$ and $\tilde{\mathscr{F}}$ are Borel, and that $\hat{\mathscr{F}}$ is smooth if and only if $\tilde{\mathscr{F}}$ is smooth. It follows that if there should be only countably many points in $\tilde{\mathscr{F}}$, $\tilde{\mathscr{F}}$ and $\hat{\mathscr{F}}$ would be smooth. At present only nine algebraically distinct nontype I factors have been identified (see [22], [23]).

The decomposition of a von Neumann algebra into factors induces a measure on $\hat{\mathscr{F}}$. We say that the algebra is *centrally smooth* if the complement of a null Borel set is countably separated. In §5, we show that such algebras are those of the form

$$\sum_{m=0}^{\infty} \int \mathfrak{B}(\xi) d\mu_m(\xi) \otimes \mathfrak{P}_m$$

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where the μ_m are countably separated measures on $\hat{\mathscr{F}}$, almost every $\mathfrak{B}(\xi)$ is a factor in the equivalence class ξ, and the $\mathfrak{P}_m$ are various abelian von Neumann algebras. The proof uses a double integral technique of Mackey [16, Theorem 2.11]. In an attempt to make the latter more accessible, we have included a careful exposition of the theory in §4. Our approach gives some information even when the quotient measure is not countably separated (see Lemmas 4.4 and 4.5).

In §6 we consider the global pathology that might occur should $\hat{\mathscr{F}}$ be non-smooth. Introducing the notion of global type, it becomes apparent that a von Neumann algebra of global type II could not be centrally smooth.

We are indebted to R. V. Kadison for several fruitful conversations on the material in §6, and to both him and J. Ringrose for ideas that resulted in Lemma 2.5. We have recently a manuscript from J. Feldman [6], in which he has proved the analogue of Theorem 2.8 for the representation and state spaces of separable C^*-algebras.

2. **Subsets of $\mathscr{A}$.** Let $\mathfrak{L}_n$ be the bounded linear operators on $\mathfrak{H}_n$. The weak, σ-weak, strong, and σ-strong topologies generate the same Borel structure on $\mathfrak{L}_n$, and the algebraic operations are Borel.

Let G_n be the group of unitaries on $\mathfrak{H}_n$ with the strong topology. G_n is a polonais topological group [3, Lemma 4], and defining $\phi_n: G_n \otimes \mathscr{A}_n \to \mathscr{A}_n$ by $\phi_n(U, \mathfrak{A}) = U\mathfrak{A}U^{-1}$, we obtain a transformation group.

LEMMA 2.1. *ϕ_n is Borel.*

Proof. Let $\mathfrak{A} \to A_i(\mathfrak{A})$ be Borel choice functions on $\mathscr{A}_n$ with $A_i(\mathfrak{A})$ weakly dense in $\mathfrak{A}_1$ for each $\mathfrak{A}$ [5, Corollary of Theorem 2]. If $f \in \mathfrak{L}_{n*}$,

$$\| f \mid \phi_n(U, \mathfrak{A}) \| = \sup\{ |f(UA_i(\mathfrak{A})U^*)| : i = 1, 2, \cdots\}.$$

As multiplication is Borel, $(U, \mathfrak{A}) \to f(UA_i(\mathfrak{A})U^*)$ is Borel. Thus

$$(U, \mathfrak{A}) \to \| f \mid \phi_n(U, \mathfrak{A}) \|$$

is Borel, and from [5, Theorem 1], ϕ_n is Borel.

Let $\mathfrak{I}_n$ be the scalar multiples of the identity operator on $\mathfrak{H}_n$. $\mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{I}_\infty$ may be realized as a map θ of $\mathscr{A}$ into $\mathscr{A}_\infty$ as follows. For each n, choose an infinite sequence of isometries U_{in} of $\mathfrak{H}_n$ into $\mathfrak{H}_\infty$, for which the projections $E_{in} = U_{in}U_{in}^*$ are orthogonal and $\Sigma_i E_{in} = I$. Define θ_n': $\mathfrak{L}_n \to \mathfrak{L}_\infty$ by $\theta_n'(A) = \Sigma_i U_{in} A U_{in}^*$. θ_n' is a σ-weakly continuous isomorphism of $\mathfrak{L}_n$ into $\mathfrak{L}_\infty$ which preserves the identity. If $\mathfrak{A} \in \mathscr{A}_n$, let $\theta_n(\mathfrak{A}) = \{\theta_n'(A): A \in \mathfrak{A}\}$. As θ_n' is σ-weakly continuous, it is the adjoint of a map $\theta_{n*}': \mathfrak{L}_{\infty *} \to \mathfrak{L}_{n*}$. As θ_n' is an isometry, we have for $f \in \mathfrak{L}_{\infty *}$,

$$\| f \mid \theta_n(\mathfrak{A}) \| = \| \theta_{n*}' f \mid \mathfrak{A} \|,$$

and from [5, Theorem 1], θ_n is Borel. We let $\theta = \bigcup_n \theta_n$.

THEOREM 2.2. *If $\mathfrak{A} \in \mathscr{A}$, the spatial and algebraic equivalence classes $[\mathfrak{A}]$ and $[[\mathfrak{A}]]$ containing $\mathfrak{A}$ are Borel.*

Proof. If $\mathfrak{A} \in \mathscr{A}_n$, let $G_n(\mathfrak{A})$ be the stabilizer subgroup of $\mathfrak{A}$ in G_n, i.e., those unitary U with $U\mathfrak{A}U^* = \mathfrak{A}$. As $G_n(\mathfrak{A})$ consists of the $U \in G_n$ with $U\mathfrak{A}U^* \subseteq \mathfrak{A}$ and $U^*\mathfrak{A}U \subseteq \mathfrak{A}$, $G_n(\mathfrak{A})$ is closed in G_n. From [3, Lemma 3], there exists a Borel set T in G_n intersecting each left coset of $G_n(\mathfrak{A})$ in one point. We have

$$[\mathfrak{A}] = \phi_n(G_n \times \{\mathfrak{A}\}) = \phi_n(T \times \{\mathfrak{A}\}).$$

As ϕ_n is Borel and one-to-one on the standard Borel space $T \times \{\mathfrak{A}\}$, $[\mathfrak{A}]$ is Borel (see [18, Theorem 3.2]). von Neumann algebras with purely infinite commutants on a separable Hilbert space are algebraically isomorphic if and only if they are spatially isomorphic, hence

$$[[\mathfrak{A}]] = \theta^{-1}([\theta(\mathfrak{A})]),$$

and $[[\mathfrak{A}]]$ is Borel.

Let $\hat{\mathscr{A}}$ and $\tilde{\mathscr{A}}$ be the spatial, respectively algebraic, equivalence classes in $\mathscr{A}$, with the quotient Borel structures.

COROLLARY 2.3. *Points in $\hat{\mathscr{A}}$ and $\tilde{\mathscr{A}}$ are Borel.*

COROLLARY 2.4. *The set $\mathscr{A}_I$ of von Neumann algebras of type* I *is Borel.*

Proof. Let $\mathscr{B}$ denote the properly infinite type I von Neumann algebras. To within algebraic equivalence, there are only countably many such algebras. Thus $\mathscr{B}$ is a countable union of algebraic equivalence classes, and is Borel. A von Neumann algebra $\mathfrak{A}$ is of type I if and only if $\theta(\mathfrak{A})'$ is properly infinite and type I. Thus

$$\mathscr{A}_I = \theta^{-1}(\mathscr{B}'),$$

and as $'$ is a Borel isomorphism of $\mathscr{A}$ into itself [5, Theorem 3], $\mathscr{A}_I$ is Borel.

Let $\mathscr{F}_n$ be the factors on $\mathfrak{H}_n$, and $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}_n$. From [5, Corollary 3 of Theorem 3] $\mathscr{F}_n$ is a Borel subset of $\mathscr{A}_n$, hence $\mathscr{F}$ is a Borel subset of $\mathscr{A}$. We wish to show that $\mathscr{F}_{fin}$, the finite factors, is also Borel.

If $\mathfrak{A}$ is a von Neumann algebra and $A \in \mathfrak{A}$, let $\kappa_{\mathfrak{A}}(A)$ be the weak closure of finite convex sums of elements of the form UAU^*, with U a unitary in $\mathfrak{A}$.

LEMMA 2.5. *Let $\mathfrak{A}$ be a factor and $E \neq 0, I$ be a projection in $\mathfrak{A}$. $\mathfrak{A}$ is infinite if and only if $0 \in \kappa_{\mathfrak{A}}(E)$ or $0 \in \kappa_{\mathfrak{A}}(I - E)$.*

Proof. If $\mathfrak{A}$ is finite, and τ is the normalized trace on $\mathfrak{A}$, τ must be constant on each of the sets $\kappa_{\mathfrak{A}}(E)$ and $\kappa_{\mathfrak{A}}(I - E)$. As $\tau(E)$ and $\tau(I - E)$ are nonzero, $0 \notin \kappa_{\mathfrak{A}}(E)$ and $0 \notin \kappa_{\mathfrak{A}}(I - E)$.

If $\mathfrak{A}$ is infinite, say that $I - E$ is infinite. For any n, we may choose partial isometries $U_1, \cdots, U_n$ with $U_i^* U_i = E$ and $U_i U_i^* = E_i$ orthogonal projections with $E_i \leqq I - E$. Let

$$V_i = U_i + U_i^* + [I - (E + E_i)].$$

Then V_i is unitary, and

$$\left\| \frac{1}{n} \sum_{i=1}^{n} V_i E V_i^* \right\| = \frac{1}{n}.$$

As n is arbitrary, $0 \in \kappa_{\mathfrak{A}}(E)$. Similarly, if E is infinite, $0 \in \kappa_{\mathfrak{A}}(I - E)$. As one or the other is infinite, we are done.

Lemma 2.6. *There exist Borel choice functions $\mathfrak{A} \to A_i(\mathfrak{A})$ and $\mathfrak{A} \to U_i(\mathfrak{A})$ on $\mathscr{A}$ with the $A_i(\mathfrak{A})$ strongly dense in $\mathfrak{A}_{SA,1}$, the self-adjoint elements $A \in \mathfrak{A}$ with $\| A \| \leqq 1$, and the $U_i(\mathfrak{A})$ strongly dense in the unitaries of $\mathfrak{A}$, for all $\mathfrak{A} \in \mathscr{A}$.*

Proof. From [5, Theorem 2], there exist Borel $\mathfrak{A} \to C_j(\mathfrak{A})$ with the $C_j(\mathfrak{A})$ weakly dense in $\mathfrak{A}_1$. As the adjoint operation is weakly continuous, $\mathfrak{A} \to B_j(\mathfrak{A}) = \frac{1}{2}[C_j(\mathfrak{A})^* + C_j(\mathfrak{A})]$ is Borel, and the $B_j(\mathfrak{A})$ are weakly dense in $\mathfrak{A}_{SA\ 1}$. Let $A_i(\mathfrak{A})$ be an enumeration of the finite sums of the form $\Sigma r_j B_j(\mathfrak{A})$, where the r_j are positive rationals with $\Sigma_j r_j = 1$. For each $\mathfrak{A}$, the strong closure of the $A_i(\mathfrak{A})$ is convex, hence it is weakly closed and coincides with $\mathfrak{A}_{SA,1}$.

Define f on the closed interval $[-1,1]$ by $f(t) = \exp(i\pi t)$, and g on the unit circle by letting $g(z)$ be the unique real t with $-1 \leqq t < 1$ and $\exp i\pi t = z$. If $A \in \mathfrak{A}_{SA,1}$, $f(A)$ is unitary, and if U is unitary, $g(U) \in \mathfrak{A}_{SA,1}$ and $f(g(U)) = (f \circ g)(U) = U$. Thus g maps $\mathfrak{A}_{SA,1}$ onto the unitaries of $\mathfrak{A}$. As $A \to f(A)$ is strongly continuous [12, p. 232], the operators $U_i(\mathfrak{A}) = f(A_i(\mathfrak{A}))$ are strongly dense in the unitaries. From the argument used in [5, Theorem 5], $\mathfrak{A} \to U_i(\mathfrak{A})$ is Borel.

Lemma 2.7. *There exists a Borel choice function $\mathfrak{A} \to E(\mathfrak{A})$ on $\mathscr{A}$ with $E(\mathfrak{A})$ a projection, $E(\mathfrak{A}) \neq 0$, I for $\mathfrak{A} \neq \mathfrak{I}_n$, $n = 1, 2, \cdots \infty$.*

Proof. If A is a self-adjoint operator on $\mathfrak{H}_n$, let

$$m(A) = \inf \{Ax \cdot x : \| x \| = 1\},$$

$$M(A) = \sup \{Ax \cdot x : \| x \| = 1\},$$

$$\omega(A) = M(A) - m(A).$$

Letting x_i be dense in the unit ball of $\mathfrak{H}_n$,

$$m(A) = \inf\{Ax_i \cdot x_i : i = 1, 2, \cdots\},$$

hence m, and similarly M, are Borel on $\mathfrak{L}_{n,SA}$. $\omega(A) = 0$ if and only if A is a scalar multiple of I. Let $\mathfrak{A} \to A_i(\mathfrak{A})$ be Borel choice functions on $\mathfrak{A}$ with $A_i(\mathfrak{A})$ weakly dense in $\mathfrak{A}_{SA,1}$. If $\omega(A_i(\mathfrak{A})) = 0$ for all i, $\mathfrak{A} = \mathfrak{I}_n$ for some n. Define a Borel choice function B by $B(\mathfrak{I}_n) = 0$ for all n, and if $\mathfrak{A} \neq \mathfrak{I}_n$, $B(\mathfrak{A}) = A_i(\mathfrak{A})$, where i is the first integer with $\omega(A_i(\mathfrak{A})) \neq 0$. Let $E(\mathfrak{A}) = f^{\mathfrak{A}}(B(\mathfrak{A}))$, where $f^{\mathfrak{A}}$ is the characteristic function of the closed interval

$$\left[\frac{1}{2}m(B(\mathfrak{A})) + \frac{1}{2}M(B(\mathfrak{A})), 1\right].$$

Then $E(\mathfrak{A}) \neq 0$, I for $\mathfrak{A} \neq \mathfrak{I}_n$, and it suffices to show that $\mathfrak{A} \to f^{\mathfrak{A}}(B(\mathfrak{A}))$ is Borel.

Let $\mathfrak{C}$ be the continuous real-valued functions on $[-2, 1]$ with the uniform norm. The map

$$\mathfrak{C} \times \mathfrak{L}_{n,SA,1} \to \mathfrak{L}_n : (f, A) \to f(A)$$

is continuous in the first variable, and Borel in the second (see the proof of [5, Theorem 5]), hence it is jointly Borel (see [15, Lemma 9.2], [13,§27V]). Given $a \in [-1, 1]$ and a positive integer k, define $f_k^a \in \mathfrak{C}$ by $f_k^a(t) = 0$ for $t \leqq a - 1/k$, $f_k^a(t) = 1$ for $t \geqq a$, and letting $f_k^a(t)$ be linear on intermediate points. $a \to f_k^a$ is a continuous map of $[-1, 1]$ into $\mathfrak{C}$, hence letting

$$\mu(\mathfrak{A}) = \frac{1}{2}m(B(\mathfrak{A})) + \frac{1}{2}M(B(\mathfrak{A})),$$

$\mathfrak{A} \to f_k^{\mu(\mathfrak{A})}(B(\mathfrak{A}))$ is Borel for each k. Fixing $\mathfrak{A}$, the $f_k^{\mu(\mathfrak{A})}$ are uniformly bounded and converge point-wise to $f^{\mathfrak{A}}$, hence $f_k^{\mu(\mathfrak{A})}(B(\mathfrak{A})) \to f^{\mathfrak{A}}(B(\mathfrak{A}))$ weakly. Thus $\mathfrak{A} \to f^{\mathfrak{A}}(B(\mathfrak{A}))$ is a limit of Borel functions, and is itself Borel.

THEOREM 2.8. *$\mathscr{F}_{fin}$ is a Borel subset of $\mathscr{F}$.*

Proof. Choose $\mathfrak{A} \to U_i(\mathfrak{A})$ and $\mathfrak{A} \to E(\mathfrak{A})$ as in Lemmas 2.6 and 2.7. Let d be a metric for the weak topology on $\mathfrak{L}_{\infty,1}$. If $\mathfrak{A} \in \mathscr{A}_\infty$ and $0 \in \kappa_{\mathfrak{A}}(E(\mathfrak{A}))$, then given $\varepsilon > 0$, there exist finitely many unitaries $V_j \in \mathfrak{A}$ and non-negative reals t_j with $\Sigma t_j = 1$ and

$$d(\textstyle\sum t_j V_j E(\mathfrak{A}) V_j^*, 0) < \varepsilon.$$

As multiplication is strongly continuous on bounded sets, and the adjoint operation is strongly continuous on the unitaries, we may choose non-negative rationals r_i with $\sum r_i = 1$ and

$$d(\textstyle\sum r_i U_i(\mathfrak{A}) E(\mathfrak{A}) U_i(\mathfrak{A})^*, 0) < \varepsilon.$$

Letting

$$\Delta_1 = \{\mathfrak{A} \in \mathscr{A}_\infty : \inf d(\textstyle\sum r_i U_i(\mathfrak{A}) E(\mathfrak{A}) U_i(\mathfrak{A})^*, 0) = 0\},$$

the inf being taken over all such finite rational convex sums, Δ_1 is Borel, and consists of those $\mathfrak{A} \in \mathscr{A}_\infty$ for which $0 \in \kappa_{\mathfrak{A}}(E(\mathfrak{A}))$. Similarly the set Δ_2 of $\mathfrak{A} \in \mathscr{A}_\infty$ with $0 \in \kappa_{\mathfrak{A}}(I - E(\mathfrak{A}))$ is Borel. From Lemma 2.5,

$$\mathscr{F}_{fin} = \left(\bigcup_{n<\infty} \mathscr{F}_n\right) \cup [\mathscr{F}_\infty - (\Delta_1 \cup \Delta_2)] \cup \{\mathfrak{I}_\infty\},$$

and $\mathscr{F}_{fin}$ is Borel.

3. **Coupling and the index spaces.** If x is a vector in $\mathfrak{H}_n$ and $\mathfrak{A} \in \mathscr{A}_n$, let $[\mathfrak{A}x] \in \mathfrak{A}'$ be the projection on the smallest linear space containing x, invariant under $\mathfrak{A}$. If $\mathfrak{A} \in \mathscr{F}_{fin}$, let $\tau_{\mathfrak{A}}$ be the normal trace on $\mathfrak{A}$ with $\tau_{\mathfrak{A}}(I) = 1$. If $\mathfrak{A} \in \mathscr{F}_{fin} \cap \mathscr{F}'_{fin}$, and x is a nonzero vector in the underlying Hilbert space, $\tau_{\mathfrak{A}}([\mathfrak{A}'x])/\tau_{\mathfrak{A}'}([\mathfrak{A}x])$ does not depend on x, and is known as the *coupling* $C(\mathfrak{A})$ of $\mathfrak{A}$. Two algebraically isomorphic algebras in $\mathscr{F}'_{fin} \cap \mathscr{F}_{fin}$ are spatially isomorphic if and only if they have the same coupling (see [2, Chapter III, §6.4]).

LEMMA 3.1. *If $x \in \mathfrak{H}_n$, $\mathfrak{A} \to [\mathfrak{A}x]$ is a Borel map of $\mathscr{A}_n$ into $\mathfrak{L}_n$.*

Proof. If $E \in \mathfrak{L}_n$ is a projection and $y \in \mathfrak{H}_n$,

$$\| (I - E)y \| = \inf\{ \| y - z \| : z \in E\mathfrak{H}_n\}.$$

This is due to the fact that the closest vector in $E\mathfrak{H}_n$ to y is Ey (see [11, §11]). Let $A_i(\mathfrak{A})$ be Borel choice functions on $\mathscr{A}_n$, weakly dense in $\mathfrak{A}_1$ for each $\mathfrak{A}$, and $B_j(\mathfrak{A})$ be the finite rational combinations of the $A_i(\mathfrak{A})$. The strong closure of the $B_j(\mathfrak{A})$ is linear, hence weakly closed, and contains $\mathfrak{A}_1$. Thus the $B_j(\mathfrak{A})$ are strongly dense in $\mathfrak{A}$, and for $y \in \mathfrak{H}_n$,

$$\| (I - [\mathfrak{A}x])y \| = \inf\{ \| y - B_j(\mathfrak{A})x \|\}.$$

Thus

$$\mathfrak{A} \to [\mathfrak{A}x]y \cdot y = \| y \|^2 - \|(I - [\mathfrak{A}x])y \|^2$$

is Borel, and the lemma follows.

LEMMA 3.2. *If $\mathfrak{A} \to A(\mathfrak{A}) \in \mathfrak{A}_1$ is a Borel choice function on $\mathscr{F}_{fin}$, then $\mathfrak{A} \to \tau_{\mathfrak{A}}(A(\mathfrak{A}))$ is also Borel.*

Proof. It suffices to prove that the function is Borel on $\mathscr{F}_{fin} \cap \mathscr{A}_n$. From Lemma 2.6, we may let $\mathfrak{A} \to U_i(\mathfrak{A})$ be Borel with $U_i(\mathfrak{A})$ strongly dense in the unitaries of $\mathfrak{A}$. Let d be a metric for the weak topology on $\mathfrak{L}_{n,1}$. Fixing $\mathfrak{A}$, $\tau_{\mathfrak{A}}(A(\mathfrak{A}))I$ is the unique element in $\kappa_{\mathfrak{A}}(A(\mathfrak{A})) \cap \mathfrak{I}_n$ [2, p. 272]. Thus given a closed subset F of the complex plane, if $\tau_{\mathfrak{A}}(A(\mathfrak{A})) \in F$, then for any $\varepsilon > 0$ there exist non-negative rationals r_i with $\Sigma r_i = 1$ and

$$d(\textstyle\sum r_i U_i(\mathfrak{A}) A(\mathfrak{A}) U_i(\mathfrak{A})^*, FI) \leqq \varepsilon.$$

Conversely if for each integer j there is a finite rational convex sum B_j satisfying the above inequality for $\varepsilon = 1/j$, let B_{j_k} be a weakly convergent subsequence. We have $\tau_{\mathfrak{A}}(A(\mathfrak{A}))I = \lim B_{j_k} \in FI$. Letting $T(\mathfrak{A}) = \tau_{\mathfrak{A}}(A(\mathfrak{A}))$,

$$T^{-1}(F) = \{\mathfrak{A}: \inf d(\textstyle\sum r_i U_i(\mathfrak{A})A(\mathfrak{A})U_i(\mathfrak{A})^*, FI) = 0\},$$

the inf taken over all finite convex rational sums, hence $T^{-1}(F)$, and T, are Borel.

THEOREM 3.3. *The coupling function $\mathfrak{A} \to C(\mathfrak{A})$ on $\mathscr{F}_{fin} \cap \mathscr{F}'_{fin}$ is Borel.*

Proof. Let x_n be a fixed nonzero vector in $\mathfrak{H}_n$. As $\mathfrak{A} \to \mathfrak{A}'$ is Borel, we have from the above lemmas that

$$\mathfrak{A} \to C(\mathfrak{A}) = \tau_{\mathfrak{A}}([\mathfrak{A}' x_n])/\tau_{\mathfrak{A}'}([\mathfrak{A} x_n])$$

is Borel on $\mathscr{F}_{fin} \cap \mathscr{F}'_{fin} \cap \mathfrak{A}_n$.

From Corollary 2.3, Borel sets in $\hat{\mathscr{F}}$ and $\tilde{\mathscr{F}}$ separate points. We recall that a Borel space is *countably separated* if there exists a countable family of Borel sets, or equivalently, complex-valued Borel functions, separating points. As $\mathscr{F}$ is standard, should $\hat{\mathscr{F}}$ or $\tilde{\mathscr{F}}$ be countably separated, it would be analytic [18, p. 141].

THEOREM 3.4. *$\hat{\mathscr{F}}$ is countably separated if and only if $\tilde{\mathscr{F}}$ is countably separated.*

Proof. Let π_1 and π_2 be the quotient maps of $\mathscr{F}$ onto $\hat{\mathscr{F}}$ and $\tilde{\mathscr{F}}$, respectively, and define α by $\alpha \circ \pi_1 = \pi_2$. α is Borel, as if B is Borel in $\tilde{\mathscr{F}}$, $\pi_1^{-1}(\alpha^{-1}(B)) = \pi_2^{-1}(B)$ is Borel in $\mathscr{F}$. Extend coupling to $\mathscr{F}$ by letting $C(\mathfrak{A}) = 0$ for $\mathfrak{A} \notin \mathscr{F}_{fin} \cap \mathscr{F}'_{fin}$. C is Borel (Theorems 2.8 and 3.3), and constant on spatial equivalence classes, hence it defines a Borel function $\hat{C}$ on $\hat{\mathscr{F}}$. Similarly the commutant operation $'$ defines a Borel isomorphism on $\hat{\mathscr{F}}$, which we again indicate by $'$. Let $\mathscr{F}_{inf}$ be the infinite factors.

Suppose there exist Borel functions f_i, $i = 1, 2, \cdots$, separating points in $\tilde{\mathscr{F}}$. Then we claim the functions $f_i \circ \alpha$, $f_i \circ \alpha \circ '$, and $\hat{C}$ together separate points in $\hat{\mathscr{F}}$. As two factors with infinite commutants are spatially isomorphic if and only if they are algebraically isomorphic, the $f_i \circ \alpha$ separate points in $\hat{\mathscr{F}}'_{inf}$. It follows that the $f_i \circ \alpha \circ '$ separate points in $\hat{\mathscr{F}}_{inf}$. $\hat{C}$ together with the $f_i \circ \alpha$ separate points in $\hat{\mathscr{F}}_{fin} \cap \hat{\mathscr{F}}'_{fin}$. The $f_i \circ \alpha$ separate $\hat{\mathscr{F}}_{fin}$ from $\hat{\mathscr{F}}_{inf}$, hence all points are separated.

Let T be the Borel set $\{\mathfrak{A} \in \mathscr{F}_{fin} \cap \mathscr{F}'_{fin} : C(\mathfrak{A}) = 1\} \cup [\mathscr{F}_{inf} \cap \mathscr{F}'_{inf}]$, and $\hat{T} = \pi_1(T)$. α is a one-to-one Borel map of $\hat{T}$ onto $\tilde{\mathscr{F}}$. We shall prove that α is a Borel isomorphism on $\hat{T}$. It will follow that if $\hat{\mathscr{F}}$ is countably separated, then so is $\tilde{\mathscr{F}}$.

If B is a subset of $\mathscr{F}$, let B^s and B^a be the saturations of B with respect to spatial and algebraic equivalence, respectively. Using the notation of §2, if B is Borel

$$B^s = \bigcup \phi_n(G_n \times (B \cap \mathscr{A}_n))$$

and

$$B^a = \theta^{-1}(\theta(B)^s)$$

are analytic. If D is a Borel subset of $\hat{T}$,

$$\pi_2^{-1}(\alpha(D)) = (\pi_1^{-1}(D))^a$$

and

$$\pi_2^{-1}(\tilde{\mathscr{F}} - \alpha(D)) = \pi_2^{-1}(\alpha(\hat{T} - D)) = (\pi_1^{-1}(\hat{T} - D))^a$$

are disjoint analytic sets. As their union is the standard space $\mathscr{F}$, both must be Borel (see [13, §35, III]), hence $\alpha(D)$ is Borel in $\tilde{\mathscr{F}}$.

4. **Direct integral theory.** We begin with a brief summary of the constructive theory in order to introduce our terminology. Details and omitted proofs may be found in [2, Chapter II].

As suggested in [7, pp. 83–84] and [19, p. 634], one may regard a field of objects over a Borel space as a cross-section in an appropriate "bundle". We shall instead use "coherences" to map fields into bundles with constant fiber. (We essentially follow [17].) It is then unnecessary to introduce the bundle terminology.

By a *measure* μ on a Borel space $(Z, \mathscr{S})$, we mean a real, non-negative, finite, countably additive function on $\mathscr{S}$. If $x: Z \to \mathfrak{H}_n$ is weakly Borel, let $x(\mu)$ be the class of weakly Borel functions of Z into $\mathfrak{H}_n$, equal to x μ-almost everywhere. Let $\bar{L}_n^2(\mu) = \bar{L}_n^2(Z, \mathscr{S}, \mu)$ be the weakly Borel functions $x: Z \to \mathfrak{H}_n$ with

$$\int \| x(\zeta) \|^2 d\mu(\zeta) < \infty.$$

$\bar{L}_n^2(\mu)$ is a linear space with Hilbert pseudo-norm $\| x \| = [\int \| x(\zeta) \|^2 d\mu(\zeta)]^{1/2}$, and null space the functions equal to 0 μ-almost everywhere. Let $L_n^2(\mu) = L_n^2(Z, \mathscr{S}, \mu)$ be the quotient Hilbert space.

If $A: Z \to \mathfrak{L}_n$ is a uniformly bounded, weakly Borel function, define $\bar{A}$ on $\bar{L}_n^2(\mu)$ by $(\bar{A}(x))(\zeta) = A(\zeta)x(\zeta)$. $\bar{A}$ preserves null functions, and induces a map $A(\mu)$ on $L_n^2(\mu)$ with $\| A(\mu) \| = \text{ess.}\sup \| A(\zeta) \|$. If $\mathfrak{A}: Z \to \mathscr{A}_n$ is Borel, let $\mathfrak{A}(\mu) = \int \mathfrak{A}(\zeta) d\mu(\zeta)$ be all operators of the form $A(\mu)$, where $A: Z \to \mathfrak{L}_n$ is a uniformly bounded weakly Borel function with $A(\zeta) \in \mathfrak{A}(\zeta)$ for all ζ. $\mathfrak{A}(\mu)$ is a von Neumann algebra with $\mathfrak{A}(\mu)' = \mathfrak{A}'(\mu)$.

Let $\mathfrak{H}_0 = \mathfrak{L}_0 = \mathscr{A}_0 = \{0\}$. As it will be necessary to consider four varieties of fields, we introduce the "fibers":

$$\mathscr{H} = \{\mathfrak{H}_0, \mathfrak{H}_1, \cdots, \mathfrak{H}_\infty\},$$

$$\mathfrak{H}_u = \bigcup_{n=0}^{\infty} \mathfrak{H}_n,$$

$$\mathfrak{L}_u = \bigcup_{n=0}^{\infty} \mathfrak{L}_n,$$

$$\mathscr{A}_u = \bigcup_{n=0}^{\infty} \mathscr{A}_n = \mathscr{A} \cup \mathscr{A}_0,$$

where we regard the last three unions as disjoint. Let $\mathscr{H}$ have the discrete Borel structure, $\mathfrak{H}_u$ and $\mathfrak{L}_u$ have, the structures generated by the weak (or, equivalently, strong) structures on the $\mathfrak{H}_n$ and $\mathfrak{L}_n$ and $\mathscr{A}_u$ have the structure generated by those on the $\mathscr{A}_n$.

If $\mathfrak{H} : Z \to \mathscr{H}$ is Borel and $Z_n = \{\zeta : \mathfrak{H}(\zeta) = \mathfrak{H}_n\}$, define $\mathfrak{H}(\mu) = \int \mathfrak{H}(\zeta) d\mu(\zeta)$ to be $\sum_{n=1}^{\infty} L_n^2(\mu_n)$, where μ_n is the restriction of μ to Z_n. $\mathfrak{H}(\mu)$ consists of all sequences $x(\mu) = (x(\mu_n))$, where $x : Z \to \mathfrak{H}_u$ is Borel with $x(\zeta) \in \mathfrak{H}_n$ for $\zeta \in Z_n$ and

$$\| x(\mu) \|^2 = \sum_{n=1}^{\infty} \int \| x(\zeta) \|^2 d\mu_n(\zeta) = \int \| x(\zeta) \|^2 d\mu(\zeta) < \infty .$$

Similarly, if $\mathfrak{A} : Z \to \mathscr{A}_u$ is Borel, and $Z_n = \{\zeta : \mathfrak{A}(\zeta) \in \mathscr{A}_n\}$, we define $\mathfrak{A}(\mu) = \int \mathfrak{A}(\zeta) d\mu(\zeta)$ to be $\sum_{n=1}^{\infty} \mathfrak{A}(\mu_n)$. Assuming the Z_n for $\mathfrak{H}$ and $\mathfrak{A}$ coincide, $\mathfrak{A}(\mu)$ is defined on $\mathfrak{H}(\mu)$. It consists of all sequences $A(\mu) = (A(\mu_n))$, where $A : Z \to \mathfrak{L}_u$ is uniformly bounded and Borel, and $A(\zeta) \in \mathfrak{A}(\zeta)$ for all ζ.

If Z is a set, a *field of Hilbert spaces* $\mathfrak{H}$ on Z is a map $\zeta \to \mathfrak{H}(\zeta)$ of Z into a collection of separable Hilbert spaces. Letting $Z_n = \{\zeta : \dim \mathfrak{H}(\zeta) = n\}$ a *coherence* γ for $\mathfrak{H}$ is a map $\zeta \to \gamma(\zeta)$, where $\gamma(\zeta)$ is an isometry of $\mathfrak{H}(\zeta)$ onto $\mathfrak{H}_n$ for $\zeta \in Z_n$. A *vector field* x in $\mathfrak{H}$ is a map $\zeta \to x(\zeta) \in \mathfrak{H}(\zeta)$, an *operator field* A on $\mathfrak{H}$ is a map $\zeta \to A(\zeta) \in \mathfrak{L}(\mathfrak{H}(\zeta))$, and a *field of von Neumann algebras* $\mathfrak{A}$ on $\mathfrak{H}$ is a map $\zeta \to \mathfrak{A}(\zeta) \in \mathscr{A}(\mathfrak{H}(\zeta))$. If Z has a Borel structure and γ is a coherence for $\mathfrak{H}$, we say these fields are γ-Borel when

$$\begin{aligned} \mathfrak{H}^{\gamma}(\zeta) &= \gamma(\zeta)\mathfrak{H}(\zeta), \\ x^{\gamma}(\zeta) &= \gamma(\zeta)x(\zeta), \\ A^{\gamma}(\zeta) &= \gamma(\zeta)A(\zeta)\gamma(\zeta)^{-1}, \\ \mathfrak{A}^{\gamma}(\zeta) &= \gamma(\zeta)\mathfrak{A}(\zeta)\gamma(\zeta)^{-1}, \end{aligned}$$

are Borel maps of Z into $\mathscr{H}$, $\mathfrak{H}_u$, $\mathfrak{L}_u$, and $\mathscr{A}_u$, respectively. As $\mathfrak{H}$ is γ-Borel if and only if the sets Z_n are Borel, the coherence is irrelevant, and we say $\mathfrak{H}$ is *Borel* if the latter is true. If μ is a measure on Z, we write $\int^{\gamma} \mathfrak{H}(\zeta) d\mu(\zeta)$ and $\int^{\gamma} \mathfrak{A}(\zeta) d\mu(\zeta)$ for $\int \mathfrak{H}^{\gamma}(\zeta) d\mu(\zeta)$ and $\int \mathfrak{A}^{\gamma}(\zeta) d\mu(\zeta)$, respectively.

Let I be a finite or countable set. For each sequence $m = (m_i)_{i \in I}$ of non-negative integers, let σ_m be an isometry of $\sum \mathfrak{H}_{m_i}$ onto $\mathfrak{H}_{\Sigma m_i}$. If $(\mathfrak{H}^i, \gamma_i)$ are Borel coherent fields (i.e., each $\mathfrak{H}^i$ is Borel and has coherence γ_i), and ζ is such that $\dim \mathfrak{H}^i(\zeta) = m_i$, define for $(x_i) \in \sum \mathfrak{H}^i(\zeta)$

$$\left(\sum \gamma_i(\zeta) \right)(x_i) = \sigma_m(\gamma_i(\zeta) x_i).$$

Then $(\sum \mathfrak{H}_i, \sum \gamma_i)$ is a Borel coherent field. If $\mathfrak{A}_i$ are γ_i-Borel, then $\sum \mathfrak{A}_i$ is $\sum \gamma_i$-Borel, and

$$\int^{\Sigma \gamma_i} \sum \mathfrak{A}_i(\zeta) d\mu(\zeta) \cong \sum \int^{\gamma_i} \mathfrak{A}_i(\zeta) d\mu(\zeta). \tag{1}$$

Without going into details (see the proof of Lemma 4.5), we remark that the underlying isometry is defined by

$$(x_i)^{\Sigma\gamma_i}(\mu) \to (x_i^{\gamma_i}(\mu))$$

where x_i is a sequence of γ_i-Borel vector fields with $\sum \int \| x_i \|^2 d\mu < \infty$.

Let $\tau_{m,n}$ be a fixed isometry of $\mathfrak{H}_m \otimes \mathfrak{H}_n$ onto $\mathfrak{H}_{mn}$ for $m, n \geqq 0$ (we let $0 \cdot \infty = 0$). If $(\mathfrak{H}, \gamma)$ and $(\mathfrak{K}, \delta)$ are Borel coherent fields, and ζ is such that $\dim \mathfrak{H}(\zeta) = m$, $\dim \mathfrak{K}(\zeta) = n$, let $\gamma(\zeta) \otimes \delta(\zeta)$ be the unique linear isometry of $\mathfrak{H}(\zeta) \otimes \mathfrak{K}(\zeta)$ onto $\mathfrak{H}_m \otimes \mathfrak{H}_n$ satisfying for $u \in \mathfrak{H}(\zeta)$, $v \in \mathfrak{K}(\zeta)$, $\gamma(\zeta) \otimes \delta(\zeta)(u \otimes v) = \gamma(\zeta)(u) \otimes \delta(\zeta)(v)$. Let $\gamma \otimes \delta(\zeta) = \tau_{m,n} \circ (\gamma(\zeta) \otimes \delta(\zeta))$. Then $(\mathfrak{H} \otimes \mathfrak{K}, \gamma \otimes \delta)$ is a Borel coherent field. If $\mathfrak{A}$ and $\mathfrak{B}$ are γ and δ-Borel respectively, $\mathfrak{A} \otimes \mathfrak{B}$ is $\gamma \otimes \delta$-Borel. If $\mathfrak{B}^\delta(\zeta) = \mathfrak{B}_0$ for all ζ,

$$\int^{\gamma \otimes \delta} \mathfrak{A} \otimes \mathfrak{B}(\zeta) d\mu(\zeta) \cong \int^{\gamma} \mathfrak{A}(\zeta) d\mu(\zeta) \otimes \mathfrak{B}_0. \tag{2}$$

Let $\mathfrak{H}_n$ underlie $\mathfrak{B}_0$, v_j be arbitrary vectors in $\mathfrak{H}_n$, and $y_j(\zeta) = \delta(\zeta)^{-1}(v_j)$. The spatial isometry of (2) is the closure of the map

$$\sum (x_i \otimes y_j)^{\gamma \otimes \delta}(\mu) \to \sum x_i^{\gamma}(\mu) \otimes v_j,$$

where the sums are finite, and the x_i are γ-Borel vector fields with $\int \| x_i \|^2 d\mu < \infty$ If $\mathfrak{A}(\zeta) = \mathfrak{L}_1$ (the complex numbers) for all ζ, it follows that

$$\int \mathfrak{B}_0 d\mu(\zeta) \cong L^\infty(\mu) \otimes \mathfrak{B}_0, \tag{3}$$

where $L^\infty(\mu)$ acts on $L^2(\mu)$ by multiplication.

Suppose that Z_1 and Z_2 are Borel spaces with measure μ_1 and μ_2 respectively. A *measure isomorphism* (T, N_1, N_2) of Z_1 and Z_2 is a Borel isomorphism T of $Z_1 - N_1$ onto $Z_2 - N_2$, where N_i are Borel in Z_i, $\mu_i(N_i) = 0$, such that the μ_2-null sets in $Z_2 - N_2$ are the images of the μ_1-null sets in $Z_1 - N_1$.

LEMMA 4.1. *Say that Z_i, $i = 1,2$ are standard Borel spaces with Borel coherent fields $(\mathfrak{H}_i, \gamma_i)$, γ_i-Borel fields $\mathfrak{A}_i$, and measures μ_i. If there is a measure isomorphism (T, N_1, N_2) of Z_1 and Z_2 with $\mathfrak{A}_1(\zeta_1) \cong \mathfrak{A}_2(T(\zeta_1))$ for all $\zeta \in Z_1 - N_1$, then $\int^{\gamma_1} \mathfrak{A}_1(\zeta_1) d\mu_1(\zeta_1) \cong \int^{\gamma_2} \mathfrak{A}_2(\zeta_2) d\mu_2(\zeta_2)$.*

We say a family of vectors $\{x_\alpha\}$ *spans* a topological vector space $\mathfrak{X}$ if the linear space generated by the x_α is dense in $\mathfrak{X}$. Rephrasing some of the results in [2, Chapter II], we have

LEMMA 4.2. *Suppose that $(\mathfrak{H}, \gamma)$ is a Borel coherent field on a Borel space Z. Given a measure μ on Z, say g_i are bounded Borel functions with $g_i(\mu)$ spanning $L^\infty(\mu)$ in the weak* topology. If x_j is a sequence of uniformly bounded γ-Borel vector fields in $\mathfrak{H}$ with $x_j(\zeta)$ spanning $\mathfrak{H}(\zeta)$ for all ζ, then the vectors $g_i x_j^\gamma(\mu)$*

span $\mathfrak{H}^\gamma(\mu)$. If $\mathfrak{A}$ is a γ-Borel field of von Neumann algebras on Z, and A_k is a sequence of uniformly bounded γ-Borel operator fields in $\mathfrak{A}$, with $A_k(\zeta)$, generating $\mathfrak{A}(\zeta)$, the operators $A_k^\gamma(\mu)$, together with the $g_i(\mu)I$, generate $\mathfrak{A}^\gamma(\mu)$.

Proof. Let x be a γ-Borel field in $\mathfrak{H}$ with $x^\gamma(\mu) \in \mathfrak{H}^\gamma(\mu)$, and $x^\gamma(\mu) \perp g_i x_j^\gamma(\mu)$ for all i, j. Then

$$\begin{aligned} 0 &= x^\gamma(\mu) \cdot g_i x_j^\gamma(\mu) \\ &= \sum_n \int x^\gamma(\zeta) \cdot g_i(\zeta) x_j^\gamma(\zeta) d\mu_n(\zeta) \\ &= \sum_n \int [x(\zeta) \cdot x_j(\zeta)] \bar{g}_i(\zeta) d\mu_n(\zeta) \\ &= \int [x(\zeta) \cdot x_j(\zeta)] \bar{g}_i(\zeta) d\mu(\zeta) \end{aligned}$$

for all i implies $x(\zeta) \cdot x_j(\zeta) = 0$ a.e. It follows that $x(\zeta) = 0$ a.e., hence $x(\mu) = 0$.

Let $\mathfrak{B}$ be the von Neumann algebra generated by the $A_k^\gamma(\mu)$ and $g_i(\mu)I$. It suffices to prove that $\mathfrak{B}' \subseteq \mathfrak{A}^\gamma(\mu)'$. If $B' \in \mathfrak{B}'$, B' commutes with the $g_i(\mu)I$. From the decomposition theory, it follows that there is a uniformly bounded γ-Borel operator field $\zeta \to C(\zeta) \in \mathfrak{L}(\mathfrak{H}(\zeta))$ with $B' = C^\gamma(\mu)$. As B' commutes with $A_k(\mu)$, $C(\zeta)$ commutes with $A_k(\zeta)$ a.e., hence $C(\zeta) \in \mathfrak{A}(\zeta)'$ a.e., and $B' \in \mathfrak{A}'(\mu) = \mathfrak{A}(\mu)'$.

If Z is a Borel space, let $M(Z)$ be the measures on Z, together with the Borel structure defined by the functions $\mu \to \int f d\mu$, with f a bounded Borel function on Z. If Z is standard, let Z have a compact metrizable topology generating its structure, and let $\mathfrak{C}(Z)$ be the continuous functions. $M(Z)$ is Borel isomorphic to the positive cone of $\mathfrak{C}(Z)^*$ with the weak* topology, and thus is standard.

LEMMA 4.3. *Say that Z is standard, and $\mathfrak{H} : Z \to \mathscr{H}$ is Borel. Then $\mu \to \mathfrak{H}(\mu)$ is a Borel field on $M(Z)$, and there exists a coherence $\gamma(\mu)$ for $\mathfrak{H}(\mu)$ with the following properties:*

(1) *If $x : Z \to \mathfrak{H}_u$ is a uniformly bounded Borel field in $\mathfrak{H}$, $\mu \to x(\mu)$ is a γ-Borel field in $\mu \to \mathfrak{H}(\mu)$, uniformly bounded on $M(Z)_1$.*

(2) *If $A : Z \to \mathfrak{L}_u$ is a uniformly bounded Borel field on $\mathfrak{H}$, $\mu \to A(\mu)$ is a uniformly bounded γ-Borel field on $\mu \to \mathfrak{H}(\mu)$.*

(3) *If $\mathfrak{A} : Z \to \mathscr{A}_u$ is a Borel field on $\mathfrak{H}$, $\mu \to \mathfrak{A}(\mu)$ is a γ-Borel field in $\mu \to \mathfrak{H}(\mu)$.*

Proof. Let Z have a compact metrizable topology generating its structure, and let $\mathfrak{C}(Z)$ be the continuous functions on Z. Choose g_i uniformly dense in $\mathfrak{C}(Z)$. Then for each $\mu \in M(Z)$, the $g_i(\mu)$ span $L^\infty(\mu)$ in the weak* topology. Let $e_{n_1}, e_{n_2}, \cdots$ be an orthonormal basis for $\mathfrak{H}_n$, $n \geqq 1$, and define $x_{nj}(\zeta) = e_{nj}$ for $\zeta \in Z_n$, $x_{nj}(\zeta) = 0$ elsewhere. From Lemma 4.2, the vectors $g_i x_{nj}(\mu)$ span $\mathfrak{H}(\mu)$. Furthermore, for any combination of subscripts,

$$(4) \qquad \mu \to g_i x_{nj}(\mu) \cdot g_{i'} x_{n'j'}(\mu) = \begin{cases} \int g_i \bar{g}_{i'} d\mu & \text{if } n = n', \quad j = j', \\ 0 & \text{otherwise,} \end{cases}$$

is Borel on $M(Z)$. Let $v_p(\zeta)$ be an enumeration of the fields $g_i(\zeta)x_{nj}(\zeta)$. A simultaneous orthonormalization procedure (see [2, p. 139]) on the $v_p(\mu)$, $\mu \in M(Z)$, yields vector fields

$$(5) \qquad w_k(\mu) = \sum_{1 \leqq p \leqq k} h_p^k(\mu) v_p(\mu)$$

where $h_p^k(\mu)$ is a Borel scalar function on $M(Z)$, and the $w_k(\mu)$, $k \leqq \dim \mathfrak{H}(\mu)$ form an orthonormal basis for $\mathfrak{H}(\mu)$, $w_k(\mu) = 0$ for $k > \dim \mathfrak{H}(\mu)$. From (5) and (4), $\mu \to \| w_k(\mu) \|^2$ is Borel, hence the set M_m of μ with $\dim \mathfrak{H}(\mu) = m$ is Borel, and $\mu \to \mathfrak{H}(\mu)$ is Borel. If $\mu \in M_m$, define $\gamma(\mu)\colon \mathfrak{H}(\mu) \to \mathfrak{H}_m$ by

$$\gamma(\mu)y = \sum_k (y \cdot w_k(\mu)) e_{mk}.$$

A vector field $\mu \to y(\mu)$ will be γ-Borel if and only if $y(\mu) \cdot w_k(\mu)$ is Borel for all k, hence from (5), if and only if $y(\mu) \cdot g_i x_{nj}(\mu)$ are all Borel.

Let x, A, and $\mathfrak{A}$ be as described above. Since

$$\mu \to x(\mu) \cdot g_i x_{nj}(\mu) = \sum_n \int_{Z_n} x(\zeta) \cdot e_{nj} \bar{g}_i(\zeta) d\mu(\zeta)$$

is Borel, $\mu \to x(\mu)$ is γ-Borel, and if $\| x(\zeta) \| \leqq K$, $\| x(\mu) \| \leqq K\mu(Z)^{1/2}$. Turning to A, if $\| A(\zeta) \| \leqq L$, $\| A(\mu) \| \leqq L$. To show that $\mu \to A^\gamma(\mu)$ is Borel, it suffices to prove that on M_m

$$\mu \to A^\gamma(\mu) e_{mi} \cdot e_{mj} = A(\mu) w_i(\mu) \cdot w_j(\mu)$$

is Borel. This follows from (5) as

$$\begin{aligned} \mu &\to A(\mu) g_i x_{nj}(\mu) \cdot g_{i'} x_{nj'}(\mu) \\ &= \int_{Z_n} (A(\zeta) e_{nj} \cdot e_{nj'}) g_i(\zeta) \bar{g}_{i'}(\zeta) d\mu(\zeta) \end{aligned}$$

is Borel.

From [5], we may select uniformly bounded Borel fields $A_j(\zeta)$ in $\mathfrak{A}(\zeta)$ generating $\mathfrak{A}(\zeta)$ at each ζ. From Lemma 4.2, the fields $A_j^\gamma(\mu)$ and $g_i(\mu)I$ generate $\mathfrak{A}^\gamma(\mu)$ at each μ, hence $\mu \to \mathfrak{A}^\gamma(\mu)$ is a Borel map of $M(Z)$ into $\mathscr{A}_u$ (see discussion at the end of §3 in [5]).

Let R be an equivalence relation on a Borel space X, and π be the quotient map of X onto the set of equivalence classes X/R. Providing X/R with the quotient structure, we say that R is *smooth* if X/R is countably separated. If μ is a measure on X, define the *quotient measure* ν on X/R by $\nu(T) = \mu(\pi^{-1}(T))$. We have included the following version of the decomposition theorem for μ (see [1], [21], [9], [20]) as it gives partial results for non smooth relations.

LEMMA 4.4. *Suppose that R is an equivalence relation on a standard Borel space Z, μ is a measure on Z, and ν is the quotient measure on Z/R. Then there exists a Borel map $\xi \to \mu_\xi$ of Z/R into $M(Z)_1$ such that if f is a bounded Borel function on Z, and h is a ν-integrable function on Z/R, then*

$$\int h \circ \pi(\zeta) f(\zeta) d\mu(\zeta) = \int h(\xi) \int f(\zeta) d\mu_\xi(\zeta) d\nu(\xi). \tag{6}$$

If R is smooth, each μ_ξ may be chosen concentrated in $\pi^{-1}(\xi)$.

Proof. If Z is countably or uncountably infinite, it is Borel isomorphic to the one-point compactification of the integers, or to the Cantor set, respectively. Thus we may let Z have a zero-dimensional, compact, metrizable topology. Let Σ be the algebra of sets generated by a countable basis of compact open sets. Σ is countable, and any decomposition of a set in Σ into non empty disjoint sets in Σ must be finite. It follows that any finitely additive, non-negative function on Σ is a measure, and as Σ generates the Borel structure on Z, extends uniquely to a measure on Z (see [10, p. 54]).

For each Borel set S in Σ, $C \to \mu(S \cap \pi^{-1}(C))$ is absolutely continuous with respect to ν, hence there is a non-negative Borel function g_S on Z/R with

$$\mu(S \cap \pi^{-1}(C)) = \int_C g_S(\xi) d\nu(\xi). \tag{7}$$

If S and S' are disjoint sets in Σ,

$$\mu((S \cup S') \cap \pi^{-1}(C)) = \mu(S \cap \pi^{-1}(C)) + \mu(S' \cap \pi^{-1}(C)),$$

hence

$$\int_C g_{S \cup S'}(\xi) d\nu(\xi) = \int_C [g_S(\xi) + g_{S'}(\xi)] d\nu(\xi)$$

for all Borel C, and

$$g_{S \cup S'}(\xi) = g_S(\xi) + g_{S'}(\xi)$$

for all ν-almost all ξ. If $S = Z$ in (7),

$$\nu(C) = \int_C g_Z(\xi) d\nu(\xi),$$

hence $g_Z(\xi) = 1$ ν-almost everywhere. Letting the g_S be zero on a Borel ν-null set, we may assume $S \to g_S(\xi)$ is finitely additive, and extends to a measure $\mu_\xi \in M(Z)_1$.

The family of Borel sets T in Z for which $\xi \to \mu_\xi(T)$ is Borel and

$$\mu(T \cap \pi^{-1}(C)) = \int_C \mu_\xi(T) d\nu(\xi) \tag{8}$$

for Borel C in Z/R, is monotonic and contains Σ. Thus from [10, p. 27], the family consists of all Borel sets in Z. If f and h are Borel characteristic functions, $\xi \to \int f d\mu_\xi$ is Borel, and (6) is true. Linearity implies the same for simple f and h, i.e., finite linear combinations of Borel characteristic functions. If f and h are as described in the lemma, choose simple f_n and h_n with $f_n \to f$, $h_n \to h$ pointwise, and $|f_n| \leqq |f|$, $|h_n| \leqq |h|$. As f is bounded, it is μ_ξ-integrable, and $\int f_n d\mu_\xi \to \int f d\mu_\xi$. Thus $\xi \to \int f d\mu_\xi$ is Borel, and as f was arbitrary, $\xi \to \mu_\xi$ is Borel. As $\xi \to \int |f| d\mu_\xi$ is bounded, $|h(\xi)| \int |f| d\mu_\xi$ is a ν-integrable function dominating the $h_n(\xi) \int f_n d\mu_\xi$. Similarly, $|h \circ \pi(\zeta) f(\zeta)|$ is μ-integrable and dominates $h_n \circ \pi(\zeta) f(\zeta)$. Taking limits, we obtain (6).

If D is Borel in Z/R, we have from (8)

$$\int_C \chi_D(\xi) d\nu(\xi) = \nu(D \cap C) = \int_C \mu_\xi(\pi^{-1}(D)) d\nu(\xi).$$

Thus

$$\mu_\xi(\pi^{-1}(D)) = \chi_D(\xi) \tag{9}$$

ν-almost everywhere. If Δ is a countable separating algebra of sets in Z/R, let N be Borel with $\nu(N) = 0$ and (9) valid for $D \in \Delta$ and $\xi \notin N$. If $\xi \notin N$, choose $D_1 \supseteq D_2 \supseteq \cdots$ in Δ with $\bigcap D_i = \{\xi\}$. Then $\mu_\xi(\pi^{-1}(D_i)) = 1$ implies that $\mu_\xi(\pi^{-1}(\xi)) = 1$, i.e., μ_ξ is concentrated in $\pi^{-1}(\xi)$. Letting $\mu_\xi = 0$ for $\xi \in N$, we are done.

LEMMA 4.5 ([16, THEOREM 2.11]). *Say that Z is standard Borel space with a Borel field $\mathfrak{A}: Z \to \mathscr{A}_u$. If R is an equivalence relation on Z, and μ is a measure on Z, let $\mu = \int \mu_\xi d\nu(\xi)$ be a decomposition of the type described in Lemma* 4.4. *Then there is a spatial isomorphism*

$$\int \mathfrak{A}(\zeta) d\mu(\zeta) \cong \int^\beta \mathfrak{A}(\mu_\xi) d\nu(\xi)$$

where $\beta(\xi) = \gamma(\mu_\xi)$, γ a coherence for $\mu \to \mathfrak{H}(\mu)$ as described in Lemma 4.3.

Proof. Let $\mathfrak{H}: Z \to \mathscr{H}$ be the underlying field of Hilbert spaces for $\mathfrak{A}$. $\mathfrak{H}$ is Borel as $Z_n = \{\zeta : \mathfrak{A}(\zeta) \in \mathscr{A}_n\}$ is Borel. We must define an isometry

$$U : \mathfrak{H}(\mu) \to \int \mathfrak{H}^\beta(\mu_\xi) d\nu(\xi).$$

Let $\mathfrak{B}$ be the uniformly bounded Borel fields $x : Z \to \mathfrak{H}_u$ in $\mathfrak{H}$, and for each $\lambda \in M(Z)$, let $\mathfrak{B}(\lambda)$ be the corresponding subspace of $\mathfrak{H}(\lambda)$. From Lemma 4.3, if $x \in \mathfrak{B}$, $\lambda \to x(\lambda)$ is a uniformly bounded γ-Borel field in $\mathfrak{H}(\lambda)$ on $M(Z)_1$. Thus $\xi \to x(\mu_\xi)$ is uniformly bounded and β-Borel in $\mathfrak{H}(\mu_\xi)$, and

$$x(\mu_\xi)^\beta(\nu) \in \int \mathfrak{H}^\beta(\mu_\xi)d\nu(\xi).$$

We have

$$\begin{aligned}\| x(\mu_\xi)^\beta(\nu) \|^2 &= \int \| \beta(\xi)\, x(\mu_\xi) \|^2 \, d\nu(\xi) \\ &= \int \| x(\mu_\xi) \|^2 d\nu(\xi) \\ &= \iint \| x(\zeta) \|^2 d\mu_\xi(\zeta) d\nu(\xi) \\ &= \int \| x(\zeta) \|^2 d\mu(\zeta) \\ &= \| x(\mu) \|^2.\end{aligned}$$

Defining U on $\mathfrak{B}(\mu)$ by $U(x(\mu)) = x(\mu_\xi)^\beta(\nu)$, U is an isometry.

As in the proof of Lemma 4.3, we may select uniformly bounded Borel functions g_i on Z which span $L^\infty(\lambda)$ for all $\lambda \in M(Z)$. The functions $h_i(\xi) = \int g_i(\zeta)d\mu_\xi(\zeta)$ are bounded and Borel on Z/R. They span $L^\infty(\nu)$, as if $h \in L^1(\nu)$ and $h \perp h_i$ for all i,

$$0 = \int h\bar{h}_i d\nu = \int h \circ \pi(\zeta)\bar{g}_i(\zeta)d\mu(\zeta),$$

i.e., $h \circ \pi = 0$ μ-almost everywhere, and $h = 0$ ν-almost everywhere. If h is a uniformly bounded Borel function on Z/R, we assert that the class $h \circ \pi(\mu_\xi)$ is that of the constant $h(\xi)$ for ν-almost all ξ. If C is Borel in Z/R, letting the h of (6) be χ_C, and then $\chi_C h$,

$$\int_C \int h \circ \pi(\zeta)\bar{g}_i(\zeta)d\mu_\xi(\zeta)d\nu(\xi) = \int_C h(\xi) \int \bar{g}_i(\zeta)d\mu_\xi(\zeta)d\nu(\xi),$$

hence there is a Borel set N in Z/R with $\nu(N) = 0$ and

$$\int h \circ \pi(\zeta)\bar{g}_i(\zeta)d\mu_\xi(\zeta) = \int h(\xi)\bar{g}_i(\zeta)d\mu_\xi(\zeta)$$

for all $\xi \notin N$. As the g_i span $L^\infty(\mu_\xi)$, $h\circ \pi(\zeta) = h(\xi)$ for μ_ξ-almost all ζ, $\xi \notin N$.

Again from the proof of Lemma 4.3, we may choose uniformly bounded Borel vector fields v_j in $\mathfrak{H}$ with $v_j(\lambda)$ spanning $\mathfrak{H}(\lambda)$ for all $\lambda \in M(Z)$. From Lemma 4.2, the vectors

$$h_i(\nu)v_j(\mu_\xi)^\beta(\nu) = U(h_i \circ \pi(\mu)v_j(\mu))$$

span $\int \mathfrak{H}^\beta(\mu_\xi)d\nu(\xi)$. As the $v_j(\mu)$ span $\mathfrak{H}(\mu)$, $\mathfrak{B}(\mu)$ and its image are dense, and U extends to the desired isometry.

If A is a uniformly bounded Borel field in $\mathfrak{A}$, we have from Lemma 4.2 that $\xi \to A(\mu_\xi)$ is a uniformly bounded β-Borel field in $\mathfrak{A}(\mu_\xi)$, and

$$A(\mu_\xi)^\beta(\nu) \in \int^\beta \mathfrak{A}(\mu_\xi)d\nu(\xi).$$

If x is in $\mathfrak{B}$, $(\tilde{A}x)(\zeta) = A(\zeta)x(\zeta)$ is another field in $\mathfrak{B}$,

$$\begin{aligned} UA(\mu)x(\mu) &= U(\tilde{A}x)(\mu) \\ &= [(\tilde{A}x)(\mu_\xi)]^\beta(\nu) \\ &= [\beta(\xi)A(\mu_\xi)x(\mu_\xi)](\nu) \\ &= [A(\mu_\xi)^\beta x(\mu_\xi)^\beta](\nu) \\ &= [A(\mu_\xi)^\beta(\nu)]x(\mu_\xi)^\beta(\nu), \end{aligned}$$

hence $UA(\mu)U^{-1} = A(\mu_\xi)^\beta(\nu)$. As

$$U(h_i \circ \pi(\mu)x(\mu)) = h_i(\nu)x(\mu_\xi)^\beta(\nu),$$

$Uh_i \circ \pi(\mu)IU^{-1} = h_i(\nu)I$.

Letting $B_k(\zeta)$ be an enumeration of the fields $f_i(\zeta)I$ and $A_j(\zeta)$ described in the proof of Lemma 4.3, the $B_k(\lambda)$ generate $\mathfrak{A}(\lambda)$ for each $\lambda \in M(Z)$. From Lemma 4.2, the $B_k(\mu_\xi)^\beta(\nu)$, together with the operators $h_i(\nu)I$, generate $\int^\beta \mathfrak{A}(\mu_\xi)d\nu(\xi)$. As these are the images of the $B_k(\mu)$ and the $h_i \circ \pi(\mu)I$, respectively, U defines a spatial isomorphism of $\int \mathfrak{A}(\zeta)d\mu(\zeta)$ onto $\int^\beta \mathfrak{A}(\mu_\xi)d\nu(\xi)$.

5. **Central decompositions.** If $\mathfrak{A}$ is a von Neumann algebra on a separable Hilbert space, a *central decomposition* $(Z, \mathscr{S}, \mu, \mathfrak{F})$ of $\mathfrak{A}$ is a standard Borel space $(Z, \mathscr{S})$ with a measure μ and a Borel field $\mathfrak{F} : Z \to \mathscr{F}$ with

$$\mathfrak{A} \cong \int \mathfrak{F}(\zeta)d\mu(\zeta).$$

Such a decomposition always exists, and if $(Z_0, \mathscr{S}_0, \mu_0, \mathfrak{F}_0)$ is another such decomposition, there is a measure isomorphism (T, N, N_0) of Z and Z_0 (see §4) with $\mathfrak{F}(\zeta) \cong \mathfrak{F}_0(T(\zeta))$ for all $\zeta \in Z - N$. Letting π_1 be the quotient map of $\mathscr{F}$ onto $\hat{\mathscr{F}}$, $\pi_1 \circ \mathfrak{F}$ and $\pi_1 \circ \mathfrak{F}_0$ map μ and μ_0, respectively, onto null-set equivalent measures $\hat{\mu}$ and $\hat{\mu}_0$ on $\hat{\mathscr{F}}$. We say that $\mathfrak{A}$ is *centrally smooth* if there is a Borel set $P \subseteq \hat{\mathscr{F}}$ with $\mu(P) = 0$ and $\hat{\mathscr{F}} - P$ countably separated. Enlarging P by a Borel null-set, we may assume $\hat{\mathscr{F}} - P$ is standard (see [18, Theorem 6.1]), hence we call such measures *standard*.

If Z is a Borel space, let $M_0(Z)$ be the continuous measures on Z, i.e., those measures for which all one-point sets have zero mass. For cardinals m with $1 \leqq m \leqq \infty = \aleph_0$, let $M_m(Z)$ be the measures totally concentrated in m points. Given an arbitrary measure μ, there exist unique measures μ^c and μ^d, with

$\mu = \mu^c + \mu^d$, where μ^c is continuous, and μ^d is discrete, i.e., totally concentrated in points.

If $\mu \in M(Z)$, define

$$\| \mu \| = \sup \{ | \mu(\{\zeta\}) | : \zeta \in Z \}.$$

LEMMA 5.1. *If Z is countably separated, then $\mu \to \| \mu \|$ is Borel on $M(Z)$.*

Proof. Choose a sequence of Borel sets S_k separating Z. Let $S_k^0 = S_k$, $S_k^1 = Z - S_k$, $S(\emptyset) = Z$, and for any sequence $i_1, \cdots, i_n$ of zeros and ones, let

$$S(i_1, \cdots, i_n) = S_1^{i_1} \cap \cdots \cap S_n^{i_n}.$$

If $\mu \in M(Z)$, define

$$\| \mu \|_n = \max \mu(S(i_1, \cdots, i_n)), \qquad i_1, \cdots, i_n = 0, 1.$$

$\| \mu \|_n$ is a decreasing sequence, hence its limit exists. As $\mu \to \| \mu \|_n$ is Borel, it will suffice to prove that for all $\mu \in M(Z)$, $\| \mu \| = \lim \| \mu \|_n$.

If S is Borel, define the measure $\mu | S$ by $(\mu | S)(T) = \mu(S \cap T)$. If $\lim \| \mu \|_n = \delta > 0$, define a sequence $j_0, j_1, \cdots$ as follows. Let $j_0 = \emptyset$. Suppose that $j_0, \cdots, j_r$ have been defined with

$$\lim \| \mu | S(j_1, \cdots, j_r) \|_n = \delta. \tag{10}$$

As the sequences

$$a_n = \| \mu | S(j_1, \cdots, j_r, 0) \|_{n-1},$$
$$b_n = \| \mu | S(j_1, \cdots, j_r, 1) \|_{n-1}$$

are decreasing, and

$$\| \mu | S(j_1, \cdots, j_r) \|_n = \max \{ a_n, b_n \},$$

$\lim \| \mu | S(j_1, \cdots, j_r) \|_n$ must coincide with either $\lim a_n$ or $\lim b_n$. This provides us with j_{r+1} satisfying (10). As the sets S_k separate Z, $\bigcap_r S(j_1, \cdots, j_r)$ has at most one point. On the other hand,

$$\mu(S(j_1, \cdots, j_r)) \geqq \| \mu | S(j_1, \cdots, j_r) \|_n \geqq \delta;$$

hence the intersection must contain a point ζ with $\mu(\{\zeta\}) \geqq \delta$. We conclude that $\| \mu \| \geqq \lim \| \mu \|_n$. The converse inequality is trivial.

LEMMA 5.2. *If Z is countably separated, the sets $M_m(Z)$ are Borel in $M(Z)$, and the maps $\mu \to \mu^c$ and $\mu \to \mu^d$ are Borel.*

Proof. If $\delta > 0$ and $\mu \in M(Z)$, there are at most finitely many ζ with $\mu(\{\zeta\}) \geqq \delta$. Thus there always exist $\zeta \in Z$ with $\mu(\{\zeta\}) = \| \mu \|$. We call such ζ *maximal μ-atoms.* Define $S(i_1, \cdots, i_n)$ as in the proof of Lemma 5.1. For each $\mu \in M(Z)$, define

$i_0(\mu) = \varnothing$. Suppose that $i_0(\mu), \cdots, i_n(\mu)$ have been defined with $S(i_1(\mu), \cdots, i_n(\mu))$ containing a maximal μ-atom. Let $i_{n+1}(\mu) = 0$ or 1 be the least integer with $S(i_1(\mu), \cdots, i_{n+1}(\mu))$ containing a maximal μ-atom. Define

$$\mu^a = \lim \mu \,|\, S(i_1(\mu), \cdots, i_n(\mu)), \tag{11}$$

where limit is taken in the sense of convergence of the corresponding integrals of each bounded Borel function. μ^a is the "part" of μ concentrated in some maximal μ-atom. It will suffice to show that $\mu \to \mu^a$ is Borel. For then, letting $c_1(\mu) = \mu - \mu^a$, and $c_{n+1}(\mu) = c_1(c_n(\mu))$,

$$\begin{aligned} M_0(Z) &= \{\mu : c_1(\mu) = \mu\}, \\ \bigcup_{1 \leqq k \leqq n} M_k(Z) &= \{\mu : c_n(\mu) = 0\} \qquad (n < \infty), \\ \bigcup_{1 \leqq k \leqq \infty} M_k(Z) &= \{\mu : \lim c_n(\mu) = 0\}, \\ \mu^c &= \lim c_n(\mu). \end{aligned}$$

If $n = 0$, $\mu \to \mu \,|\, S(i_1(\mu), \cdots, i_n(\mu)) = \mu$ is Borel. Suppose that we have proved $\mu \to \mu \,|\, S(i_1(\mu), \cdots, i_n(\mu))$ is Borel. Then the maps

$$\mu \to \mu \,|\, S(i_1(\mu), \cdots, i_n(\mu), j) = \mu \,|\, S(i_1(\mu), \cdots, i_n(\mu)) \cap S_n^j, \qquad j = 0, 1,$$

are Borel. From Lemma 5.1,

$$g(\mu) = \| \mu \| - \| \mu \,|\, S(i_1(\mu), \cdots, i_n(\mu), 0) \|$$

is Borel. Letting $h(0) = 1$ and $h(t) = 0$ for real $t \neq 0$,

$$\begin{aligned} &\mu \to \mu \,|\, S(i_1(\mu), \cdots, i_n(\mu), i_{n+1}(\mu)) \\ &\qquad = h(g(\mu))\mu \,|\, S(i_1(\mu), \cdots, i_n(\mu), 0) + [1 - h(g(\mu))]\mu \,|\, S(i_1(\mu), \cdots, i_n(\mu), 1) \end{aligned}$$

is Borel. From (11) it follows that $\mu \to \mu^a$ is Borel.

Let $\mathfrak{P}_0 = L^\infty([0,1], \nu)$ where ν is Lebesque measure. If $1 \leqq m \leqq \infty = \aleph_0$, let $\mathfrak{P}_m = L\ (J_m^\infty, \nu_m)$, where J_m are the first m integers, and ν_m is the measure with $\nu_m(i) = 2^{-i}$. The $\mathfrak{P}_m$ are von Neumann algebras when they are represented on the corresponding L^2-spaces. If μ is a standard continuous or discrete measure acting on a Borel space Z, $L^\infty(Z, \mu)$ represented on $L^2(Z, \mu)$ is spatially isomorphic to one of these algebras.

THEOREM 5.3. *If $\mathfrak{A}$ is a centrally smooth von Neumann algebra, there exists a Borel subset W of $\hat{\mathscr{F}}$, a Borel map $\mathfrak{B} : W \to \mathscr{F}$ with $\pi_1 \circ \mathfrak{B}(\xi) = \xi$, and standard measures ν_m on W for which*

$$\mathfrak{A} \cong \sum_{m=0}^{\infty} \int \mathfrak{B}(\xi) d\nu_m(\xi) \otimes \mathfrak{P}_m.$$

The measures ν_m with $m > 0$ may be chosen disjoint.

Proof. Let $(Z,\mathscr{S},\mu,\mathfrak{F})$ be a central decomposition for $\mathfrak{A}$, and $\hat{\mu}$ be the induced measure on $\hat{\mathscr{F}}$. As $\hat{\mu}$ is standard, we may choose a Borel set $W\subseteq\hat{\mathscr{F}}$ with $\hat{\mu}(\hat{\mathscr{F}}-W)=0$, and W standard in the relative structure. Decreasing W by a Borel null set, we may let $\mathfrak{B}: W\to\mathscr{F}$ be a Borel cross-section for π_1 (see [18, Theorem 6.3]). Letting Z_0 be the $\pi_1\circ\mathfrak{F}$ inverse image of W, the Z_0 quotient structure must correspond to the relative structure on W, as the latter is countably separated (see the proof of [18, Theorem 5.1]). From Lemmas 4.4 and 4.5, we have a Borel map $\xi\to\mu_\xi$ of W into $M(Z_0)_1$ with μ_ξ concentrated in $\pi_1^{-1}(\xi)$ and

$$\mathfrak{A}\cong\int^{\beta}\mathfrak{F}(\mu_\xi)d\hat{\mu}(\xi).$$

As $\mathfrak{F}(\mu_\xi)\cong\mathfrak{F}(\mu_\xi^c)\oplus\mathfrak{F}(\mu_\xi^d)$, we have from Lemma 4.1 and (1)

$$\mathfrak{A}\cong\int^{\beta^c}\mathfrak{F}(\mu_\xi^c)d\mu(\xi)\oplus\int^{\beta^d}\mathfrak{F}(\mu_\xi^d)d\mu(\xi), \tag{12}$$

where $\beta^c(\xi)=\gamma(\mu_\xi^c)$ and $\beta^d(\xi)=\gamma(\mu_\xi^d)$. Let ν_0 be the restriction of $\hat{\mu}$ to $W_0=\{\xi:\mu_\xi^c\neq 0\}$ and for $m\geqq 1$, let ν_m be the restriction of $\hat{\mu}$ to $\{\xi:\mu_\xi^d\in M_m(Z_0)\}$. Then from Lemma 4.1 and (3),

$$\mathfrak{F}(\mu_\xi^c)\cong\mathfrak{B}(\xi)\otimes\mathfrak{P}_0$$

for ν_0-almost all ξ, and for $m\geqq 1$,

$$\mathfrak{F}(\mu_\xi^d)\cong\mathfrak{B}(\xi)\otimes\mathfrak{P}_m$$

for ν_m-almost all ξ. The theorem follows from (12) and (2).

It is readily verified that the measure classes of the ν_m are uniquely determined by $\mathfrak{A}$, and that any such system of standard measures arises in this manner from a centrally smooth von Neumann algebra. The latter is in contrast with the group representation situation, in which "noncanonical" standard measures can occur (see [18, p. 164]).

6. **Global structure.** Let $\mathfrak{A}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$, $\mathfrak{Z}$ the Boolean algebra of central projections in $\mathfrak{A}$. We say $E,F\in\mathfrak{Z}$ are *spatially equivalent*, $E\cong F$, *if* $\mathfrak{A}E$ and $\mathfrak{A}F$ are spatially isomorphic (a corresponding theory may be formulated using algebraic isomorphisms). Letting $M(\mathfrak{A})$ be the operators T on $\mathfrak{H}$ with $T\mathfrak{A}T^*\subseteq\mathfrak{A}$ and $T^*\mathfrak{A}T\subseteq\mathfrak{A}$, we have $E\cong F$ if and only if there is a partial isometry $U\in M(\mathfrak{A})$ with $U^*U=E$ and $UU^*=F$. $\mathfrak{Z}$, together with the equivalence $\cong$, may be thought of as a global analogue of the lattice of all projections in $\mathfrak{A}$ with the usual equivalence. More precisely, $(\mathfrak{Z},\cong)$ is a dimension lattice in the sense of Loomis [14], and we may examine all of the concepts normally associated with such a lattice.

Projections E, $F\in\mathfrak{Z}$ are *globally disjoint* if there do not exist E_1, $F_1\in\mathfrak{Z}$ with $0\neq E_1\leqq E$, $0\neq F_1\leqq F$ and $E_1\cong F_1$. E is *globally central* if E and $I-E$

are globally disjoint. We let $\mathfrak{Z}_G$ be the collection of such projections. If $E \in \mathfrak{Z}_G$ and $F \cong E$, $F \leqq E$.

LEMMA 6.1. *$\mathfrak{Z}_G$ consists of the central projections left fixed by the spatial automorphisms of $\mathfrak{A}$.*

Proof. If E is globally central, and V induces a spatial automorphism of $\mathfrak{A}$, then $V^*EV \cong E$ and $VEV^* \cong E$ imply $V^*EV = E$. If E is not globally central choose a nonzero partial isometry $U \in M(\mathfrak{A})$ with $U^*U \leqq E$, $UU^* \leqq I - E$. Then the unitary $V = U + U^* + [I - (U^*U + UU^*)]$ induces a spatial automorphism of $\mathfrak{A}$ that does not leave E fixed.

If $E \in \mathfrak{Z}$, we let $G(E)$ be the minimal projection in $\mathfrak{Z}_G$ with $E \leqq G(E)$. We say $\mathfrak{A}$ is a *global factor* if $\mathfrak{Z}_G = \{0, I\}$, and $\mathfrak{A}$ is *globally multiplicity free* if $\mathfrak{Z}_G = \mathfrak{Z}$. $\mathfrak{A}$ is of *global type I* if there is a projection $E \in \mathfrak{Z}$ with $G(E) = I$ and E globally multiplicity free. $\mathfrak{A}$ is *globally finite* if $E \in \mathfrak{Z}$ and $E \cong I$ imply $E = I$. $\mathfrak{A}$ is *globally semi-finite* if there is a projection $E \in \mathfrak{Z}$ with $G(E) = I$ and $\mathfrak{A}E$ finite. $\mathfrak{A}$ is of *global type* II if $\mathfrak{A}$ is globally semi-finite, and there does not exist $E \in \mathfrak{Z}$ with $E \neq 0$ and $\mathfrak{A}E$ of global type I. Finally, $\mathfrak{A}$ is of *global type* III if there does not exist $E \in \mathfrak{Z}$ with $E \neq 0$ and $\mathfrak{A}E$ globally semi-finite.

It is a simple exercise to check that the centrally smooth global factors are those of the form $\mathfrak{F} \otimes \mathfrak{P}_m$, $\mathfrak{F}$ a factor, and that such an algebra has global type I if $m \geqq 1$, and global type III if $m = 0$. Thus a global factor (or in fact, any von Neumann algebra) of global type II would not be centrally smooth. The existence of such an algebra would imply that $\hat{\mathscr{F}}$ and $\tilde{\mathscr{F}}$ are nonsmooth, and the existence of uncountably many nonalgebraically isomorphic factors would follow. We are currently examining the von Neumann algebras associated with the regular representations of countable discrete groups. We remark that it is not difficult to prove that if such a group has all of its automorphism classes (other than that of the identity) infinite, then the corresponding algebra is a global factor. The rationals under addition form such a group.

We have been unable to give a global characterization for the centrally smooth von Neumann algebras. In particular, we do not know if the analogue of Guichardet's theorem for multiplicity free representations [8] is true. Specifically, is any globally multiplicity free von Neumann algebra on a separable Hilbert space centrally smooth?

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