SOME PROPERTIES OF PSEUDO-COMPLEMENTS OF RECURSIVELY ENUMERABLE SETS

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Introductory remarks. Those first order systems which exhibit some real mathematical pretensions fall into what is called in [1] the class of arithmetical logics; it is there demonstrated that that any ω-consistent and adequate arithmetical logic is incomplete and brought out that the undecidable sentence can always be taken to be a closed well-formed formula which truly expresses that $n_0 \notin S$ where n_0 is an integer and S a nonrecursive recursively enumerable set.

Thus, we are led to consider those sets of integers whose members are probably (in a system T) in the complement of a given recursively enumerable set S, or, as we shall call them, the pseudo-complements of S, a notion introduced by Davis in [3]. It is to be observed that being a pseudo-complement of S is not a purely extensional property; that is to say, the pseudo-complement of an re (recursively enumerable) set S is not simply a function of S as a set, but also of the particular representation of S in the system T. Different representations of the one set S may give rise to markedly different pseudo-complements even with respect to the same theory T.

In this paper we shall explore some of the properties of pseudo-complements of re sets in re consistent extensions of Peano arithmetic. Also, since our definition of a pseudo-complement function provides a natural setting for Davis' theorems, we state his results to achieve comprehensiveness.

We prove a separation theorem, Theorem 6, to the effect that if A and B are disjoint re sets, they can be so represented that B is the pseudo-complement of A. From this it easily follows that all re sets are pseudo-complements. The fact that the pseudo-complement of the pseudo-complement is always empty [3] distinguishes sharply the enumeration of the re sets given by a pseudo-complement function from the standard enumerations. Given two numbers, one occurring in a standard enumeration and the other produced by a pseudo-complement function, the problem arises of deciding if these numbers represent

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the same re set. This can be expressed as the decision problem of a set, and the 1-degree of unsolvability of this set is computed.

Given a representation of an re set S, the set of numbers belonging neither to S nor to its pseudo-complement constitutes the undecidable region: if n is a number in that region, then an appropriate sentence of the form $n \notin S$ is true but undecidable. To enlarge the pseudo-complement of a set S is to diminish the region of undecidability. We investigate this question with respect to creative sets and pairs of effective inseparable re sets bearing in mind that the nuclei of any consistent Rosser system are effectively inseparable [16, p. 99]. A basic theorem, Theorem 10, yields new information about EI (effectively inseparable) re sets.

Notation and preliminaries. The sources of our notation for logical symbolism and the symbolism of recursive functions are Martin Davis' *Computability and unsolvability* [1], and S. C. Kleene's *Introduction to metamathematics* [10]. We use $(x)_i$ interchangeably with $iGlx$, according to the definition of the latter given in Davis' book.

We rely heavily on [7] for notation, terminology, and development of the consistent re (recursively enumerable) extensions of Peano arithmetic $\mathscr{P}$, as well as for the arithmetization of the general metamathematics of such theories via a primitive recursive extension $\mathscr{M}$ of arithmetic [7, p. 52]. Accordingly we assume a PR-extension $\mathscr{M}$ at least as strong as the $\mathscr{M}$ of [8, p. 272], in which the basic results of recursive function theory can be established. Since we rarely have need of PR-extensions $\mathscr{M}'$, $\mathscr{M}' \neq \mathscr{M}$, we shall indicate the result of the Gödel elimination transformation, [7, p. 53], [8, pp. 266–267], by simply marking an accent on the formula to be transformed. For example, if F is a formula of $\mathscr{M}$, then F' is the formula of $\mathscr{P}$ which is the image of F under the elimination transformation.

Also, we assume an a priori Gödel numbering of our formalism, of the sort exemplified on [7]. In this formulation, the formal objects (basic symbols, terms, formulas, proofs, etc.) of a theory are numbers ab initio.

Many of the notations of [7] are taken over directly, with trivial changes. In particular, we write "$[A,K]$" for the theory $\mathscr{A}$ employing a finite set K of nonlogical constants, K always including a symbol for equality and having the set A and all instances of the induction axiom as nonlogical axioms. From now on in expressions of the form "Given a pair $(\mathscr{A},\alpha)$," which often occur in the hypotheses of theorems, it is understood that $\mathscr{A}$ is an re simply consistent extension of $\mathscr{P}$ and α is an RE-formula which numerates the axioms A of $\mathscr{A}$ in $\mathscr{P}$, [7, pp. 51, 53].

We write "$\dot{\mathrm{T}}\mathrm{hm}_\alpha$" and "$\dot{\mathrm{P}}\mathrm{f}_\alpha$" for the

$$\mathrm{P}\underset{\cdot}{\mathrm{r}}_\alpha \text{ and } \mathrm{P}\underset{\cdot}{\mathrm{r}}\mathrm{f}_\alpha$$

of [7], with trivial changes. We set $\dot{\mathrm{T}}\mathrm{hm}_\alpha = x \approx \bar{0}$ if α is not a formula of $\mathscr{M}$, a primitive recursive condition.

We stipulate that in a given formula of a theory $\mathscr{A}$ distinct English letters denote distinct formal variables, identifying occurrences of the first six formal variables $v_1, v_2, v_3, v_4, v_5, v_6$ with x, y, z, u, v, w respectively unless it is explicitly stated otherwise. Finally, it is assumed throughout that Peano arithmetic is weakly ω-consistent [7, p. 53].

Formal versions of the enumeration and iteration theorems. In [5] we saw at length that the enumeration theorem for primitive recursive predicates is formalizable in $\mathscr{M}$. Our rendition reads: For each $n \geqq 0$

$$\vdash_{\mathscr{M}} \forall x \, \forall_p [\dot{\mathrm{P}}\mathrm{rD}(p,x) \wedge \dot{\mathrm{L}}\mathrm{h}(x) \approx \bar{n} + \bar{2}$$
$$\supset (\exists z \, \forall t [\exists y \dot{\mathrm{V}}\mathrm{al}_n^p(x, t \dot{*} \bar{2}^{\cdot y}, \bar{0}) \equiv \exists y \dot{T}_n(z, (t)_{.\bar{1}}, (t)_{.\bar{2}}, \cdots, (t)_{.\bar{n}}, y)])].$$

Here, the formulas $\dot{\mathrm{P}}\mathrm{rD}(p,x)$ and $\mathrm{Val}_n^p(x, t \dot{*} \bar{2}^{\cdot y}, \bar{0})$ represent in $\mathscr{M}$ the assertions "p is a primitive recursive description of x" and "the value of x for the arguments $(t)_1, \cdots, (t)_n$ is 0," respectively; $\dot{T}_n(z, s_1, \cdots, x_n, y)$ binumerates the Kleene T-predicate in $\mathscr{M}$. The theorem expresses in $\mathscr{M}$ the well-known statement: If f is a primitive recursive function of $n+1$ variables, there is a number z_0 such that for all $t_1, \cdots, t_n$

$$\bigvee_y \; [f(t_1, \cdots, t_n, y) = 0] \leftrightarrow \bigvee_y \; T_n(z_0, t_1, \cdots, t_n, y).$$

For our purposes the following version of the iteration theorem [10, p. 347], [1, p. 147], suffices:

$$\vdash_{\mathscr{M}} \exists y \dot{T}_{m+n}(z, v_1, \cdots, v_{m+n}, y) \equiv \exists y \dot{T}_n(\dot{S}_n^m(z, v_1, \cdots, v_m),\; v_{m+1}, \cdots, v_{m+n}, y)$$

where $\dot{S}_n^m$ is the representative in m of Kleene's well-known S_n^m function, and z, y are distinct variables distinct from each of $v_1, \cdots, v_{m+n}$.

By use of these two theorems we can find numbers e and a such that for each m and n

$$\vdash_{\mathscr{P}} \exists y \mathscr{T}(\bar{m}, x, y) \vee \exists y \mathscr{T}(\bar{n}, x, y) \;\equiv\; \exists y \mathscr{T}(\overline{S_1^2(e,m,n)}, x, y),$$
$$\vdash_{\mathscr{P}} \exists y \, \mathscr{T}(\bar{m}, x, y) \wedge \exists y \mathscr{T}(\bar{n}, x, y) \;\equiv\; \exists y \mathscr{T}(\overline{S_1^2(a,m,n)}, x, y),$$

where $\mathscr{T}_n(z, x_1, \cdots, x_n, y) = (\dot{T}_n(z, x_1, \cdots, x_n, y))'$ and $z, x_1, \cdots, x_n, y$ are distinct variables. We write $m \vee n$ for $S_1^2(e,m,n)$ and $m \wedge n$ for $S_1^2(a,m,n)$ and note that $\{m \vee n\} = \{m\} \cup \{n\}$ and $\{m \wedge n\} = \{m\} \cap \{n\}$.

The definition of a pseudo-complement function of a theory $\mathscr{A} = [A, K]$. Given a pair $(\mathscr{A}, \alpha)$ the formula

$$\dot{\mathrm{T}}\mathrm{hm}_\alpha(\dot{\mathrm{S}}\mathrm{b}(\overline{\neg \exists y \mathscr{T}(z,x,y)} \,|\, {}^{\dot{z}}_{\dot{n}m_z} \, {}^{\dot{x}}_{\dot{n}m_x})),$$

where $\dot{z}, \dot{x}$ are the arithmetized representatives of v_3, v_1 respectively in $\mathscr{M}$, affirms in $\mathscr{M}$ that x is in $\mathscr{A}$ provably in the complement of the zth re set. Set $C(u,z,x,y) = \dot{T}(\dot{S}_1^1(u,z),x,y)'$. Now

$$\vdash_{\mathscr{P}} \dot{\mathrm{Thm}}_\alpha(\dot{\mathrm{Sb}}(\overline{\neg\exists y \mathscr{T}(z,x,y)}\big|^{\dot{z}\ \ \dot{x}}_{\dot{n}m_z\ \dot{n}m_x}))' \equiv \exists y C(\bar{e},z,x,y)$$

for some numeral $\bar{e}$. Let

$$f(\alpha) = K(\min_t(\mathrm{Pf}_P(\dot{\mathrm{Thm}}_\alpha(\dot{\mathrm{Sb}}(\overline{\neg\exists y \mathscr{T}(z,x,y)}\big|^{\dot{z}\ \ \dot{x}}_{\dot{n}m_z\ \dot{n}m_x}))' \equiv \exists y C(\overline{K(t)}, z, x, y),\ L(t))))$$

if α is a 1-ary RE-formula; $=0$ otherwise, where K, L are recursive pairing functions, [1, pp. 44–46]. Since the set of 1-ary RE-formulas is recursive, f is general recursive. If n and t are any integers,

$$\vdash_{\mathscr{M}} \dot{\mathrm{Sb}}(\overline{\neg\exists y \mathscr{T}(z,x,y)}\big|^{\dot{z}\ \ \dot{x}}_{\dot{n}m_{\bar{n}}\ \dot{n}m_{\bar{t}}}) \approx \overline{\neg\exists y \mathscr{T}(\bar{n},\bar{t},y)},$$

which by the above implies

$$\vdash_{\mathscr{M}} \dot{\mathrm{Th}}_\alpha(\dot{\mathrm{Sb}}\ \overline{(\neg\exists y \mathscr{T}(z,x,y)}\big|^{\dot{z}\ \ \dot{x}}_{\dot{n}m\ {}_{\bar{n}}nm_t}) \equiv \dot{\mathrm{Thm}}_\alpha(\overline{\neg\exists y \mathscr{T}(\bar{n},\bar{t},y))}$$

giving

$$\vdash_{\mathscr{P}} \dot{\mathrm{Thm}}_\alpha(\overline{\neg\exists y \mathscr{T}(\bar{n},\bar{t},y))} \equiv \exists y \mathscr{T}(\overline{S_1^1(f(\alpha),n)},\bar{t},y).$$

Using weak ω-consistency of $\mathscr{P}$, we infer $\bigvee_y \mathrm{Pf}_A(\neg\exists y \mathscr{T}(\bar{n},\bar{t},y),y) \leftrightarrow \bigvee_y^1 T(S_1^1(f(\alpha), n,t,y))$. These preliminaries put us in a position to give the following.

DEFINITION. *Given a pair* $(\mathscr{A},\alpha)$, *the function* $N_\alpha(x) = S_1^1(f(\alpha),x)$ *is a pseudo-complement function of the theory* $\mathscr{A}$ *with respect to* α.

Since $\{N_\alpha(n)\} = \{x \mid \vdash_{\mathscr{A}} \neg\exists y \mathscr{T}(\bar{n},\bar{x},y)\}$ it is immediate that if α and α' are RE formulas which enumerate $\mathscr{A}$ in $\mathscr{P}$, then

$$\bigwedge_n \ [\{N_\alpha(n)\} = \{N_{\alpha'}(n)\}].$$

Using this definition and the formalized enumeration and iteration theorems the following theorems of Davis [3] on pseudo-complements are easily and rigorously established for any theory $\mathscr{A} = [A,K]$.

THEOREM 1. *Given a recursive set* R, *there is an integer* n *such that* $\bar{R} = \{m\}$, $R = \{N_\alpha(n)\}$.

THEOREM 2. $\{N_\alpha(n \vee m)\} = \{N_\alpha(n) \wedge N_\alpha(m)\}$.

THEOREM 3. $\{N_\alpha(n) \vee N_\alpha(m)\} \subset \{N_\alpha(n \wedge m)\}$.

REMARK. Theorems 2 and 3 are for pseudo-complements the analogues of the traditional DeMorgan rules.

THEOREM 4. $\{N_\alpha(N_\alpha(m))\} = \varnothing$.

This theorem is a consequence of the second Gödel incompleteness theorem. Indeed, our definition of a pseudo-complement function of a theory $\mathscr{A}$ is designed to rigorously achieve this theorem; thus, it is readily shown that for all integers

n and $x \vdash_{\mathscr{A}} \neg \exists y \mathscr{T}(\overline{N_\alpha(n)}, \bar{x}, y) \supset \dot{\mathrm{C}}\mathrm{on}_\alpha$. Consequently, if there were numbers n and x such that $\vdash_{\mathscr{A}} \neg \exists y \mathscr{T}(\overline{N_\alpha(n)}, \bar{x}, y)$, we would have $\vdash_{\mathscr{A}} \dot{\mathrm{C}}\mathrm{on}_\alpha$; but, by 5.6 of [7] not-$\vdash_{a} \dot{\mathrm{C}}\mathrm{on}_\alpha$ for any pair $(\mathscr{A}, \alpha)$.

THEOREM 5. $\{N_\alpha(n)\} \subset \{N_\alpha(n \wedge N_\alpha(m))\} \subset \overline{\{n\}}$.

The extent of pseudo-complements. In this section we shall demonstrate that, in a suitable sense, all re sets are pseudo-complements. Putnam and Smullyan have proved the theorem: In every consistent axiomatizable extension of the theory R of [17] each pair (A,B) of disjoint re sets is exactly separable [13], [16, p. 139]. Thus, the theorem holds for re consistent extensions of $\mathscr{P}$. Their proof makes use of the theorem of Muchnik and, independently, Smullyan that every effectively inseparable pair of re sets is doubly universal [16, pp. 112, 116]. The following lemma makes precise a sense in which Smullyan's proof of this latter theorem is constructive.

LEMMA 1. *Let $(\{i_1\}, \{i_2\})$ be a pair of* re EI (*effectively inseparable*) *sets. There is a binary recursive function β such that if $\{m\} \cap \{n\} = \varnothing$, $\beta(m,n)$ is a number of* 1-1 *recursive reduction of $(\{m\}, \{n\})$ to $(\{i_1\}, \{i_2\})$.*

Proof. An inspection of the chain of results leading to Smullyan's proof reveals that what is involved in the construction of the reducing function are various operations of union, intersection, composition, and definition by induction. We can therefore successively compute indices by applications of the iteration theorem.

By use of the above lemma, a careful analysis of the proof of Putnam and Smullyan, and much judicious bookkeeping, involving more or less complicated uses of the iteration theorem, one arrives at a general recursive function $F_\alpha(\bar{m}, \bar{n}, x, y, z, u)$ of α, m, n such that a given pair $(\mathscr{A}, \alpha)$, if $\{m\} \cap \{n\} = \varnothing$, $\exists y \exists z \exists u F_\alpha(\bar{m}, \bar{n}, x, y, z, u)$ is a formula which exactly separates $(\{m\}, \{n\})$ in $\mathscr{A}$. While it is obvious that the proof of [13] is constructive, the gain here is this: by employing the techniques developed in [7], the method is rendered uniform, that is, the same effective method applies to all re consistent extensions of arithmetic. Moreover, the method is such that for *any* $\alpha, m, n, F_\alpha(\bar{m}, \bar{n}, x, y, z, u)$ is a formal analogue of the predicates Smullyan calls constructive arithmetic [16, p. 31]. Thus, $\exists y \exists z \exists u F_\alpha(\bar{m}, n, x, y, z, u)$ is always provably equivalent in arithmetic to an RE-formula. Consequently, for *any* $\alpha, m, n \vdash_{\mathscr{P}} \exists y \exists z \exists u F_\alpha(\bar{m}, \bar{n}, x, y, z, u) \equiv \exists y \mathscr{T}(\bar{e}, x, y)$ for some numeral $\bar{e}$. We define $\pi(\alpha, m, n) = K(\min_t(\mathrm{Pf}_P(\exists y \exists z \exists u F_\alpha(\bar{m}, \bar{n}, x, y, z, u) \equiv \exists y \mathscr{T}(\overline{K(t)}, x, y), L(t))))$. This proves the following:

THEOREM 6. *There is a general recursive function $\pi(\alpha, m, n)$ such that given a pair $(\mathscr{A}, \alpha)$ if $\{m\} \cap \{n\} = \varnothing$, then $\{m\} = \{x \mid \vdash_{\mathscr{A}} \exists y \mathscr{T}(\overline{\pi(\alpha, m, n)}, \bar{x}, y)\}$ and $\{n\} = \{N_\alpha(\pi(\alpha, m, n))\}$; $\{m\} = \{\pi(\alpha, m, n)\}$ if $\mathscr{A}$ is weakly ω-consistent.*

By choosing any number r of the set $\{r\} = \emptyset$, we have

COROLLARY. *All* re *sets are pseudo-complements, that is, given a pair* $(\mathscr{A},\alpha)$, *if* $S = \{m\}$, *then* $S = \{N_\alpha(\pi(\alpha, r, m))\}$.

Under the usual conditions on $\mathscr{A} = [A, K]$ and α, the function N_α is an enumerating function of all the re sets. The property expressed in Theorem 4 makes the enumeration given by $\{N_\alpha(z)\}$ strikingly different from the standard enumerations of the re sets, or from any derived from the standard types by the usual uses of the iteration theorem.

As the enumerations given by, for example, the predicates $\bigvee_y T(z, x, y)$ and $\bigvee_y T(N_\alpha(z), x, y)$ proceed, we should like to know which pairs, with one member drawn from each enumeration, name the same set. More precisely, we have the following decision problem: given numbers m, n to decide if $\{m\} = \{N_\alpha(n)\}$.

Let $Ps_\alpha = \{x \mid \{K(x)\} = \{N_\alpha(L(x))\}\}$, where K, L are recursive pairing functions. We now compute the 1-degree of unsolvability of Ps_α. For this purpose we assume familiarity with the elementary properties of and relations between the scale of degrees of unsolvability and the Kleene arithmetical hierarchy, as is presented, for example, in [14]. We borrow notation and a convention or two from the same source. (Nevertheless, although Rogers treats re sets as the ranges of partial recursive functions, we shall continue to work with them as the domains of such functions.) In particular, the notations "$A \leqq_1 B$" and "$A \equiv B$" signify that the set A is 1-1 reducible to the set B and A is recursively isomorphic to B, respectively. As usual, we write "$A \equiv_1 B$" for "$A \leqq_1 B \wedge \leqq_1 A$." Using $'$ to designate the jump-operator we set $S^{(0)} = \emptyset$, $S^{(n+1)} = (S^{(n)})'$, and also use $S^{(n)}$ as in [14] to denote the 1-degree of the set $S^{(n)}$.

DEFINITION. $\mathrm{Comp} = \{x \mid \bigwedge_z [\bigvee_y T(K(x), z, y) \leftrightarrow \sim \bigvee_y T(L(x), z, y)]\}$.

LEMMA 2. $\mathrm{Comp.} \equiv \overline{S^{(2)}}$.

Proof. This was shown in Davis' thesis [2] as far as Turing reducibility is concerned. It is very easy to extend Davis' proof so that $A \equiv_1 B$. By Myhill's theorem [12], $A \equiv B$.

THEOREM 7. $Ps_\alpha \equiv \overline{S^{(2)}}$.

Proof. $x \in Ps_\alpha \leftrightarrow \bigwedge_z [\bigvee_y T(K(x), z, y) \leftrightarrow \bigvee_y T(N_\alpha(L(x)), z, y)] \leftrightarrow \bigwedge_z [(\bigvee_{y_1} T(K(x), z, y_1) \wedge \bigvee_{y_2} T(N_\alpha(L(x)), z, y_2)) \vee (\bigwedge_{y_3} \sim T(K(x), z, y_3))) \wedge \bigwedge_{y_4} \sim T(N_\alpha(L(x)), z, y_4))] \leftrightarrow \bigwedge_z \bigwedge_{y_3} \bigwedge_{y_4} \bigvee_{y_1} \bigvee_{y_2} [(T(K(x), z, y_1) \wedge T(N_\alpha(L(x)), z, y_2)) \vee (\sim T(K(x), z, y_3) \wedge \sim T(N_\alpha(L(x)), z, y_4))] \leftrightarrow \bigwedge_z \bigvee_y R(x, z, y)$ for some recursive R. Hence, $Ps_\alpha \leqq_1 \overline{S^{(2)}}$.

Now, $\bigvee_y T(u, z, y) \leftrightarrow \bigvee_y T_2(e, u, z, y) \leftrightarrow \bigvee_y T(\phi(u), z, y)$ for some primitive recursive ϕ. Let n_0 be an integer such that $\{n_0\} = I$, the (nonnegative) integers. We define $h(0, x) = \phi(x)$, $h(k+1, x) = \phi(h(k, x))$. Since $x \in \mathrm{Comp}$ if and only if

$\{K(x) \wedge L(x)\} = \varnothing$ and $\{K(x) \vee L(x)\} = I$ we have $x \in \mathrm{Comp} \leftrightarrow J(h(x,n_0),$ $\pi(\alpha, K(x) \wedge L(x),\ K(x) \vee L(x))) \in Ps_\alpha$. Also the function ϕ is strictly monotonic increasing so $x < x' \rightarrow h(x', n_0) < h(x, n_0)$. Thus, $x \neq x' \rightarrow J(h(x', n_0),$ $\pi(\alpha, K(x') \wedge L(x'),\ K(x') \vee L(x'))) \neq J(h(x,n_0),\ \pi(\alpha, K(x) \wedge L(x),\ K(x) \vee L(x)))$ where J is the function J of [1, p. 43]. Thus, $\mathrm{Comp} \leqq_1 Ps_\alpha$, and $Ps_\alpha \equiv_1 \overline{S^{(2)}}$, giving $Ps_\alpha \equiv \overline{S^{(2)}}$. Q.E.D.

Pseudo-complements, creative sets, and effectively inseparable sets. It has already been mentioned in the Introduction that to enlarge the pseudo-complement of an re set C is to diminish the region of undecidability of that set. If C is creative, the procedure of enlarging its pseudo-complement can be continued throughout the constructive transfinite, as we shall show; in fact, we shall demonstrate that if (E, F) is a pair of re EI (effectively inseparable) sets we can effectively generate two series of re sets, each series strictly ascending from E and F, respectively and extending throughout the constructive transfinite, and have the pair (A, B) occupying the xth place so represented that B is the pseudo-complement of A.

Suppose, then, that (E,F) is a pair of re EI sets, and m, n are numbers such that $\{m\} = E$, $\{n\} = F$. Let $k(x,y)$ be a separating function for (E,F). By means of an adaptation of an argument of John Myhill, we can assume k is 1-1 [16, p. 114].

In the immediate sequel, P and Q are the functions defined in [1, p. 188], which we follow in writing $[z]_1(x_1,\cdots,x_n) = U(\min_y T_n(z,x_1,\cdots,x_n,y))$. Towards an application of the recursion theorem [10, p. 348], [1, p. 175], we require, $\bigvee_y T([z]_1(P(x)),t,y) \vee t = k([z]_1(P(x)),n) \leftrightarrow \bigvee_y T_3\,(e_1, z, x, t, y)$ $\leftrightarrow \bigvee_y T(S_1^2(e_1, z, x), t, y)$ for some number e_1 if $x = 2^{(x)_1} \wedge (x)_1 \neq 0$. $\bigvee_p \bigvee_y T([z]_1(Q(x,p)),t,y) \leftrightarrow \bigvee_y T_3(e_2,z,x,t,y) \leftrightarrow \bigvee_y T(S_1^2(e_2,z,x),t,y)$ for some number e_2 if $x = 3 \cdot 5^{(x)_3}$. We set up the scheme

$$\psi(z,x) = \begin{cases} m & \text{if } x = 1, \\ S_1^2(e_1,z,x) & \text{if } x = 2^{(x)_1} \wedge (x)_1 \neq 0, \\ S_1^2(e_2,z,x) & \text{if } x = 3 \cdot 5^{(x)_3}, \\ 17 & \text{otherwise.} \end{cases}$$

Kleene's recursion theorem guarantees and provides a solution z_0 of the equation $\psi(z,x) = [z]_1(x)$. We define $g(x) = [z_0]_1(x)$, and see that g is primitive recursive.

LEMMA 3.

$$x \in \mathcal{O} \rightarrow E \subset \{g(x)\}, \quad x = 1 \rightarrow E = \{g(x)\},$$

$$x <_0 y \rightarrow \{g(x)\} \subset \{g(y)\},\ \{g(x)\} \neq \{g(y)\}.$$

Proof. By a routine inductive argument over $\mathcal{O}$, using the fact that k is a separating function for (E,F).

Let ϕ be a primitive recursive function such that $\bigwedge_n[\{n\} = \{\phi(n)\} \wedge n < \phi(n)]$. Consider the equivalences, $\bigvee_y T([z]_1(P(x)),t,y) \vee t = k(\phi(g(P(x))), [z]_1(P(x))) \leftrightarrow \bigvee_y T_3(e_3,z,x,t,y) \leftrightarrow \bigvee_y T_3(e_3,z,x,t,y) \wedge u = u \leftrightarrow \bigvee_y T_4(e_5,u,z,x,t,y) \leftrightarrow \bigvee_y T(S_1^3(e_5, u, z, x), t, y)$ for some numbers e_3, e_5 if $x = 2^{(x)_1} \wedge (x)_1 \neq 0$. $\bigvee_y \bigvee_q T([z]_1(Q(x,q)),t,y) \leftrightarrow \bigvee_y T_3(e_4,z,x,t,y) \leftrightarrow \bigvee_y T_3(e_4,z,x,t,y) \wedge u = u \leftrightarrow \bigvee_y T_4(e_6,u,z,x,t,y) \leftrightarrow \bigvee_y T(S_1^3(e_6,u,z,x),t,y)$ for some numbers e_4,e_6 if $x = 3 \cdot 5^{(x)_3}$. We define $\psi(z,x)$ by cases

$$\psi(z,x) = \begin{cases} n & \text{if } x = 1, \\ S_1^3(e_5,n,z,x) & \text{if } x = 2^{(x)_1} \wedge (x)_1 \neq 0, \\ S_1^3(e_6,n,z,x) & \text{if } x = 3 \cdot 5^{(x)_3}, \\ 19 & \text{otherwise.} \end{cases}$$

The recursion theorem provides us with a number z_1 such that $\psi(z_1,x) = [z_1]_1(x)$; we set $h(x) = [z_1]_1(x)$, and note that h is primitive recursive. Also, $\bigwedge_x[x \in \mathcal{O} \wedge x \neq 1 \rightarrow n < h(x)]$.

LEMMA 4. $\bigwedge_x \bigwedge_y[x <_0 y \rightarrow F \subset \{h(x)\} \subset \{h(y)\}]$.

Proof. By induction over $\mathcal{O}$.

LEMMA 5. $\bigwedge_x[x \in \mathcal{O} \rightarrow \{g(x)\} \cap \{h(x)\} = \varnothing\}]$.

Proof. By induction over $\mathcal{O}$. If $x = 1$, then $\{g(x)\} = \{m\}$ and $\{h(x)\} = \{n\}$ and $\{m\} \cap \{n\} = \varnothing$. Suppose $x = 2^{(x)_1}$, $(x)_1 \neq 0$, and assume the lemma holds for all y such that $y <_0 x$. Hence, $t \in \{g(x)\} \cap \{h(x)\} \rightarrow k(g(P(x)),n) = k(\phi(g(P(x))), h(P(x))) \vee k(\phi(g(P(x))), h(P(x))) \in \{g(y)\}$ for $y <_0 x \vee k(g(P(x)),n) \in \{h(y)\}$ for $y <_0 x$ using Lemma 4.

We cannot have $k(g(P(x)),n) = k(\phi(g(P(x))), h(P(x)))$ because $g(P(x)) < \phi(g(P(x)))$ and k is 1-1. Suppose for some $y <_0 x$, $k(\phi(g(P(x))), h(P(x))) \in \{g(y)\}$. Since $y <_0 x \rightarrow y \leqq_0 P(x), k(\phi(g(P(x))), h(P(x))) \in \{g(P(x))\}$. Hence, $k(\phi(g(P(x))), h(P(x))) \in \{\phi(g(P(x)))\}$. Using the induction hypothesis, $k(\phi(g(P(x))), h(P(x))) \notin \{\phi(g(P(x)))\}$, since $\{\phi(g(P(x)))\} = \{g(P(x))\}$.

Suppose $k(\phi(g(P(x))),n) \in \{h(y)\}$ for $y <_0 x$. Let y_0 be the least y in the $<_0$ ordering such that $y <_0 x$ and $k(g(P(x)),n) \in \{h(y)\}$ [11, p. 409]. Then $y_0 \neq 1$ since $\{h(1)\} = n$ and $\{m\} \subset \{g(P(x))\}$. By the minimal property of y_0 in the $<_0$ ordering, it is obvious that we may assume y_0 is of the form $2^{(y_0)_1}$. Then $k(g(P(x)),n) \notin \{h(P(y_0))\}$ and therefore $k(g(P(x)),n) = k(\phi(g(P(y_0))), h(P(y_0)))$, using Lemma 4. Consequently, $n = h(P(y_0))$ and $g(P(x)) = \phi(g(P(y_0)))$. This implies $P(y_0) = 1$ and that $g(P(x)) = \phi(m)$. But since $\{\phi(m)\} = \{m\}$, we must have $P(x) = 1$ by Lemma 3. This entails $m = g(1) = \phi(m)$, which is false.

Suppose $x = 3 \cdot 5^{(x)_3}$. Then $t \in \{g(x)\} \cap \{h(x)\} \rightarrow \bigvee_p \bigvee_q (t \in \{g(Q(x,p))\} \cap$

$\{h(Q(x,q))\})$. It follows that $\bigvee_p(t \in \{g(Q(x,p))\} \cap \{h(Q(x,p))\}$, which contradicts the induction hypothesis. The lemma is proved. Q.E.D.

We define $p_\alpha(x) = \pi(\alpha, g(x), h(x))$ and have established

THEOREM 8. *Given a pair* $(\mathscr{A}, \alpha)$ *with* $\mathscr{A}$ *weakly* ω*-consistent, if* (E, F) *is a pair of* re EI *sets, there is a recursive function* p_α *such that*

1. $x \in \mathcal{O} \to E \subset \{p_\alpha(x)\}$, $F \subset \{N_\alpha(p_\alpha(x))\}$.
2. $x = 1 \to E = \{p_\alpha(x)\}$, $F = \{N_\alpha(p_\alpha(x))\}$.
3. $x <_0 y \to \{p_\alpha(x)\} \subset \{p_\alpha(y)\} \wedge \{N_\alpha(p_\alpha(x))\} \subset \{N_\alpha(p_\alpha(y))\}$.

(*The inclusion is in the strict sense.*)

The above proof is easily modified (and, in fact, simplified) to yield the following.

THEOREM 9. *Given a pair* $(\mathscr{A},\alpha)$ *with* $\mathscr{A}$ *weakly* ω*-consistent, a creative set* C*, an* re *set* $S \subset \bar{C}$*, and numbers* m,n *such that* $\{m\} = C$, $\{n\} = S$*, then there is a recursive function* p_α *such that*

1. $x \in \mathcal{O} \to \{p_\alpha(x)) = C$, $\{N_\alpha(p_\alpha(x))\} \subset \bar{C}$.
2. $x <_0 y \to \{N_\alpha(p_\alpha(x))\} \subset \{N_\alpha(p_\alpha(y))\}$, $\{N_\alpha(p_\alpha(x))\} \neq \{N_\alpha(p_\alpha(y))\}$.
3. $\{N_\alpha(p_\alpha(I))\} = S$.

Theorem 3 tells us that if (A,B) is a pair of re sets, the union of their pseudo-complements is always a subset of the pseudo-complement of their intersection $(\{N_\alpha(m) \vee N_\alpha(n)\} \subset \{N_\alpha(m \wedge n)\})$. The question arises: under what conditions do equality or strict inequality hold? It is easily shown, using Theorem 6, that if (A,B) is any pair of recursive sets, there are numbers m,n such that $A = \{m\}$, $B = \{n\}$ and $\{N_\alpha(m) \vee N_\alpha(n)\} = \{N_\alpha(m \wedge n)\}$. It is not hard to show that if (E,F) is a recursively inseparable re pair there are numbers m,n such that $E = \{m\}$, $F = \{n\}$ and $\{N_\alpha(m \wedge n)\} - \{N_\alpha(m) \vee N_\alpha(n)\}$ is not re.

We now prove

THEOREM 10. *There is a uniform effective procedure whereby if* (E,F) *is a pair of* re EI *sets, we can find numbers* m,n *such that given any pair* $(\mathscr{A},\alpha)$ *we have* $E \subset \{x \mid \vdash_{\mathscr{A}} \exists y\, \mathscr{T}(\bar{m},\bar{x},y)\}$, $F \subset \{x \mid \vdash_{\mathscr{A}} \exists y \mathscr{T}(\bar{n},\bar{x},y)\}$, $\{N_\alpha(m \wedge n)\} = I$*, and can generate an infinite* re *subset of* $\{N_\alpha(m \wedge n)\} - \{N_\alpha(m) \vee N_\alpha(n)\}$*. In addition, whenever* $\mathscr{A}$ *is weakly* ω*-consistent, then* $\{m\} = E$ *and* $\{n\} = F$.

Proof. We can take enumerating functions f,g of E,F respectively such that their respective defining formulas G,H in $\mathscr{P}$ are RE-formulas. In fact, G and H can be taken to be formal analogues predicates which are constructive arithmetic in the sense of Smullyan [16, p. 31], as follows from a formalization of results of Davis [1, p. 113], [4] or Smullyan [16]. Then, by the fruitful argument of Rosser [15] the formula $\exists y[G(y,x) \wedge \forall z[z \leqq y \supset \neg H(z,x)]]$ separates (E,F) in $\mathscr{P}$ and a fortiori in $\mathscr{A}$ and $E = \{x \mid \vdash_{\mathscr{A}} \exists y[G(y,\bar{x}) \wedge \forall z[z \leqq y \supset \neg H(z,\bar{x})]]\}$ if $\mathscr{A}$ is weakly ω-consistent. By the symmetry of the Rosser construction the formula $\exists y[H(y,x) \wedge \forall z[z \leqq y \supset \neg G(z,x)]]$ separates (F,E) in $\mathscr{P}$ and $\mathscr{A}$; and again if $\mathscr{A}$ is weakly ω-consistent $F=\{x \mid \vdash_{\mathscr{A}} \exists y[H(y,\bar{x}) \wedge \forall z[z \leqq y \subset \neg G(z,\bar{x})]]$. Let m,n be numbers, which certainly can be found, such that

$$\vdash_{\mathscr{P}} \exists y[G(y,x) \wedge \forall z[z \leqq y \supset \neg H(z,x)]] \equiv \exists y \mathscr{T}(\bar{m},x,y),$$

$$\vdash_{\mathscr{P}} \exists y[H(y,x) \wedge \forall z[z \leqq y \supset \neg G(z,x)]] \equiv \exists y \mathscr{T}(\bar{n},x,y).$$

It is easy to verify that

$$\vdash_{\mathscr{P}} \exists y[G(y,x) \wedge \forall z[z \leqq y \supset \neg H(z,x)]] \supset \neg \exists y[H(y,x) \wedge \forall z[z \leqq y \supset \neg G(z,x)]].$$

Hence, $\vdash_{\mathscr{P}} \exists y \mathscr{T}(\bar{m}, x, y) \supset \neg \exists y \mathscr{T}(\bar{n}, x, y)$. Of course, this implies $\vdash_{\mathscr{A}} \exists y \mathscr{T}(\bar{m},x,y) \supset \neg \exists y \mathscr{T}(\bar{n},x,y)$, giving $\vdash_{\mathscr{A}} \forall x[\exists y \mathscr{T}(\bar{m},x,y) \vee \neg \exists y \mathscr{T}(\bar{n},x,y)]$. We conclude that $\{N_\alpha(m \wedge n)\} = I$.

A theorem due to Smullyan [16, p. 126] guarantees the existence of a recursive permutation i with Gödel number i such that $i(K_1) = E$, $i(K_2) = F$, where K_1, K_2 are the re EI Kleene sets:

$$K_1 = \left\{ x \mid \bigvee_y (T(K(x),x,y) \wedge \bigwedge_{z \leqq y} \sim T(L(x),x,z)) \right\},$$

$$K_2 = \left\{ x \mid \bigvee_y (T(L(x),x,y) \wedge \bigwedge_{z \leqq y} \sim T(K(x),x,z)) \right\}.$$

Let j be a Gödel number of i^{-1}.

$$\begin{aligned} x \in i^{-1}(\{w\}) &\leftrightarrow \bigvee_t \bigvee_{y_1} \bigvee_{y_2} [T(j,t,y_1) \wedge T(w,t,y_2) \wedge U(y_1) = x] \\ &\leftrightarrow \bigvee_y T(\phi(j,w),x,y) \end{aligned}$$

where ϕ is primitive recursive. Now, we define sets S_1, S_2 by

$$x \in S_1 \leftrightarrow x \in i^{-1}(\{N_\alpha(n)\}) \leftrightarrow x \in \{\phi(j, N_\alpha(n))\},$$

$$x \in S_2 \leftrightarrow x \in i^{-1}(\{N_\alpha(m)\}) \leftrightarrow x \in \{\phi(j, N_\alpha(m))\}.$$

Then, $K_1 \subset S_1$, $K_2 \subset S_2$, $S_1 \cap K_2 = \varnothing$, $S_2 \cap K_1 = \varnothing$. Let $z_1 = \pi(\alpha, k_2, \phi(j, N_\alpha(n)))$, $z_2 = \pi(\alpha, k_1, \phi(j, N_\alpha(m)))$ where $\{k_1\} = K_1$ and $\{k_2\} = K_2$. Then $K_2 = \{x \mid \vdash_{\mathscr{A}} \exists y \mathscr{T}(\bar{z}_1, \bar{x}, y)\}$, $K_1 = \{x \mid \vdash_{\mathscr{A}} \exists y \mathscr{T}(\bar{z}_2, \bar{x}, y)\}$, $\{N_\alpha(z_1)\} = S_1$, $\{N_\alpha(z_2)\} = S_2$. In addition, if $\mathscr{A}$ is weakly ω-consistent, $\{z_1\} = K_2$, $\{z_2) = K_1$. Consider the function

$$\begin{aligned} f(\alpha,z,x) &= \min_y[\mathrm{Pf}_P[\dot{\mathrm{T}}\mathrm{hm}_\alpha(\dot{\mathrm{S}}\mathrm{b}(\overline{\neg \exists y \mathscr{T}(z, x, y)} \,|^{\dot{z}}_{\dot{n}m_{\bar{z}}} {}^{\dot{x}}_{\dot{n}m_{\bar{x}}}))', y]] \\ &\qquad \text{if } \alpha \text{ is a 1-ary RE-formula} \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

f is partial recursive, and so for some number e and primitive recursive function θ, $f(\alpha,z,x) = U(\min_y T_3(e,\alpha,z,x,y)) = U(\min_y T(\theta(\alpha,z),x,y))$. When α is as we have assumed throughout, notice that by weak ω-consistency of $\mathscr{P}$,

$$\bigvee_y T(\theta(\alpha,z_1),x,y) \leftrightarrow \bigvee_y T(N_\alpha(z_1),x,y),$$

$$\bigvee_y T(\theta(\alpha,z_2),x,y) \leftrightarrow \bigvee_y T(N_\alpha(z_2),x,y).$$

Consider the number $J(\theta(\alpha, z_2), \theta(\alpha, z_1))$ which we call t. Suppose $t \in \{\theta(\alpha, z_1)\} \cap \{\theta(\alpha,z_2)\}$. We define $y_1 = \min_y T(\theta(\alpha,z_1),t,y)$, $y_2 = \min_y T(\theta(\alpha,z_2),t,y)$. Now, $f(\alpha,z_1,t) = U(y_1)$, $f(\alpha,z_2,t) = U(y_2)$. Thus,

$$\mathrm{Pf}_P[(\dot{\mathrm{Thm}}_\alpha(\dot{\mathrm{Sb}}(\overline{\neg\exists y \mathscr{T}(z,x,y)}|^{\dot{z}}_{\dot{n}m_{\bar{z}_1}}{}^{\dot{x}}_{\dot{n}m_{\bar{t}}})))', U(y_1)]$$

and

$$\mathrm{Pf}_P[(\dot{\mathrm{Thm}}_\alpha(\dot{\mathrm{Sb}}(\overline{\neg\exists y \mathscr{T}(z,x,y)}|^{\dot{z}}_{\dot{n}m_{\bar{z}_2}}{}^{\dot{x}}_{\dot{n}m_{\bar{t}}})))', U(y_2)].$$

Since

$$\{x \mid \vdash_{\mathscr{A}} \exists y \mathscr{T}(\bar{z}_1,\bar{x},y)\} \neq \{x \vdash_{\mathscr{A}} \exists y \mathscr{T}(\bar{z}_2,\bar{x},y)\}, \quad z_1 \neq z_2,$$

and

$$z_1 \neq z_2 \rightarrow \dot{\mathrm{Sb}}(\overline{\neg\exists y\, \mathscr{T}(z,x,y)}\,|^{\dot{z}}_{\dot{n}m_{\bar{z}_1}}{}^{\dot{x}}_{\dot{n}m_{\bar{t}}}) \neq \dot{\mathrm{Sb}}(\overline{\neg\exists y\, \mathscr{T}(z,x,y)}|^{\dot{z}}_{\dot{n}m_{\bar{z}_2}}{}^{\dot{x}}_{\dot{n}m_{\bar{t}}}) \rightarrow$$

$$U(y_1) \neq U(y_2) \rightarrow y_1 \neq y_2.$$

Assume, without loss of generality, that $y_1 < y_2$. Then $\bigvee_y[T(\theta(\alpha,z_1),t,y) \wedge \bigvee_{z \leqq y} \sim T(\theta(\alpha,z_2),t,z)]$. That is, $\bigvee_y[T(L(t)t,y) \wedge \bigwedge_{z\leqq y} \sim T(K(t),t,z)] \leftrightarrow t \in K_2$. But $\{\theta(\alpha,z_1)\} \cap K_2 = \varnothing$. Hence $t \notin \{\theta(\alpha,z_1)\} \cap \{\theta(\alpha,z_2)\}$. Similarly, if $y_2 < y_1$. Thus, $t \notin \{\theta(\alpha, z_1)\} \cap \{\theta(\alpha, z_2)\}$. Suppose $t \in \{\theta(\alpha, z_1)\} - \{\theta(\alpha, z_2)\}$. Then $\bigvee_y[T(\theta(\alpha, t_1), t, y) \wedge \bigwedge_{z\leqq y} \sim (T(\theta(\alpha, z_2), t, z)] \leftrightarrow \bigvee_y[T(L(t), t, y) \wedge \bigwedge_{z \leqq y} \sim T(K(t), t, y)] \leftrightarrow t \in K_2$. But $t \in \{\theta(\alpha, z_1)\} \rightarrow t \notin K_2$. Contradiction. Hence $t \notin \{\theta(\alpha,z_1)\} - \{\theta(\alpha,z_2)\}$. By symmetry (simply interchange 1 and 2 and K and L), $t \notin \{\theta(\alpha,z_2)\} - \{\theta(\alpha,z_1)\}$. Thus, $t \notin \{\theta(\alpha,z_1)\} \cup \{\theta(\alpha,z_2)\}$. Consequently, $i(t) \notin \{N_\alpha(m) \vee N_\alpha(n)\}$. Since this procedure is capable of indefinite iteration, that is, we add t to one of S_1, S_2 and go through the business again, and arrive at $i(t') \notin \{N_\alpha(m) \vee N_\alpha(n)\}$, the theorem follows. Q.E.D.

Remark. It is the use of $z_1 = \pi(\alpha, k_2, \phi(j, N_\alpha(n)))$ and $z_2 = \pi(\alpha, k_1, \phi(j, N_\alpha(m)))$ that makes the argument of Theorem 10 valid for systems $\mathscr{A}$ which are not weakly ω-consistent. If we had taken $\{n_1\} = S_1$, $\{n_2\} = S_2$ and

$$g(\alpha,z,x) = \min_y[\mathrm{Pf}_P[\dot{\mathrm{Thm}}_\alpha(\dot{\mathrm{Sb}}(\overline{\exists y \mathscr{T}(z,x,y)}|^{\dot{z}}_{\dot{n}m_{\bar{z}}}{}^{\dot{x}}_{\dot{n}m_{\bar{x}}}))',y]],$$

then for some number e and primitive recursive function η

$$g(\alpha,z,x) = U(\min_y T_3(e,\alpha,z,x,y)) = U(\min_y T(\eta(\alpha,z),x,y)).$$

But, we cannot assert that $S_1 = \{\eta(\alpha,n_1)\}$, $S_2 = \{\eta(\alpha,n_2)\}$ if $\mathscr{A}$ is not weakly ω-consistent, that $\{\eta(\alpha,n_1)\} \cap K_2 = \varnothing$ and $\{\eta(\alpha,n_2)\} \cap K_1 = \varnothing$. Thus, we cannot carry through the argument of Theorem 10.

A simple alteration in the argument of Theorem 10 yields a general assertion concerning re EI sets. By taking in place of the sets $\{N_\alpha(n)\}$, $\{N_\alpha(m)\}$, any re sets $\{m\}$, $\{n\}$ such that $E \subset \{m\}$, $F \subset \{n\}$, $E \cap \{n\} = \varnothing$, $F \cap \{m\} = \varnothing$, successively computing Gödel numbers of recursive operations defined in the course of the proof of Theorem 10, and a couple of further uses of the iteration theorem, we easily arrive at the following theorem which extends Theorem 12(b) of [16].

THEOREM 11. *There is a primitive recursive function $\gamma(x,y)$ such that given a pair (E,F) of* re EI *sets and a pair $(\mathscr{A},\alpha)$ and an i such that $i(K_1) = E$, $i(K_2) = F$, $\gamma(\alpha,i)$ is a number of a recursive function $\delta_\alpha(x,y,v,w)$ such that if $\{n_1\} = E$, $\{n_2\} = F$, then whenever $\{m\} \cap F = \varnothing$, $\{n\} \cap E = \varnothing$, we have $\delta_\alpha(n_1,n_2,m,n) \notin \{m\} \cup E \cup \{n\} \cup F$.*

REMARK. This last theorem provides a uniform effective method whereby given a consistent re extension $\mathscr{A}$ of $\mathscr{P}$ and a pair (E,F) of re EI sets one can construct a recursive function k such that if $\{m\} \cap F = \varnothing$, $\{n\} \cap E = \varnothing$, then $k(m,n) \notin E \cup F \cup \{m\} \cup \{n\}$. We do not require that $\{m\}$ and $\{n\}$ be disjoint.

It easily follows that the difference $\{N_\alpha(m \wedge n)\} - \{N_\alpha(m) \vee N_\alpha(n)\}$ of Theorem 10 is not recursively enumerable.

Since the first part of the proof of Theorem 10 applies to any pair of disjoint re sets we extract the following.

THEOREM 12. *Given a pair (A,B) of disjoint* re *sets, there are numbers m and n such that given any pair $(\mathscr{A}, \alpha)$, $A \subset \{x \mid \vdash_{\mathscr{A}} \exists y \mathscr{T}(\bar{m}, \bar{x}, y)\}$, $B \subset \{x \mid \vdash_{\mathscr{A}} \exists y \mathscr{T}(\bar{n}, \bar{x}, y)\}$ and $\{N_\alpha(m \wedge n)\} = I$; if $\mathscr{A}$ is any weakly ω-consistent extension of $\mathscr{P}$, we also have $A = \{m\}$, $B = \{n\}$.*

Finally, we relate the well-known diagonal set $K = \{x \mid x \in \{x\}\}$ to its analogue for pseudo-complements, the set $L_\alpha = \{x \mid x \in \{N_\alpha(x)\}\}$.

THEOREM 13. *Given a pair $(\mathscr{A},\alpha)$ with $\mathscr{A}$ weakly ω-consistent, the pair (K, L_α) is* EI *with the separating function $\delta_\alpha(m,n) = \pi(\alpha,n,m)$.*

Proof. Consider m,n such that $K \subset \{m\}$, $L_\alpha \subset \{n\}$, $\{m\} \cap \{n\} = \varnothing$. Then $z_0 = \pi(\alpha,m,n) \in \{n\} \leftrightarrow z_0 \in \{z_0\} \leftrightarrow z_0 \in K$. But, $K \cap \{n\} = \varnothing$, so $\pi(\alpha,n,m) \notin \{n\}$, since K is creative. $z_0 \in \{m\} \leftrightarrow z_0 \in \{N_\alpha(z_0)\} \leftrightarrow z_0 \in L_\alpha$. But $L_\alpha \cap \{m\} = \varnothing$, so $\pi(\alpha, n, m) \notin \{m\}$. Consequently, $\pi(\alpha\ n,\ m) \notin \{m\} \cup \{n\}$.

COROLLARY. *Given any pair $(\mathscr{A},\alpha)$, the set L_α is creative.*

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