ON THE RADON-NIKODYM DERIVATIVES
OF MEASURABLE TRANSFORMATIONS[1]

BY

DOROTHY MAHARAM

1. Introduction.

1.1. Let T be a 1-1 transformation of a measure space (X, m); we assume T measurability-preserving, but not necessarily measure-preserving. For each $n = 1, 2, \cdots$, the Radon-Nikodym derivative $\omega_n(x)$ is determined (almost everywhere) so that $\int_A \omega_n(x)\, dm(x) = m(T^n A)$ for every measurable set $A \subset X$. The main theorem of this paper (Theorem 3, 5.1) is that, if $m(X)$ is finite, we may discard a *fixed* null set from X in such a way that, for all remaining points x, the set of n's for which $\omega_n(x) \geq \alpha$ has a well-defined relative density in the set of n's for which $\omega_n(x) \geq \beta$, whenever $0 \leq \beta \leq \alpha$. We also show (Theorem 4, 5.2) that this relative density has some useful properties provided T is incompressible. If we were willing to let the discarded null set depend on α and β (or, what comes to the same thing, to restrict attention to only countably many α's and β's), the theorem would be relatively straightforward. The need for obtaining a *single* "bad" null set is responsible for most of the complications of the proof; but it also makes the relative density much more useful. The author hopes to use it in a later paper, to obtain conditions for the existence of an invariant measure, and to construct one when one exists.

The main theorems are deduced from two others (Theorems 1 and 2, 2.1 and 4.1) which may also be of interest. We assume more about T (that it is measure-preserving), and obtain conclusions about the relative frequencies of n's for which $f(T^n x) \geq \alpha$, f being a positive and summable function (the hypothesis actually employed is a little weaker than this). We also include some examples (3.5, 5.3) to show that the hypotheses in our theorems cannot be omitted.

Both Theorems 1 and 3 are easily reduced to the case in which T is incompressible, and for the greater part of the paper we work with incompressible transformations. The general idea of the proof of Theorem 1 is to apply the Halmos-Hopf ergodic theorem, not to f itself, but to a suitably "smooth" modification g of f (or rather, for technical convenience, of $1/f$). Our construction of g requires severe restrictions on X, but has the effect that, once we obtain the desired relations for *rational* α and β, we can infer them for *all* α and β. Finally the re-

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strictions on X are removed one at a time, in §3, using standard techniques. Once the existence of the relative density has been established (Theorem 1), its properties (Theorem 2) follow less laboriously, via ergodic theory; and the deduction of Theorems 3 and 4 is straightforward. We remark that, while the proof of Theorem 1 could be greatly simplified by assuming that T is ergodic, this would not help with Theorem 3, even if we restrict it to ergodic T; for Theorem 3 is obtained by applying Theorem 1 to a nonergodic transformation.

1.2. *Notation.* We suppose throughout that (X, m) is a measure space, the measure m being countably additive, σ-finite, nonnegative and complete (subsets of null sets are measurable). T is a 1-1 map of X onto X such that both T and T^{-1} preserve measurability and null sets (but not necessarily the measure). A set $E \subset X$ is "invariant" if $E = TE$. Clearly every null set E is contained in the invariant null set $\bigcup \{T^i E \mid i = 0, \pm 1, \pm 2, \cdots\}$. These expressions also exemplify a notation-simplifying device we shall often use: the omission of brackets in expressions like $T^i E$, mE, etc.

Throughout the paper, we make the convention that $0/0 = 0$.

The characteristic function of a set E is denoted by $\mathcal{X}(E)$ or $\mathcal{X}E$; its value at x is $\mathcal{X}(E; x)$. The cardinal number of E is $|E|$. The empty set is $\varnothing$; the set of positive integers is $\mathcal{N}$, and the symbol n always denotes a positive integer. If $A \subset \mathcal{N}$ (and $n \in \mathcal{N}$), A_n denotes the set $\{j \mid j \in A, j \leq n\}$. If $A \subset B \subset \mathcal{N}$, we define the "relative density of A in B", written $d(A, B)$, to be the limit as $n \to \infty$ (if it exists) of $d_n(A, B) = |A_n| / |B_n|$. (Thus $d(\varnothing, \varnothing) = 0$, because of our convention about $0/0$.) In particular, if $A(x) = \{n \mid \omega_n(x) \geq \alpha\}$ and $B(x) = \{n \mid \omega_n(x) \geq \beta\}$, where $x \in X$ and $0 \leq \beta \leq \alpha \ (< \infty)$, we write $d(A(x), B(x))$ (if it exists) as $d(\alpha, \beta; x)$. Again, if f is a real-valued function on X, and $A(f, x) = \{n \mid fT^n x \geq \alpha\}$, $B(f, x) = \{n \mid fT^n x \geq \beta\}$, we write $d_n(A(f, x), B(f, x))$ as $D_n(\alpha, \beta; x)$, and its limit $d(A(f, x), B(f, x))$ as $D(\alpha, \beta; x)$[2].

1.3. *Measurability.* A subset E of the product $X \times Y$ of two measure spaces will be called "fully measurable" if

(i) E is measurable in $X \times Y$,

(ii) the section $x = $ constant of E (that is, $\{y \mid (x, y) \in E\}$) is measurable in Y for each $x \in X$,

(iii) the sections $y = $ constant of E are measurable in X for all $y \in Y$. Similarly, a function f on $X \times Y$ is "fully measurable" if f is measurable and also $f(x, y)$ is measurable in y for each fixed x, and in x for each fixed y. These notions extend to products of more than two factors; we require measurability when any subset of the variables is fixed, as a function of the remaining variables.

Given a measurable set $E \subset X \times Y$, we can always remove a null set from E so that the remaining set E_1 is fully measurable. For there is a null set $N_1 \subset X$ outside

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[2] Of course $D(\alpha, \beta; x)$ depends also on f; but as we consider only one f at a time we need not incorporate this dependence into the notation.

which the x-section of E is measurable, and similarly there is a null set $N_2 \subset Y$ outside which the y-section is measurable; we remove $E \cap \{(N_1 \times Y) \cup (X \times N_2)\}$. By applying this to each of the sets $\{(x, y) | f(x, y) < \rho\}$ for ρ rational, we see: *given a measurable function f on $X \times Y$, we can alter f on a null set so that it becomes fully measurable*. We shall make frequent use of this later.

Except in 3.1 below, where nonmeasurable sets may occur, all sets and functions arising will be measurable. Usually the verifications of measurability are routine arguments and so omitted. The following results will be useful in dealing with less routine situations. In all of them, f is an extended-real measurable function on the measure space (X, m), and A is a fixed measurable subset of X.

(1) If $B(y) = \{x | f(x) > f(y)\}$, for each $y \in Y$, then $m(A \cap B(y))$ is a measurable function of y.

For each $\alpha \geqq 0$ we must prove that the set $C_\alpha = \{y \mid y \in X, m(A \cap B(y)) > \alpha\}$ is measurable. For each rational number ρ, write $L_\rho = \{x \mid x \in X, f(x) < \rho\}$, $U_\rho = \{x \mid x \in X, f(x) > \rho\}$. It is easy to verify that $C_\alpha = \bigcup\{L_\rho \mid m(A \cap U_\rho) > \alpha\}$, a countable union of measurable sets.

(2) If $D(y) = \{x | f(x) \leqq f(y)\}$, for each $y \in Y$, then $m(A \cap D(y))$ is a measurable function of y.

For we can write A as the union of pairwise disjoint sets A_n ($n \in \mathcal{N}$), each measurable and of finite measure, and have $m(A \cap D(y)) = \sum_{n=1}^{\infty} \{m(A_n) - m(A_n \cap B(y))\}$, a sum of measurable functions, by (1).

A similar argument now shows:

(3) If $E(y) = \{x | f(x) = f(y)\}$, then $m(A \cap E(y))$ is a measurable function of y.

2. Relative densities for functions.

2.1. In this section we begin the proof of:

THEOREM 1. *Suppose T is measure-preserving, and f is a positive measurable real function on X such that, for every $\alpha > 0$, $m\{x | f(x) \geqq \alpha\} < \infty$. Then there exists an invariant null set N such that, whenever $x \in X - N$ and $0 \leqq \beta \leqq \alpha (< \infty)$, then $D(\alpha, \beta; x)$ exists.*

The proof begins by establishing the theorem in a special case.

LEMMA 1. *Theorem 1 is true if T is incompressible and X is a linear interval (perhaps infinite) with Lebesgue measure.*

2.2. **Proof of Lemma 1.** Since X is now "σ-normal" in the sense of [4], we can apply [4, Theorem 6] to express X as a union of pairwise disjoint measurable invariant sets,

$$X = N_0 \cup \bigcup \{Z_n | n \in \mathcal{N}\},$$

where N_0 is null and where (Z_n, m) is isometric (that is, isomorphic in a measure-preserving way) to a product $(X_n, v_n) \times (Y_n, \mu_n)$ of σ-normal measure spaces, in such a way that (i) each "fibre" $(x_n \times Y_n, \mu_n)$ ($x_n \in X_n$) is invariant under T i(or,

more accurately, under the transformation which corresponds to T under the isomorphism between Z_n and $X_n \times Y_n$; we still denote this by T), and (ii) T is ergodic and measure-preserving on it. It will evidently suffice to prove the theorem for each Z_n, so we may replace X by $X_n \times Y_n$.

Now, as shown in [4, Theorem 6], there are only three possibilities for (Y_n, μ_n):

(a) It consists of a finite number of points, permuted by T. on $X_n \times Y_n$, and the assertion of Theorem 1 is trivial.

Here T is periodic all $x_n \in X_n$.

(b) It consists of points p_i, $i = 0, \pm 1, \pm 2, \cdots$, and $T(x_n, p_i) = (x_n, p_{i+1})$ for Because T is incompressible, this case cannot arise here.

(c) The final possibility is that (Y_n, μ_n) is isometric to a linear interval (perhaps infinite) with Lebesgue measure.

Since (X_n, ν_n) is σ-finite, we can express it as a countable union of pairwise disjoint measurable sets U_{nj} with $\nu_n(U_{nj}) < \infty$; and it will suffice to prove the theorem for each $U_{nj} \times Y_n$.

Thus it is enough to prove the Lemma under the further assumptions:

(1) $(X, m) = (U, \nu) \times (Y, \mu)$, where $\nu(U) < \infty$; (Y, μ) is a linear interval of the form $[0, a)$, where $0 < a \leq \infty$, with Lebesgue measure; each $u \times Y$ is invariant under T, and T restricted to $(u \times Y, \mu)$ is ergodic and measure-preserving.

2.3.   The given function f, of Theorem 1, is now a function of the two variables $u \in (U)$, $y \in (Y)$. For each $\gamma \geq 0$ we write

$$R(\gamma) = \{(u, y) \mid f(u, y) > \gamma\} = f^{-1}(\gamma, \infty),$$

$$S(\gamma) = \{(u, y) \mid f(u, y) \geq \gamma\} = f^{-1}[\gamma, \infty).$$

By hypothesis on f, all these sets are measurable and (except perhaps for $\gamma = 0$) of finite measure. As shown in 1.3, we may alter f on an (invariant) null set (which does not alter the assertion of Theorem 1) and make it fully measurable. The reasoning in 1.3 also shows that we may here arrange that all the sets $(u \times Y) \cap R(\gamma)$, $(u \times Y) \cap S(\gamma)$ are not only measurable but also of *finite* μ-measure if $\gamma > 0$.

We introduce the following notation. If E is any subset of $U \times Y$ meeting the fibre $u \times Y$ in a measurable set, then $E_u = \{y \mid (u, y) \in E\}$ (so that $u \times E_u = (u \times Y) \cap E$), and $\mu_u(E) = \mu(E_u)$.

Now we define real-valued functions h, H, k on X by:

$$h(u, y) = \mu_u R(f(u, y)) = \mu\{z \mid f(u, z) > f(u, y)\},$$

$$H(u, y) = \mu_u S(f(u, y)) = \mu\{z \mid f(u, z) \geq f(u, y)\},$$

and

$$k(u, y) = \mu\{z \mid 0 \leq z \leq y, f(u, z) = f(u, y)\}.$$

These functions are measurable, from 1.3(1)—(3), and in fact fully measurable. We observe that k is continuous in y, for fixed u. Clearly also

(1) $$0 \leqq k(u, y) \leqq H(u, y) - h(u, y).$$

Now define

(2) $$g(u, y) = h(u, y) + k(u, y) + 1/f(u, y),$$

a fully measurable finite positive function. We shall prove that, for each real number α and for each $u \in U$,

(3) $$\mu\{y \mid y \in Y, \ g(u, y) = \alpha\} = 0.$$

Throughout the proof of (3), α and u are fixed. The first step is to show

(4) $$\text{if } f(u, y) < f(u, z) \text{ then } g(u, y) > g(u, z).$$

For we have $H(u, z) \leqq h(u, y)$, and therefore, from (1) and (2),

$$g(u, z) \leqq H(u, z) + 1/f(u, z) < h(u, y) + 1/f(u, y) \leqq g(u, y).$$

It follows that the values of g determine those of f, and we may find a positive real number β such that

(5) $$\text{if } g(u, y) = \alpha, \text{ then } f(u, y) = \beta.$$

Write $A = \{y \mid y \in Y, \ g(u, y) = \alpha\}$, $B = \{y \mid y \in Y, \ f(u, y) = \beta\}$; thus (5) asserts $A \subset B$. We must prove $\mu(A) = 0$.

Write $\lambda = \mu_u R(\beta) + 1/\beta$. We observe that, if $y \in A$, then $\beta = f(u, y)$ and therefore $R(\beta)_u = \{z \mid f(u, z) > f(u, y)\}$, giving $\lambda = h(u, y) + 1/f(u, y)$. Thus

(6) $$\text{if } y \in A, \text{ then } g(u, y) = \lambda + k(u, y).$$

Now let y_1, y_2 be any distinct members of A, say with $y_1 < y_2$. Then $g(u, y_1) = \alpha = g(u, y_2)$ and $f(u, y_1) = \beta = f(u, y_2)$. It follows from (6) that $k(u, y_1) = k(u, y_2)$. From the definition of k, we now have

$$\mu([0, y_1] \cap B) = \mu([0, y_2] \cap B),$$

both finite, and therefore $\mu([y_1, y_2] \cap B) = 0$. A fortiori, A meets every interval $[y_1, y_2]$, where $y_1, y_2 \in A$, in a null set. Hence A itself is null, and (3) is proved.

2.4. We next derive some further properties of g.

(1) If $g(u, z) \geqq g(u, y)$, then

$$0 \leqq \frac{1}{f(u, z)} - \frac{1}{f(u, y)} \leqq g(u, z) - g(u, y).$$

The first inequality comes from 2.3 (4). In proving the second, we may assume $f(u, z) < f(u, y)$. Then $H(u, y) \leqq h(u, z)$, and the desired inequality follows from 2.3 (1) and (2).

As a sort of converse to 2.3 (5), we have:

(2) Given $u \in U$ and $\beta > 0$, there exists $\gamma \geqq 0$ such that $f(u, y) \geqq \beta$ if and only if $g(u, y) \leqq \gamma$. This also holds when $\beta = 0$, if we allow γ to be infinite.

We define γ as follows. If $f(u, y) < \beta$ for all $y \in Y$, put $\gamma = 0$. Otherwise, $\gamma = \sup \{g(u, y) \,|\, f(u, y) \geqq \beta\}$. This is finite if $\beta > 0$, because 2.3(2) shows that if $f(x, y) \geqq \beta$ then $g(u, y) \leqq H(u, y) + 1/\beta \leqq \mu_u S(\beta) + 1/\beta$, a (finite) constant. If $f(u, y) \geqq \beta$, the definition of γ ensures $g(u, y) \leqq \gamma$. Conversely, suppose $g(u, y) \leqq \gamma$. Then (since $\gamma \geqq g(u, y) > 0$) there must be y's for which $f(u, y) \geqq \beta$, and hence there is a sequence $\{y_n \,|\, n \in \mathcal{N}\}$ such that $f(u, y_n) \geqq \beta$, $g(u, y_1) \leqq g(u, y_2) \leqq \cdots$, and $\lim_{n \to \infty} g(u, y_n) = \gamma$. If $g(u, y) < \gamma$, it follows that $g(u, y) < g(u, y_n)$ for some n; by 2.3(4) it follows that $f(u, y) \geqq f(u, y_n) \geqq \beta$, as required. In the remaining case, $g(u, y) = \gamma$, it follows from 2.4(1) that $\lim_{n \to \infty} 1/f(u, y_n) = 1/f(u, y) > 0$, and hence that $f(u, y) = \lim_{n \to \infty} f(u, y_n) \geqq \beta$, completing the proof.

Now define

$$(3) \qquad\qquad G(\alpha) = \{(u, y) \,|\, g(u, y) \leqq \alpha\}.$$

Thus, in accordance with our previous notation, $\mu_u G(\alpha) = \mu\{y \,|\, g(u, y) \leqq \alpha\}$. Given $u \in U$ and $\alpha \geqq 0$ (and finite), we prove

$$(4) \qquad\qquad mG(\alpha) < \infty \text{ and } \mu_u G(\alpha) < \infty.$$

Suppose first $\alpha > 0$. If $g(u, y) \leqq \alpha$, then $f(u, y) \geqq 1/\alpha$ from 2.3(2); thus $G(\alpha) \subset S(1/\alpha)$, of finite μ- and μ_u-measure. If $\alpha = 0$, the result follows a fortiori.

Finally, a result which will be very useful later:

(5) *For each fixed $u \in U$, $\mu_u G(\alpha)$ is a continuous function of α.*

For it is monotone, and 2.3(3) shows it has no jumps.

2.5. For each $u \in U$ we define

$$g_0(u) = \sup \{t \,|\, \mu_u G(t) = 0\}, \quad N^* = \{(u, y) \,|\, g(u, y) \leqq g_0(u)\}.$$

Then g_0 is easily seen to be measurable, so that N^* is fully measurable. Moreover, for each u we have

$$(1) \qquad\qquad \mu_u N^* = 0.$$

For we can take a sequence $t_1 < t_2 < \cdots$ converging to $g_0(u)$, and then have

$$N_u^* = \bigcup_{n=1}^{\infty} \{y \,|\, g(u, y) \leqq t_n\} \cup \{y \,|\, g(u, y) = g_0(u)\};$$

the last of these sets is null, by 2.3(3), and the others are null because $t_n < g_0(u)$.

It follows that $mN^* = 0$. We enlarge N^* to an invariant null set $\bar{N}$ and assert:

(2) If $(u, y) \in X - \bar{N}$, and t is any real number such that $\mu_u G(t) = 0$, then $\sum_{i=0}^{\infty} \chi(G(t); T^i(u, y)) = 0$.

For if not, we have some $T^i(u, y) \in G(t)$, where $(u, y) \in X - \bar{N}$. Then $g(T^i(u, y)) \leqq t \leqq g_0(u)$, giving $T^i(u, y) \in N^*$ and therefore $(u, y) \in \bar{N}$, a contradiction.

Next we show that (after enlarging $\bar{N}$ to a possibly larger invariant null set) we also have, for all $(u, y) \in X - \bar{N}$:

(3) If t is a real number such that $\mu_u G(t) > 0$, then

$$\sum_{i=0}^{\infty} \chi(G(t); T^i(u, y)) = \infty.$$

For, since T restricted to $u \times Y$ is ergodic, it follows that (3) holds for *fixed* u and t and for almost all y. Now, fixing merely t, we see that the set of (u, y) for which (3) fails, being measurable and meeting each $u \times Y$ in a null set, is null. Thus (3) holds for almost all (u, y) and for all *rational* t; but, from 2.4 (5), it then follows for all t.

Now suppose $0 \leq \rho \leq \sigma < \infty$. We apply the Hopf-Halmos ergodic theorem to the incompressible transformation T on $X = U \times Y$, and to the functions $\chi G(\rho)$, $\chi G(\sigma)$. A convenient form of the theorem for our purpose is given in [5, Lemma 7.2]. Both of these characteristic functions are nonnegative and summable (from 2.4 (4)). We must also check that, in the notation of [5],

$$Q(\chi G(\rho)) \subset Q(\chi G(\sigma)),$$

where $Q(\chi(E)) = \bigcup_{i=-\infty}^{\infty} T^i\{(u, y) \mid \chi(E; (u, y)) \neq 0\} = \bigcup_{i=-\infty}^{\infty} T^i E$; and this follows because $G(\rho) \subset G(\sigma)$. Thus the theorem gives the existence of a null set $N(\rho, \sigma)$ such that, on writing

$$(4) \qquad L_n(\rho, \sigma; u, y) = \frac{\sum \{\chi(G(\rho); T^i(u, y)) \mid 0 \leq i < n\}}{\sum \{\chi(G(\sigma); T^i(u, y)) \mid 0 \leq i < n\}},$$

we have

(5) $L_n(\rho, \sigma; u, y)$ converges to a (finite) limit $L(\rho, \sigma; u, y)$ as $n \to \infty$, whenever $(u, y) \in U \times Y - N(\rho, \sigma)$.

Now, from (2) and (3), the denominator in (5) (almost everywhere) either tends to ∞ or is always 0 (in which case our convention makes $L_n(\rho, \sigma; u, y) = 0$). Hence the limit $L(\rho, \sigma; u, y)$, or $L(u, y)$ for short, is invariant (except on a null set); and by enlarging $N(\rho, \sigma)$ to an invariant null set we may arrange that $L(u, y)$ is invariant everywhere on $U \times Y - N(\rho, \sigma)$. But T is ergodic on each $u \times Y$, so that for almost all y (depending on u) $L(u, y)$ must be a constant, which we denote by $L(\rho, \sigma, u)$, or $L(u)$ for short. We show that

$$(6) \qquad L(u) = \mu_u G(\rho) / \mu_u G(\sigma).$$

In fact, if $\mu_u G(\sigma) > 0$, Halmos's formulation of the ergodic theorem applies to the restriction of T to $u \times Y$, for $\chi G(\sigma)$ is invariantly positive on $u \times Y$. Hence [2, p. 160] we have

$$\int_{u \times Y} \chi(G(\rho); u, y) \, d\mu_u(y) = \int_{u \times Y} L(u, y)\chi(G(\sigma); u, y) \, d\mu_u(y);$$

that is, $\mu_u G(\rho) = L(u)\mu_u G(\sigma)$, giving (6). If, however, $\mu_u G(\sigma) = 0$, we have $L_n(u, y) = 0/0 = 0$ for almost all y (depending on u), and hence $L(u) = 0$, so that (6) still holds.

From (6), the subset of $U \times Y$ on which $L(u, y) \neq L(u)$ is measurable; and, since it meets each $u \times Y$ in a null set, it is therefore null. So, by a further enlargement of $N(\rho, \sigma)$ to an invariant null set, we have:

(7) For all $(u, y) \in U \times Y - N(\rho, \sigma)$,

$$\lim_{n \to \infty} L_n(\rho, \sigma; u, y) = L(\rho, \sigma; u, y) = \mu_u G(\rho)/\mu_u G(\sigma).$$

Note that if $\mu(Y)$ is finite, then so is $m(U \times Y)$, and we can allow ρ and / or σ to be ∞ in the foregoing; for $XG(\sigma)$ will still be summable.

2.6. We write N for the union of all the discarded invariant null sets $\bar{N}$ and $N(\rho, \sigma)$ for *rational* ρ and σ $(0 \leq \rho \leq \sigma)$, counting ∞ as "rational" if $\mu(Y)$ is finite. Of course N is itself an invariant null set. Our immediate object is to show:

(1) If $(u, y) \in U \times Y - N$, and if $0 \leq \gamma \leq \delta \leq \infty$, then $\lim_{n \to \infty} L_n(\gamma, \delta; u, y)$ exists and equals $\mu_u G(\gamma)/\mu_u G(\delta)$ [3].

We first deal with the case in which $\mu_u G(\delta) = 0$. Here 2.5(2) shows that the denominator of $L_n(\gamma, \delta; u, y)$ is also 0, and (1) holds trivially. In what follows we assume $\mu_u G(\delta) > 0$, and hence $\delta > 0$.

From 2.5(3), the denominator of $L_n(\gamma, \delta; u, y)$ is now never 0. Thus if $\gamma = \delta$, finite or infinite, (1) holds trivially, so we may assume $\gamma < \delta$.

Suppose first $\delta < \infty$. The case in which $\gamma = 0$ is again trivial (we have $G(0) = \varnothing$ and the numerator of $L_n(0, \delta; u, y)$ is 0), so we may assume $\gamma > 0$ and take rational numbers $\rho_1, \sigma_1, \rho_2, \sigma_2$ such that $0 < \rho_1 < \gamma < \sigma_1 < \rho_2 < \delta < \sigma_2$. Further, from 2.4(5), we may suppose ρ_2 so close to δ that $\mu_u G(\rho_2) > 0$. (Here u is fixed.) Then

$$L_n(\rho_1, \sigma_2; u, y) \leq L_n(\gamma, \delta; u, y) \leq L_n(\sigma_1, \rho_2; u, y),$$

so that 2.5(7) gives

$$\mu_u G(\rho_1)/\mu_u G(\sigma_2) \leq \liminf_{n \to \infty} L_n(\gamma, \delta; u, y)$$

$$\leq \limsup_{n \to \infty} L_n(\gamma, \delta; u, y)$$

$$\leq \mu_u G(\sigma_1)/\mu_u G(\rho_2).$$

We make $\rho_1, \sigma_1 \to \gamma$ and $\rho_2, \sigma_2 \to \delta$ and again use the continuity of $\mu_u G$ (2.4(5)); and (1) follows.

Now suppose $\delta = \infty$. Let i be any integer greater than γ; then the above argument gives

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[3] Here, if μY is infinite, $\mu_\blacksquare G(\infty) = \infty$. We adopt the convention that λ/∞ means 0 if λ is finite, 1 if $\lambda = \infty$.

$$0 \leq \limsup_{n \to \infty} L_n(\gamma, \infty; u, y) \leq \lim_{n \to \infty} L_n(\gamma, i; u, y) = \mu_u G(\gamma) / \mu_u G(i).$$

But g is everywhere positive, so $\bigcup_i G(i) = X$ and $\lim_{i \to \infty} \mu_u G(i) = \mu Y$. Thus if $\mu Y = \infty$, we obtain $\lim_{n \to \infty} L_n(\gamma, \infty; u, y) = 0 = \mu_u G(\gamma)/\mu_u G(\infty)$, as required. Finally, if μY is finite, we pick a rational number $\sigma > \gamma$, large enough for $\mu_u G(\sigma) > 0$. The previous results give $\lim_{n \to \infty} L_n(\gamma, \sigma; u, y) = \mu_u G(\gamma)/\mu_u G(\sigma)$, and the construction of N (see the end of 2.5 and the beginning of 2.6) gives $\lim_{n \to \infty} L_n(\sigma, \infty; u, y) = \mu_u G(\sigma)/\mu_u G(\infty)$. Multiplying, we obtain

$$\lim_{n \to \infty} L_n(\gamma, \infty; u, y) = \mu_u G(\gamma) / \mu_u G(\infty),$$

as required.

2.7. We can now easily conclude the proof of Lemma 1. The ratio we have to investigate is

$$D_n(\alpha, \beta; u, y) = \frac{\Sigma\{\chi(S(\alpha), T^i(u, y)) \mid 1 \leq i \leq n\}}{\Sigma\{\chi(S(\beta), T^i(u, y)) \mid 1 \leq i \leq n\}},$$

where $0 \leq \beta \leq \alpha < \infty$, and where we assume $(u, y) \in U \times Y - N$. Keeping u (and y) fixed, we apply 2.4 (2) to obtain $\gamma \geq 0$ (possibly infinite) such that $f(u, z) \geq \alpha$ if and only if $g(u, z) \leq \gamma$, and similarly $\delta \geq 0$ such that $f(u, z) \geq \beta$ if and only if $g(u, z) \leq \delta$. The construction used to prove 2.4(2) shows that here $\gamma \leq \delta$. Because $T^i(u, y) \in u \times Y$ for each i, we have $T^i(u, y) \in S(\alpha)$ if and only if $T^i(u, y) \in G(\gamma)$. That is, the numerator of $D_n(\alpha, \beta; u, y)$ is just

$$\sum_{i=1}^{n} \chi(G(\gamma); T^i(u, y)).$$

Similar considerations apply to the denominator, so that $D_n(\alpha, \beta; u, y)$ is just $L_n(\gamma, \delta; T(u, y))$, and by 2.6 (1) it converges to the (finite) limit $\mu_u G(\gamma)/\mu_u G(\delta)$ as $n \to \infty$.

3. Proof of Theorem 1.

3.1. Let $\mathfrak{B}$ be the σ-field of measurable sets of (X, m), and $\mathfrak{N}$ the family of null sets; we have assumed m complete, so $\mathfrak{N} \subset \mathfrak{B}$.

LEMMA 2. *Theorem 1 is true if* (i) *T is incompressible,* (ii) *X has a separating sequence which generates the measure algebra $\mathfrak{B}/\mathfrak{N}$, and* (iii) *$\mathfrak{B}/\mathfrak{N}$ has no atoms.*

By the construction in [3, p. 335], there is an isometry (a measure-preserving point-isomorphism) of (X, m) onto a subset Z of a linear interval L (finite or infinite), where Z is possibly nonmeasurable but of full outer measure in L, and has the relative measure induced by Lebesgue outer measure in L([4]). To save notation,

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(4) The result quoted here, though not explicitly stated in [3], is implicit in the proof of [3, Theorem 1]. It is assumed in [3] that $m(X) = 1$, but the extension to the σ-finite case is immediate.

we suppose $X = Z$ and that the isometry is the identity. We show that the situation in Z can be imitated in L, by standard technique, and the conclusion will follow from applying Lemma 1 to L.

The measure algebra $\mathfrak{B}/\mathfrak{N}$ of Z ($= X$) is isometric to the measure algebra $\mathfrak{B}^*/\mathfrak{N}^*$ of L (the measure class of $A \subset L$ corresponding to the measure class of $A \cap Z$). Thus T induces an isomorphism of $\mathfrak{B}^*/\mathfrak{N}^*$. Because L is an interval, this isomorphism is induced by a point-isomorphism T^* of L [3, Theorem 3][5]. That is, there is a 1-1 measure-preserving map T^* of L onto L such that, for each measurable $A \subset L$, $T(A \cap Z) = Z \cap T^*A$ modulo null sets (of Z, and so of L).

We first show that T and $T^* | Z$ agree almost everywhere. To see this, consider the intervals $C_{ni} = L \cap [i/n, (i + 1)/n)$, where $n \in \mathcal{N}$ and $i = 0, \pm 1, \pm 2, \cdots$; let N_{ni} denote the symmetric difference of the sets $Z \cap T^*C_{ni}$ and $T(Z \cap C_{ni})$. Then each N_{ni} is a null subset of Z; hence so is the union of all of them, which we denote by N^1, and hence so also is $N^2 = T^{-1}N^1 \cup N^1$. We show

(1) if $z \in Z - N^2$, then $Tz = T^*z$ and $T^{-1}z = T^{*-1}z$.

For each $n \in \mathcal{N}$ there is a well-defined integer $j = j(n, z)$ such that $z \in C_{nj}$. Then $Tz \in \bigcap_n T(Z \cap C_{nj})$. But $Tz \notin N^1$ (else $z \in N^2$), so $Tz \in \bigcap_n T^*C_{nj} \cap Z$. However (because T^* is 1-1) $\bigcap_n T^*C_{nj}$ consists of the single point T^*z; thus $Tz = T^*z$. Again, if $w = T^{-1}z$, we have $Tw \notin N^1$, so by the previous argument $Tw = T^*w$; that is, $z = T^*w$, or $T^{*-1}z = w = T^{-1}z$.

Let $N^* = \bigcup \{T^{*i}N^2 \mid i = 0, \pm 1, \pm 2, \cdots\}$, a null subset of L which is invariant under T^*. We define a new transformation $\bar{T}$ of L by:

(a) if $t \in L - N^*$, $\bar{T}t = T^*t$;

(b) $\bar{T} | N^*$ is an arbitrary permutation of N^*.

Clearly $\bar{T}$ has all the properties which we required of T^*, and from (1) it has the further property

(2) if $z \in Z - N^*$ then $\bar{T}z = Tz$ and $\bar{T}^{-1}z = T^{-1}z$. Also we have:

(3) N^* is invariant under $\bar{T}$, and $N^* \cap Z$ is invariant under T.

The first part of this assertion is immediate from the definitions of N^* and T^*, and the second follows from the first, because of (2).

Now we are given a positive measurable function f on Z such that $mS(\alpha) < \infty$ for every $\alpha > 0$, where (as before) $S(\alpha) = \{z \mid z \in Z, f(z) \geqq \alpha\}$. For each real α we take a measurable subset $S^*(\alpha)$ of L so that $Z \cap S^*(\alpha) = S(\alpha)$; when $\alpha \leqq 0$ we take $S^*(\alpha) = L$. Define $\bar{S}(\alpha) = \bigcap\{S^*(\rho) \mid \rho$ rational, $\rho < \alpha\}$; thus $\bar{S}(0) = L$, and $\alpha > \beta$ implies $\bar{S}(\alpha) \subset \bar{S}(\beta)$. It is easy to verify that $Z \cap \bar{S}(\alpha) = S(\alpha)$. We define a function $\bar{f}$ on L by:

$$\bar{f}(t) = \sup\{\alpha \mid t \in \bar{S}(\alpha)\} \qquad (t \in L),$$

and again easily verify that $\{t \mid \bar{f}(t) \geqq \alpha\} = \bar{S}(\alpha)$. It follows that $\bar{f}$ is measurable, and $\bar{f}(z) = f(z)$ for all $z \in Z$. Hence $0 < \bar{f}(t) < \infty$ for almost all $t \in L$. Moreover,

(5) Again in, [3] it is assumed that $m(L) = 1$, but the result follows for arbitrary intervals.

if $\alpha > 0$, meas $\{t \mid \bar{f}(t) \geqq \alpha\} < \infty$, because this measure is the same as the (outer) measure of its intersection with Z, which is $mS(\alpha)$, finite by hypothesis.

Thus we may apply Lemma 1 to the measure space L, the transformation T (which is clearly incompressible) and the function $\bar{f}$. There is an invariant null set $\bar{N} \subset L$ such that, if $t \in L - \bar{N}$ and $0 \leqq \beta \leqq \alpha$, then the relative density of

$$\{n \mid \bar{f}(T^n t) \geqq \alpha\} \quad \text{in} \quad \{n \mid \bar{f}(T^n t) \geqq \beta\}$$

exists. Thus the relative density of $\{n \mid f(T^n z) \geqq \alpha\}$ in $\{n \mid f(T^n z) \geqq \beta\}$ exists when $z \in Z - (\bar{N} \cup N^*)$, that is, for almost all $z \in Z$ — whenever $0 \leqq \beta \leqq \alpha$.

3.2. **LEMMA 3.** *Theorem* 1 *is true if* (i) T *is incompressible and* (ii) $\mathfrak{B}/\mathfrak{N}$ *has no atoms.*

Because m is σ-finite and there are no atoms, we can find, for each $n \in \mathcal{N}$, a covering $\mathscr{C}_n$ of X by measurable sets $C_{nj}, j \in \mathcal{N}$, each of which has positive measure less than $1/n$. Write $\mathscr{C} = \bigcup_n \mathscr{C}_n$, and let $\mathscr{F}$ denote the finitely additive field of sets generated by all the sets $T^i C$, $T^i R(\rho)$, $i = 0, \pm 1, \pm 2, \cdots$, where $C \in \mathscr{C}$, ρ is rational, and (as before) $R(\rho) = \{x \mid f(x) > \rho\}$. Let $\mathscr{D}$ be the σ-field generated by $\mathscr{F}$. We observe that $\mathscr{D}$ also contains all the sets $T^i R(\alpha)$ where α is real; for $R(\alpha) = \bigcup \{R(\rho) \mid \rho \text{ rational}, \rho > \alpha\}$. Also (1) $\mathscr{D}/\mathfrak{N}$ has no atoms; for if $D \in \mathscr{D}$ and $mD > 0$, then for large enough n and some j we have $0 < m(D \cap C_{nj}) < mD$, and clearly $D \cap C_{nj} \in \mathscr{D}$.

For each $x \in X$, define

$$\pi(x) = \bigcap \{F \mid x \in F \in \mathscr{F}\}.$$

Thus $x \in \pi(x) \in \mathscr{D}$ (because $\mathscr{F}$ is countable), and $m(\pi(x)) = 0$ (because $\mathscr{C} \subset \mathscr{F}$). It is also easily seen that if $\pi(x) \neq \pi(y)$ then $\pi(x) \cap \pi(y) = \varnothing$; in fact, $\pi(x)$ is the equivalence class of x under an obvious equivalence relation.

Next we prove, by "Borel induction", that

(2)   if $x \in D \in \mathscr{D}$, $\pi(x) \subset D$.

For let $\mathscr{E}$ be the family of all subsets E of X such that $\pi(x) \subset E$ for all $x \in E$. Clearly $\mathscr{F} \subset \mathscr{E}$, and one readily verifies that $\mathscr{E}$ is a Borel field. Thus $\mathscr{D} \subset \mathscr{E}$, as required.

Applying (2) to $X - D$, we have

(3)   if $x \notin D \in \mathscr{D}$, $\pi(x) \cap D = \varnothing$.

Another consequence of (2) is

(4)   if $D \in \mathscr{D}$, $D = \bigcup \{\pi(x) \mid x \in D\}$.

It is easily seen that $F \in \mathscr{F}$ if and only if $TF \in \mathscr{F}$, and therefore

(5) $\pi(Tx) = T\pi(x).$

And since each $R(\rho)$ is in $\mathscr{F}$, we have

(6) f is constant on each $\pi(x)$.

Let X' denote the set $\{\pi(x) \mid x \in X\}$, and let π' denote the natural map of X onto X'; that is, $\pi'(x) = \pi(x)$ (but regarded as an element of X' rather than as a subset of X). We extend π' in the usual way to a map from subsets of X to subsets of X'; that is, $\pi'(E) = \{\pi(x) \mid x \in E\}$ (so that the usual notation makes $\pi(E) = \bigcup \pi'(E)$). Let $\mathfrak{B}'$ be the family $\{\pi'(D) \mid D \in \mathscr{D}\}$, clearly a Borel field of subsets of X'; from (4), π' provides a 1-1 correspondence between $\mathscr{D}$ and $\mathfrak{B}'$. We define a countably additive σ-finite complete measure m' on X' by: $m'(B') = m(\pi'^{-1}B')$ $(B' \in \mathfrak{B}')$, and a transformation T' on X' by

$$T'(\pi'x) = \pi(Tx) \qquad (x \in X);$$

by (5) this is well defined, and we have

(7) $T' = \pi' T \pi'^{-1},$

showing that T' is 1-1, onto and measure-preserving. It is easily seen that T' is incompressible. Also the other hypotheses of Lemma 2 are satisfied: the measure algebra of X' has no atoms, from (1), and is generated by the separating sequence formed by the (countable) family of sets $\pi'F$, $F \in \mathscr{F}$. We use (6) to define a function f' on X' by: $f'(\pi x) = f(x)$; this is positive and finite, and we have

(8) $S'(\alpha) = \{x' \mid x' \in X', f'(x') \geq \alpha\} = \pi'S(\alpha),$

so that f' is measurable and $m'S'(\alpha) < \infty$ if $\alpha > 0$. Thus, by Lemma 2, the relative density of $\{n \mid f'(T'^n x') \geq \alpha\}$ in $\{n \mid f'(T'^n x') \geq \beta\}$ exists whenever $x' \in X' - N'$ and $0 \leq \beta \leq \alpha$, N' being an invariant null set. But this implies at once the corresponding statement about f, T and X (we take N to be the null set $\pi'^{-1}N'$), and Lemma 3 is proved.

3.3. LEMMA 4.   *Theorem 1 is true if T is incompressible.*

It is enough to show that the atoms of $\mathfrak{B}/\mathfrak{N}$ can be eliminated, so that Lemma 3 applies. Because m is σ-finite, there are only countably many atoms $a_1, a_2, \cdots$; their supremum corresponds to a measurable set $X_1 \subset X$, which modulo null sets must be invariant under T. Thus we may (by a routine juggling with null sets) assume that X_1 is invariant. Lemma 3 now applies to $X - X_1$, showing that the desired limit exists for almost all $x \in X - X_1$. We have only to prove that it exists for almost all $x \in X_1$, and it suffices to prove this for almost all $x \in A_1$, where A_1 corresponds to the atom a_1. Consider the sets $T^i A_1$, $i = 0, \pm 1, \pm 2, \cdots$; they must all be "atomic" also, so every two of them are (modulo null sets) either equal or disjoint. Because T is incompressible, they cannot all be disjoint, so some two are equal; and it follows that there are (modulo null sets) only finitely many

different sets $T^i A_1$, say p. Again, f must be constant (almost everywhere) on each atomic set $T^i A_1$. Hence, except for a fixed null set of x's in A_1, the sequences arising in the definition of $D(\alpha, \beta; x)$ are constant (that is, independent of x) and periodic (with period p), so the limit exists as required.

3.4. LEMMA 5. *Theorem 1 is true without restriction.*

Here T may be compressible. We can write $X = X_1 \cup X_2$ where X_1, X_2 are disjoint and invariant, where $T \mid X_2$ is incompressible, and where $T \mid X_1$ is purely dissipative: that is, there is a measurable set A such that (except for a null set) $X_1 = \bigcup \{ T^i A \mid i = 0, \pm 1, \pm 2, \cdots \}$, and the sets $T^i A$ are pairwise disjoint([6]). The assertion of Theorem 1, that $D(\alpha, \beta; x)$ exists (for $0 \leq \beta \leq \alpha$, and for almost all $x \in X$, independent of α and β), follows for almost all $x \in X_2$, by Lemma 4. We have only to show $D(\alpha, \beta; x)$ exists for almost all $x \in X_1$.

For each measurable subset B of X_1, and for each $a \in A$, write

$$\theta(B, a) = \sum_{i = -\infty}^{\infty} \chi(B; T^i a).$$

Then

(1) $$mB = \int_A \theta(B, a) dm(a).$$

For, since $\chi(B; T^i a) = \chi(T^{-i} B, a)$, the integral here is $\sum_i \int_A \chi(T^{-i} B; a) dm(a)$ $= \sum_i m(T^{-i} B \cap A) = \sum_i m(B \cap T^i A)$ (since T is measure-preserving) $= mB$.

We apply (1) to the set $B_j = \{ x \mid x \in X_1, f(x) \geq 1/j \}$, f being the function of the theorem, and j an arbitrary positive integer. By hypothesis, $mB_j < \infty$. Thus (1) shows that $\theta(B_j, a)$ is finite almost everywhere on A, say for $a \in A - N_j$ where N_j is null. Let N denote the invariant null set $\bigcup \{ T^i N_j \mid i = 0, \pm 1, \pm 2, \cdots, j \in \mathcal{N} \}$. We show:

(2) if $x \in X_1 - N$ and $\gamma > 0$, $\{ n \mid f(T^n x) \geq \gamma \}$ is finite.

For we have $x = T^p a$ for some $a \in A$ and some integer p. Taking $j > 1/\gamma$, we have $a \notin N_j$, so $\theta(B_j, a)$ is finite; and (2) follows.

Now suppose $0 \leq \beta \leq \alpha$ and $x \in X_1 - N$; we must show that $\lim_{n \to \infty} D_n(\alpha, \beta; x)$ exists. If $\beta > 0$, (2) shows that both the numerator and the denominator of $D_n(\alpha, \beta; x)$ are ultimately constant. If $0 = \beta < \alpha$, the numerator is ultimately constant while the denominator tends to ∞; and if $\alpha = \beta$, numerator and denominator are equal. Thus in all cases the limit exists, trivially.

3.5. *Examples.* Since the proof of Theorem 1 has consisted of steady generalization from special cases, it is natural to ask whether the process of generalization can be continued further. It is not hard to see that we may replace "f is

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([6]) See, for example, [5, 8.2].

positive'' by ''f is nonnegative''. On the other hand, we cannot omit the hypothesis that $m\{x \mid f(x) \geqq \alpha\}$ be finite for each $\alpha > 0$, nor may the hypothesis that T be measure-preserving be weakened to that of measurability (of T and T^{-1}). This is shown by the following examples.

EXAMPLE 1. *Theorem 1 does not remain true without the hypothesis that, for each $\alpha > 0$, $m\{x \mid f(x) \geqq \alpha\} < \infty$, even if T is ergodic and measure-preserving (and so incompressible).*

We take (X, m) to be (say) the real line with Lebesgue measure, and T to be an arbitrary ergodic measure-preserving transformation of X. Then T admits no *finite* invariant measure equivalent to m, so by [1, Theorem 2] there exists a measurable set $W \subset X$, of positive measure, which is ''weakly wandering''; that is, there exist positive integers $p(1) < p(2) < \cdots$ such that the sets W, $T^{p(j)}W$ $(j \in \mathcal{N})$ are all pairwise disjoint. Take an infinite subset $\mathcal{J}$ of $\mathcal{N}$ for which the asymptotic density $d(\mathcal{J}, \mathcal{N})$ does not exist. Put $A = \bigcup \{T^{p(j)}W \mid j \in \mathcal{N}\}$, $B = \bigcup \{T^{p(j)}W \mid j \in \mathcal{J}\}$, and define $f = 1 + \chi A + \chi B$; clearly f is measurable, positive, and bounded. But, whenever $x \in W$, $D(5/2, 3/2; x)$ fails to exist, since

$$\{n \mid f(T^n x) \geqq 5/2\} = \{n \mid n = p(j) \text{ for some } j \in \mathcal{J}\}$$

and

$$\{n \mid f(T^n x) \geqq 3/2\} = \{n \mid n = p(j) \text{ for some } j \in \mathcal{N}\},$$

so that the existence of $D(5/2, 3/2; x)$ would entail that of $d(\mathcal{J}, \mathcal{N})$.

EXAMPLE 2. *Theorem 1 does not remain true without the hypothesis that T be measure-preserving, even if T is ergodic (and measurability-preserving) and mX is finite.*

We merely modify the preceding example by replacing m by an equivalent finite measure.

4. Properties of the relative density $D(\alpha, \beta; x)$.

4.1. As in Theorem 1, let T be a 1-1 measure-preserving transformation of (X, m) onto (X, m), and let f be a positive real measurable function on X such that, for each $\alpha > 0$, $m\{x \mid f(x) \geqq \alpha\} < \infty$. Theorem 1 asserts that $D(\alpha, \beta; x)$ exists for all $x \in X - N$, where N is a fixed invariant null set and α, β are arbitrary subject to $0 \leqq \beta \leqq \alpha$. In this section we derive further properties of $D(\alpha, \beta; x)$ on the assumption that T is *incompressible*, as will, of course, be the case whenever mX is finite. (It would be interesting to know what the situation is in general.)

4.2. THEOREM 2. *Assume the hypotheses of Theorem 1, and suppose further that T is incompressible. Then*

(i) *$D(\alpha, \beta; x)$ is fully measurable (for $0 \leqq \beta \leqq \alpha$ and $x \in X - N$);*

(ii) *Defining $F(x) = \lim \sup_{n \to \infty} f(T^n x)$ $(x \in X)$, we have that F is measurable, invariant under T, and almost everywhere positive;*

(iii) *There is an invariant null set $N^* \subset X$ such that, for all $x \in X - N^*$ and*

for all α, β such that $0 < \beta \leq \alpha < F(x)$, we have $D(\alpha, \beta; x) > 0$;

(iv) $D(\alpha, \beta; x)$ *is invariant under T (for $x \in X - N^*$ and $0 \leq \beta \leq \alpha$), and each of the sets $\{n \mid fT^n(x) \geq \alpha\}$ is either infinite or empty.*

To prove (i) (for which the incompressibility of T is not required), we show first that, for fixed α and β, $D(\alpha, \beta; x)$ is a measurable function of x. This follows from the fact that $D(\alpha, \beta; x) = \lim_{n \to \infty} D_n(\alpha, \beta; x)$ where

$$D_n(\alpha, \beta; x) = \sum_{i=1}^{n} \chi(S(\alpha); T^i x) \Big/ \sum_{i=1}^{n} \chi(S(\beta); T^i x),$$

$S(\alpha)$ denoting $\{x \mid f(x) \geq \alpha\}$ as before, a ratio of measurable functions since $\chi(S(\alpha); T^i x) = \chi(T^{-i} S(\alpha); x)$ is measurable.

Next, suppose β is fixed; we prove $D(\alpha, \beta; x)$ is a measurable function of (x, α). The preceding reasoning shows that it is enough to prove that the set

$$\{(x, \alpha) \mid f(T^i x) \geq \alpha\}$$

is measurable in (x, α). But this is just the ordinate set of the graph of $f(T^i)$, measurable by Fubini's theorem. The other cases are similar.

4.3. For the proof of (ii) and (iii) we need the following (essentially known) lemma:

LEMMA 6. *For any measurable set S, write*

$$I(S) = \bigcap_i \bigcup_j T^{-(i+j)} S \qquad (i, j = 0, 1, 2, \cdots).$$

Then (T being incompressible) $I(S)$ is an invariant set which contains S except for a null set, and is (modulo null sets) the smallest such set.

It is easily verified that $I(S)$ is invariant. Now, if we put

$$J_i = \bigcup_j T^{(-i+j)} S,$$

we have $T^{-1} J_i = J_{i+1} \subset J_i$; hence, because T is incompressible, $J_i - J_{i+1}$ is null. But $J_0 - I(S) = \bigcup_i (J_i - J_{i+1})$ and is therefore also null. And trivially $J_0 \supset S$. Hence $I(S)$ does contain S except for a null set. If, finally, E is any invariant set which contains S except for a null set, E contains (almost all of) J_0 and hence $I(S)$.

4.4. Now we deduce the assertion (ii) of Theorem 2. It is clear that F is measurable and invariant. Now put $S_n = S(1/n) = \{x \mid f(x) \geq 1/n\}$ $(n = 1, 2, \cdots)$; because $f(x) > 0$ everywhere, each x of X is in some S_n. By Lemma 6, almost every x is in some $I(S_n)$. But if $x \in I(S_n)$, we readily verify that $F(x) = \limsup_{r \to \infty} f(T^r x) \geq 1/n > 0$. Thus $F(x) > 0$ almost everywhere.

4.5. In proving the assertion (iii) of Theorem 2, we first consider *fixed* real α, β such that $0 < \beta \leq \alpha$. Let $A = \{x \mid F(x) > \alpha\}$; A is invariant, because F is.

We apply the Halmos-Hopf ergodic theorem [2] to the measure space (A, m), the transformation $T | A$, and the two functions $\chi(S\alpha \cap A)$, $\chi(S\beta \cap A)$ (where $S\alpha$, as before, denotes $\{x | f(x) \geq \alpha\}$). Both these functions are summable, because $m(S\beta) < \infty$ by hypothesis, and invariantly positive, since for each $x \in A$ we have $f(T^i x) > \alpha > 0$ for some i. Thus [2, Theorem 5] applies. In the first instance this merely says again that $D(\alpha, \beta; x)$ exists almost everywhere on A. But, as remarked in [2, p. 160], we also have that $D(\alpha, \beta; x)$ is invariant on A, and that, for every invariant measurable set $E \subset A$ for which $\int_E \chi(S\beta \cap A) dm < \infty$, we have

$$(1) \qquad \int_E \chi(S\alpha \cap A) \, dm = \int_E \chi(S\beta \cap A) D(\alpha, \beta;) \, dm.$$

Now take $E = \{x | x \in A, D(\alpha, \beta; x) = 0\}$; this is an invariant subset of A, and of course $\int_E \chi(S\beta \cap A) dm \leq m(S\beta) < \infty$. So (1) gives

$$(2) \qquad m(E \cap S\alpha) = 0.$$

Now consider the set $I(S\alpha)$, in the notation of Lemma 6 (4.3). It is easily seen that $A \subset I(S\alpha)$; thus, from (2), $I(S\alpha) - E$ is an invariant set which (modulo null sets) contains $S\alpha$. But (Lemma 6) $I(S\alpha)$ is (modulo null sets) the smallest such set. Hence E is null, and $D(\alpha, \beta; x) > 0$ almost everywhere on A.

We now let α, β vary. For each pair of *rational* numbers α, β such that $0 < \beta \leq \alpha$, the foregoing gives an invariant null set E; the union of these is an invariant null set N^* such that, whenever $x \in X - N^*$ and α, β are rational numbers satisfying $0 < \beta \leq \alpha < F(x)$, then $D(\alpha, \beta; x) > 0$. But here the restriction to rational numbers can be removed. For, given $x \in X - N^*$ and arbitrary α, β such that $0 < \beta \leq \alpha < F(x)$, we take rational numbers ρ, σ such that $0 < \sigma < \beta$ and $\alpha < \rho < F(x)$. A trivial computation gives $D(\alpha, \beta; x) \geq D(\rho, \sigma; x) > 0$.

4.6. To complete the proof of Theorem 2, we first construct a suitable invariant null set N^*([7]). First, define

$$N_1 = \{x | f(x) > F(x)\}.$$

Then N_1 *is null*. For if not, a familiar argument gives a measurable subset M of N_1, of positive measure, on which $f(x) > \rho > F(x)$ for some fixed (rational) ρ. By Lemma 6 (4.3), $m(I(M) \cap M) = mM > 0$, so we may take $y \in I(M) \cap M$. From the definition of $I(M)$, we have $T^n y \in M$ for arbitrarily large values of n; for each such n, $f(T^n y) > \rho$, and therefore $F(y) \geq \rho$. But $F(y) < \rho$, since $y \in M$, giving a contradiction.

Next, write $E = \{x | f(x) = F(x)\}$, $N_2 = E - I(E)$. From Lemma 6, N_2 is null. Finally, by Theorem 1, there is a null set N_3 such that $D(\alpha, \beta; x)$ exists for all

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$x \in X - N_3$ and for all α, β such that $0 \leq \beta \leq \alpha$. We define N^* $= \bigcup \{ T^i(N_1 \cup N_2 \cup N_3) \mid i = 0, \pm 1, \pm 2, \cdots \}$, an invariant null set, and show first that it satisfies the second of the two assertions in Theorem 2(iv).

Given $x \in X - N^*$ and $\beta \geq 0$, write $\mathscr{B}(f, x) = \{ n \mid n \in \mathscr{N}, f(T^n x) \geq \beta \}$; we must show that $\mathscr{B}(f, x)$ is either infinite or empty. Now, if $\beta < F(x) = \limsup_{n \to \infty} f(T^n x)$, $\mathscr{B}(f, x)$ is clearly infinite. If $\beta > F(x)$, then $\mathscr{B}(f, x) = \varnothing$ since otherwise, for some n, we have $f(T^n x) > F(x) = F(T^n x)$, giving $T^n x \in N_1$, which contradicts $x \notin N^*$. Finally, if $\beta = F(x)$, we distinguish two cases. If $x \in I(E)$, then $T^n x \in E$ for arbitrarily large values of n; that is, $f(T^n x) = F(T^n x) = F(x) = \beta$ for infinitely many n's, and $\mathscr{B}(f, x)$ is infinite. If however $x \notin I(E)$, we show that $\mathscr{B}(f, x) = \varnothing$. For otherwise we have, for some n, $f(T^n x) \geq \beta = F(x)$. Since $T^n x \notin N_1$, we must have $f(T^n x) = F(x) = F(T^n x)$; thus $T^n x \in E$. Because $I(E)$ is invariant, this proves $T^n x \in E - I(E) = N_2$, and therefore $x \in N^*$, a contradiction. Thus the assertion is proved.

Now suppose $x \in X - N^*$ and $0 \leq \beta \leq \alpha$. Then $D(\alpha, \beta; x)$ exists and equals $\lim_{n \to \infty} |\mathscr{A}_n(f, x)| / |\mathscr{B}_n(f, x)|$, where $\mathscr{A}_n(f, x) = \{ j \mid f(T^j x) \geq \alpha, 1 \leq j \leq n \}$, and $\mathscr{B}_n$ is defined similarly. Now $|\mathscr{A}_n(f, x)|$ and $|\mathscr{A}_n(f, Tx)|$ differ by at most 1, and the same is true of $|\mathscr{B}_n(f, x)|$ and $|\mathscr{B}_n(f, Tx)|$. Hence if $\mathscr{B}(f, x)$ is infinite, $D(\alpha, \beta; x) = D(\alpha, \beta; Tx)$. In the only other case, $\mathscr{B}(f, x)$ is empty, and our convention $0/0 = 0$ gives $D(\alpha, \beta; x) = 0 = D(\alpha, \beta; Tx)$. Thus $D(\alpha, \beta; x)$ is invariant under T on $X - N^*$, and the proof is complete.

5. The Radon-Nikodym derivatives.

5.1. In the remainder of this paper, we drop the requirement that T be measure-preserving, but require instead that mX be finite.

THEOREM 3. *Let T be a measurable transformation of a measure space (X, m), and suppose $mX < \infty$. Then there is an invariant null set N such that, whenever $x \in X - N$ and $0 \leq \beta \leq \alpha$, the relative density $d(\alpha, \beta; x)$ (of the n's for which $\omega_n(x) \geq \alpha$ in those for which $\omega_n(x) \geq \beta$) exists.*

As in [5, §3], we form the measure-theoretic product space $(X^*, m^*) = (X, m) \times (Y, \mu)$, where Y is the linear interval $(0, \infty)$ with Lebesgue measure μ, and consider the transformation T^* of X^* given by

$$T^*(x, y) = (Tx, y/\omega(x)). \tag{8}$$

As proved in [5, Theorem 1], T^* is a measure-preserving transformation of X^*. Defining $f^*(x, y) = 1/y$, we observe that, for each $\alpha > 0$,

$$m^*\{(x, y) \mid f^*(x, y) \geq \alpha\} = m^*\{(x, y) \mid y \leq 1/\alpha\} = m(X)/\alpha < \infty.$$

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(8) In [5], Y was taken to be the half-closed interval $[0, \infty)$. The exclusion of 0 does not affect anything in [5], and is convenient here.

Thus Theorem 1 applies to X^*, T^* and f^*, giving an invariant null set $N^* \subset X^*$ such that, for all $(x, y) \in X^* - N^*$, and for all real α, β such that $0 \leq \beta \leq \alpha$, $D(\alpha, \beta; x, y)$ exists. All that remains is to translate this into a statement about X and T.

First, let $\mathscr{A} = \{n \mid \omega_n(x) \geq \alpha\}$, $\mathscr{B} = \{n \mid \omega_n(x) \geq \beta\}$. Now, as shown in [5, 3.1(2)], $T^{*n}(x, y) = (T^n x, y / \omega_n(x))$. Thus $f^*(T^{*n}(x, y)) = \omega_n(x)/y$, and it follows that

$$\mathscr{A} = \{n \mid f^*(T^{*n}(x, y)) \geq \alpha/y\}, \qquad \mathscr{B} = \{n \mid f^*(T^{*n}(x, y)) \geq \beta/y\}.$$

Hence

(1) $\qquad d(\alpha, \beta; x) = D(\alpha/y, \beta/y; x, y)$ whenever either exists.

Next, let $N = \{x \mid x \in X, \; N_x^* \text{ is not null}\}$, where $N_x^* = \{y \mid (x, y) \in N^*\}$. By Fubini's theorem, N is null; and N is invariant under T (because N^* is invariant under T^*). Given $x \in X - N$ and real numbers α, β such that $0 \leq \beta \leq \alpha$, pick $y \in Y - N_x^*$ (which is certainly not empty); then $D(\alpha/y, \beta/y; x, y)$ exists, and (1) shows that $d(\alpha, \beta; x)$ exists, as required.

5.2. THEOREM 4. *Let T be a measurable transformation of a measure space (X, m), and suppose that $mX < \infty$ and that T is incompressible. Then:*

(i) *$d(\alpha, \beta; x)$ is fully measurable (for $0 \leq \beta \leq \alpha$ and $x \in X - N$);*

(ii) *Defining $\lambda(x) = \limsup_{n \to \infty} \omega_n(x)$, we have that λ is measurable, invariant under T, and almost everywhere positive (here $x \in X$);*

(iii) *There is an invariant null set $N' \subset X$ such that, for all $x \in X - N'$ and for all α, β such that $0 < \beta \leq \alpha < \lambda(x)$, we have $d(\alpha, \beta; x) > 0$.*

(iv) *If $x \in X - N'$ and $0 \leq \beta \leq \alpha \; (< \infty)$, then $d(\alpha\omega(x), \beta\omega(x); x) = d(\alpha, \beta; Tx)$, and each of the sets $\{n \mid \omega_n(x) \geq \alpha\}$ is either infinite or empty.*

The proof of (i) is essentially the same as that of Theorem 2(i) of (4.2), and again does not require the incompressibility of T.

The nontrivial part of the assertion (ii) is that $\lambda(x) > 0$ a.e. This follows from [5, Theorem 4], but can also be deduced from Theorem 2 as follows. Using the notation of 5.1, we put $F^*(x, y) = \limsup_{n \to \infty} f^*(T^{*n}(x, y)) = \limsup_{n \to \infty} \omega_n(x)/y = \lambda(x)/y$; then Theorem 2(ii) gives that $F^*(x, y) > 0$ almost everywhere, giving (ii).

To prove (iii), we again apply Theorem 2 to the space X^*, transformation T^* and function f^* used to prove Theorem 3. There is a null set $N^* \subset X^*$ such that, whenever $(x, y) \in X^* - N^*$ and $0 < \beta \leq \alpha < \lambda(x)$, we have $D(\alpha/y, \; \beta/y; x, y) > 0$; that is (5.1(1)), $d(\alpha, \beta; x) > 0$. We merely arrange that N' will include $\{x \mid N_x^* \text{ is not null}\}$; then, for each $x \in X - N'$ we can pick a suitable y (as in the proof of Theorem 3), and the result follows.

Finally, on applying Theorem 2(iv) to X^*, T^* and f^*, we have that

(1) if $(x, y) \in X^* - N^*$ and $\gamma \geq 0$, then $\{n \mid f^* T^{*n}(x, y) \geq \gamma\}$ is either infinite or empty,

(2) if $(x, y) \in X^* - N^*$ and $0 \leqq \delta \leqq \gamma$, then

$$D(\gamma, \delta; x, y) = D(\gamma, \delta; T^*(x, y)).$$

We define N' as before; then if $x \in X - N'$ we can choose y so that

$$(x, y) \in X^* - N^*.$$

In (1), put $\gamma = \alpha/y$; we obtain that $\{n \mid \omega_n(x) \geqq \alpha\}$ is infinite or empty. In (2), put $\gamma = \alpha\omega(x)/y$, $\delta = \beta\omega(x)/y$; then (in view of 5.1(1)) it follows that

$$d(\alpha\omega(x), \beta\omega(x); x) = d(\alpha, \beta; Tx),$$

as required.

5.3. We remark that Theorems 3 and 4 can be generalized so as to include Theorems 1 and 2, at least in the case for which $m(X)$ is finite. In fact, given a positive function f on X, we can replace f^*, in the proofs of Theorems 3 and 4, by the function $f(x)/y$. The arguments go through, provided f is summable, and give the existence and properties of the relative densities of sets of integers of the form $\{n \mid f T^n(x)\omega_n(x) \geqq \alpha\}$. However, this gain in generality would be illusory, as the resulting theorems also follow on applying Theorems 3 and 4 to X with a new measure M given by $M(A) = \int_A f(x) \, dm(x)$.

Another, perhaps more useful modification could be made as follows. Instead of considering relative densities of sets of the form $\{n \mid f(T^n x) \geqq \alpha\}$ or $\{n \mid \omega_n(x) \geqq \alpha\}$, we could replace "$\geqq \alpha$" throughout by "$> \alpha$", in all four theorems. The proofs would apply almost unchanged.

Two further modifications can be made in Theorems 1 and 2; both involve weakening the hypothesis that f is everywhere positive. In the first place, we may relax this to requiring only that $f(x) \geqq 0$ $(x \in X)$. (We still require the finiteness assumption, that $m\{x \mid f(x) \geqq \alpha\} < \infty$ for all $\alpha > 0$.) In fact, the proofs of Theorems 1 and 2 can be adapted to this more general situation, though with some complications; and the conclusions of Theorems 1 and 2 apply unchanged, except of course for the assertion (in Theorem 2 (ii)) that $F(x) > 0$ a.e. In the second place, we may drop all requirements of positivity on f, and also on α and β, at the expense of strengthening the finiteness requirement, which we now require to hold for *all* real α. Again, the conclusions of Theorems 1 and 2 continue to hold; of course, we replace (for example) "$0 \leqq \beta \leqq \alpha$" by "$\beta \leqq \alpha$", and in Theorem 2(ii) we replace the conclusion that $F(x) > 0$ a.e. by "$F(x) > -\infty$ a.e." To see this, we have only to apply Theorems 1 and 2 to the function $e^{f(x)}$.

Finally, we observe that the hypothesis in Theorem 3 that $m(X) < \infty$ cannot be omitted, even if T is required to be incompressible. This can be seen from Example 1, 3.5, as follows. Using the same X, T and f as in Example 1, we replace the measure m by the measure μ defined by:

$$\mu(A) = \int_A f(x) \, dm(x).$$

It is easily verified that $\omega_n(x) = f(T^n x)/f(x)$, whence it follows that, whenever $x \in W$, $d(5/2, 3/2; x)$ fails to exist. Of course, if T is allowed to be compressible, the construction of a counterexample is easier; and in fact (cf. [5, 8.6]) we can arrange that, on a set of positive measure, the ω_n's are arbitrarily prescribed positive measurable functions.

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UNIVERSITY OF ROCHESTER,
    ROCHESTER, NEW YORK