SINGULAR QUADRATIC FUNCTIONALS OF n DEPENDENT VARIABLES(1)

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Introduction. In this paper singular quadratic functionals of n dependent variables are systematically studied. Necessary and sufficient conditions for the existence of a minimum are given, generalizing the results of Morse and Leighton [6]. The study of principal quadratic functionals of n dependent variables is undertaken. A necessary condition is proved and sufficient conditions are obtained for certain cases. Finally, an oscillation theorem is given for systems of second-order linear differential equations.

We shall make the usual conventions regarding notation. A repeated subscript indicates summation. We shall use the symbol $(*)$ to indicate the transpose of a matrix or vector. In general capital letters indicate matrices, and we shall employ the notation $A = \| a_{ij} \|$ for a matrix when necessary. The determinant of a matrix A will be written det A. The notation $|a|$ will be reserved to indicate the absolute value of a number a.

I. The functional. Let

$$f(x, y, y') = y^{*\prime}(x)R(x)y'(x) + 2y^{*\prime}(x)Q(x)y(x) + y^*(x)P(x)y(x), \tag{1.1}$$

where $R(x)$, $Q'(x)$, and $P(x)$ are symmetric matrices continuous in the real variable x on $[a, \infty)$, and $R(x)$ is positive definite for any fixed x on $[a, \infty)$. We consider the functional

$$J(y)\big|_a^b = \int_a^b f(x, y, y')\,dx \quad (a \leqq b < \infty). \tag{1.2}$$

Integrals employed throughout are Lebesgue integrals and their extensions.

Received by the editors June 4, 1965.

(1) This will acknowledge the partial support of the author by the U. S. Army Research Office (Durham) under Grant numbered DA-ARO(D)-31-124-G197. Reproduction in whole or part is permitted for any purpose of the United States government.

(2) The author is indebted to Professor Walter Leighton for suggesting this problem and for his suggestions in its preparation.

Following Morse and Leighton [6] we call the vector function

$$y^*(x) = (y_1(x), \cdots, y_n(x))$$

A-admissible on $[a, \infty)$ if,

1. $y(x)$ is continuous on the interval $[a, \infty)$, $y(a) = 0$, and $\lim_{x \to \infty} y(x) = 0$;
2. $y(x)$ is absolutely continuous and each term in the sum $y^{*\prime}(x) y'(x)$ is Lebesgue integrable on each closed subinterval of $[a, \infty)$.

Observe that the segment $[a, \infty)$ of the x-axis is A-admissible and that on this segment $J = 0$. We seek conditions under which

$$\lim_{x = \infty} \inf \int_a^x f(x, y, y')\, dx \geqq 0. \tag{1.3}$$

If (1.3) holds for a given class of curves, we say that $[a, \infty)$ affords a *minimum limit* to J among curves of the given class.

II. **Conjugate points.** We seek an analogue of the Jacobi necessary condition. The Euler equations and the Jacobi equations associated with the functional (1.2) take the form

$$[R(x) y'(x) + Q(x) y(x)]' - [Q(x) y'(x) + P(x)] y(x) = 0. \tag{2.1}$$

We can now define the *first conjugate point* of $x = \infty$ first given by Morse and Leighton [6] for $n = 1$ and extended by Chellevold [1] for $n = n$. If there does not exist a point α on (a, ∞) such that α is conjugate to a in the ordinary sense (cf. Morse [5, p. 9]), then we say that the first conjugate point of $x = \infty$ does not exist on $[a, \infty)$. Otherwise, there exist α and c on (a, ∞), $c < \alpha$, such that α is the first conjugate point (in the ordinary sense) to c. It is well known that c is a continuous nondecreasing function of α. Thus, $\lim_{\alpha = \infty} c(\alpha)$ exists, finite or infinite. The first conjugate point of $x = \infty$ is then defined as this limit.

Equation (2.1) can be written in matrix form as

$$[R(x) Y'(x)]' + [Q'(x) - P(x) Y(x)] = 0, \tag{EE}$$

where a solution matrix of (EE) has column vectors which are solutions of (2.1). It is easily verified using the symmetry of $P(x)$, that if $Y = Y(x)$ and $Z = Z(x)$ are two solution matrices of (EE), then

$$Y^* R Z' - Y^{*\prime} R Z = C, \tag{2.3}$$

where C is a constant matrix.

If in (2.3), we set $Z = Y$, and the constant matrix C is the zero matrix, that is to say,

$$Y^* R Y' = Y^{*\prime} R Y, \tag{2.4}$$

we follow Hartman [2] and call Y *a prepared solution.*

We assume that there exists a prepared solution $Y(x)$ of (EE) which is nonsingular on some interval $[x_0, \infty)$, $a < x_0 < \infty$. Hence the matrix function

$$\int_{x_0}^{x} (Y^*RY)^{-1}\,dx \tag{2.5}$$

can be formed. We shall employ the terminology introduced by Leighton [3] and extended by Hartman [2] and call a solution $Y = Y(x)$ of (EE) a *principal solution* if all the characteristic roots of (2.5) are unbounded. However, we shall call a solution $Y = Y(x)$ of (EE) an *antiprincipal solution* if all the characteristic roots of (2.5) are bounded (Hartman [2] calls such a solution nonprincipal).

Theorems 1, 2, and 3 stated below are due to Hartman. They play an important role in the variational theory we employ. Equivalent theorems were proved by Morse and Leighton for the case $n = 1$.

We assume for Theorems 1, 2, and 3 that there exists a prepared solution $Y(x)$ of (EE) which is nonsingular on some interval $[x_0, \infty)$.

THEOREM 1. *There exist antiprincipal prepared solutions of* (EE).

Hartman shows that if $Y(x)$ is a prepared solution of (EE) such that $\det Y(x) \neq 0$ on $[x_0, \infty)$, then

$$U(x) = Y(x)\left[I + \int_{x_0}^{x} (Y^*(x)R(x)Y(x))^{-1}dx\right]$$

is an antiprincipal prepared solution.

THEOREM 2. *There exist principal prepared solutions of* (EE).

It is shown that if $U(x)$ is an antiprincipal prepared solution then,

$$W(x) = U(x)\int_{x}^{\infty} [U^*(x)R(x)U(x)]^{-1}dx$$

is a principal prepared solution.

THEOREM 3. *The prepared solutions $W(x)$ and $Z(x)$ of* (EE) *are principal if and only if there exists a constant matrix A such that*

$$Z(x) = W(x)A \qquad (\det A \neq 0).$$

We proceed with the proof of the following theorem.

THEOREM 2.1. *If the interval $(a, \infty]$ does not contain a conjugate point of $x = \infty$, there then exists a prepared solution $Y(x)$ such that* $\det Y(x) \neq 0$ *on* (a, ∞).

The solution $U(x)$ such that $U(a) = 0$ and $U'(a) = I$ is such a solution.

We come now to an important lemma.

LEMMA 2.1. *If the largest zero of* $\det W(x)$ *is* c, *where* $W(x)$ *is a principal prepared solution of* (EE), *then for every antiprincipal prepared solution* $U(x)$, $\det U(x)$ *vanishes on* $[c, \infty)$.

Suppose $\det U(x) \neq 0$ on $[c, \infty)$. Then, by Theorems 2 and 3, we can write

$$W(x) = U(x) \int_x^{\infty} (U^*(x)R(x)U(x))^{-1}\, dxA,$$

where $\det A \neq 0$, and $x \geqq c$. The matrix $\int_c^{\infty} (U^*RU)^{-1} dx$ is positive definite, and $\det U(c) \neq 0$; hence $\det W(c) \neq 0$. From this contradiction we infer the truth of the lemma.

We note that the concept of a prepared solution of (EE) is the same as the concept of a family of conjugate solutions of (2.1) (cf. Morse [5, pp. 46–47]). This enables us to restate Morse's separation theorem in the following form:

THEOREM 4. *The number of zeros of the determinant of a prepared matrix solution of* (EE) *on a given interval* (*open or closed*) *differs from that of any other prepared matrix solution by at most* n.

We can now prove the following result.

THEOREM 2.2. *If* $W(x)$ *is a principal prepared solution of* (EE), *then the largest zero of* $\det W(x)$ *and the point conjugate to* $x = \infty$ *coincide, if either one exists.*

We suppose that the largest zero of $\det W(x)$ is $x = c$. Then by definition of the first conjugate point of $x = \infty$ and by Theorem 4, the conjugate point α cannot be greater than c. If $\alpha < c$, by Theorem 2.1 there exists a prepared solution $V(x)$ such that $\det V(x) \neq 0$ on (α, ∞). Then by Theorem 1, we can write an antiprincipal solution $U(x)$ as

$$U(x) = V(x) \left[I + \int_{x_0}^{x} (V^*(x)R(x)V(x))^{-1} dx \right] \qquad (\alpha < x_0 < c). \tag{2.5}$$

Lemma 2.1 and equation (2.5) imply that $\det V(x)$ vanishes on $[c, \infty)$. From this contradiction, we infer that $\alpha = c$.

If the first conjugate point of $x = \infty$ exists and equals α, then from the same arguments as before we conclude that α is also the largest zero of $\det W(x)$.

We next establish the analogue of the Jacobi necessary condition for the functional (1.2).

THEOREM 2.3. *If* $[a, \infty)$ *affords a minimum limit to* J *among* A*-admissible curves, the interval* $(a, \infty]$ *does not contain a point conjugate to* $x = \infty$.

Assume that $x = \alpha$ is the conjugate point of $x = \infty$ and $\alpha > a$. Then by definition of the conjugate point, there exists a solution $y(x)$ which vanishes at x_1

and x_2, where $a < x_1 < x_2 < \infty$. But from the nonsingular theory there cannot exist a point conjugate to $x = x_1$ on (x_1, x_3) where $x_3 > x_2$.

III. **The Hilbert integral.** Suppose there is no point conjugate to $x = \infty$ on $(a, \infty]$. Then by Theorem 2.1 there exists a solution, $U(x)$, of (EE) such that

$$U(a) = 0, \qquad U'(a) = I,$$

and such that $\det U(x)$ does not vanish on (a, ∞). The family of extremals

$$y = U(x)\beta \qquad (\beta \textit{ a constant vector}) \tag{3.1}$$

will form a Mayer field in the region S of the (x, y) space for which $a < x < \infty$. The slope vector $p(x, y)$ of the field at the point (x, y) is given by the equation

$$p(x, y) = U'(x)U^{-1}(x)y, \tag{3.2}$$

where $p(x, y)$ is a column vector whose ith component is $p_i(x, y)$.

The Hilbert integral corresponding to the field (3.1) is a line integral of the form

$$H = \int [f(x, y, p) - p_i f_{p_i}(x, y, p)]\, dx + f_{p_i} dy_i, \tag{3.3}$$

where $p_i = p_i(x, y)$.

The problem due to the apparent singularity of the Hilbert integral at the point $(a, 0)$ can be resolved in the same manner as Morse and Leighton [6] by transforming (3.3) from $(x, y) = (x, y_1, \cdots, y_n)$ space into $(x, \beta) = (x, \beta_1, \cdots, \beta_n)$ space using the one-to-one transformation on S given by (3.1). We summarize these findings in the following statement.

Let S^* be the point set union of S and the point $(a, 0)$ in the xy-space. Let g be a curve on a bounded subset of S^* of the form

$$x = x(t),\ y^* = y^*(t) = (y_1(t), \cdots, y_n(t)) \qquad (t_1 \leqq t \leqq t_2),$$

where $x(t)$ and $y_i(t)$ are absolutely continuous. If g lies on S, the Hilbert integral exists and depends only on the endpoints of g. If g terminates at the point $(a, 0)$, H still exists and depends only on its first endpoint provided

$$U^{-1}(x(t))\, y(t) = \beta(t)$$

is bounded as t tends to t_1.

An extension of the Weierstrass formula. Let $y(x)$ be an A-admissible curve and $U(x)$ a solution such that

$$U(a) = 0, \qquad U'(a) = I.$$

Set

$$S[y(x), a] = y^*(x)[R(x)U'(x)U^{-1}(x) + Q(x)]y(x).$$

We call $S[y(x),a]$ the *singularity function* belonging to $x = a$ and we call

$$S(x) = R(x)U'(x)U^{-1}(x) + Q(x)$$

the *singularity matrix* belonging to $x = a$.

It is readily verified that the Weierstrass E-function for J is

$$E(x,y,\lambda,\mu) = (\mu - \lambda)^* R(x)(\mu - \lambda).$$

We come to the following extension of the Weierstrass formula.

THEOREM 3.1. *If there is no point conjugate to $x = \infty$ on $(a,\infty]$, if $y(x)$ is A-admissible on $[a,\infty)$ and of class C' neighboring $x = a$, then*

$$J(y)\Big|_a^b = \int_a^b E[x,y(x),y'(x),p(x,y(x))]\,dx + S[y(b),a], \qquad a < b < \infty, \tag{3.1}$$

where $p = p(x,y)$ is the slope vector (3.2).

In view of the continuity of $y'(x)$ in the neighborhood of $x = a$, the Hilbert integral H exists when taken along $y = y(x)$ from $x = a$ to $x = b$. Let this be denoted by $H_y\big|_a^b$. We are thus led to the Weierstrass formula

$$J(y)\Big|_a^b = \int_a^b E\,dx + H_y\Big|_a^b,$$

where the arguments are those of (3.1). We shall show that

$$H_y\Big|_a^b = S[y(b),a]. \tag{3.2}$$

Let γ denote the curve $y = y(x)$ and let γ_i $(i = 1,\cdots,n)$ denote the straight lines in the (x,y) space, $\gamma_i = (b,y_1(b),\cdots,y_{i-1}(b),y,0,\cdots,0)$. Consider the curve Γ beginning at $(b,0,\cdots,0)$ and continuing along γ_1 until γ_1 intersects γ_2 at the point $(b,y_1(b),0,\cdots,0)$ and then continuing along each γ_i in a similar manner until finally γ_n is traversed and stops at the point $(b,y_1(b),\cdots,y_n(b))$. Taking the Hilbert integral along the closed curve by following Γ from $(b,0)$ to $(b,y(b))$, the curve $y = y(x)$ from $(b,y(b))$ to $(a,0)$, and the x-axis from $x = a$ to $x = b$, we find that

$$H_\Gamma \Big|_{y=0}^{y=y(e)} - H_y \Big|_{x=e}^{x=b} = 0. \tag{3.3}$$

Using the definition of the Hilbert integral we find

$$H_\Gamma = \int_\Gamma 2[R(b)p(b,y) + Q(b)y]^*\,dy = \int_\Gamma 2y^*S^*(b)\,dy,$$

where dy is understood to mean a (column) vector whose ith component is dy_i. It is readily seen using the preparedness of $U(x)$ and the symmetry of $Q(x)$, that $S^*(x) = S(x)$ and hence that

$$H_\Gamma = \int_\Gamma 2y^*S(b)dy.$$

For the rest of this proof we drop the summation convention and sum only over the indices indicated. We have then

$$\int_{\gamma_j} 2y^*S(b)dy = \sum_{k=1}^{j} \int_{\gamma_j} 2y_kS_{kj}(b)\,dy_j \tag{3.4}$$

since $dy_i = 0$ if $i \neq j$ and $y_i = 0$ if $i > j$. Integrating the right side of (3.4) gives

$$\int_{\gamma_j} 2y^*S(b)dy = \sum_{k<j} 2y_k(b)y_j(b) + y_j^2(b)S_{jj}(b). \tag{3.5}$$

Using (3.5) we see that

$$\int_\Gamma 2y^*\,S(b)dy = \sum_{j=1}^{n} \int_{\gamma_j} 2y^*\,S(b)dy = y^*(b)S(b)y(b),$$

so that

$$H_\Gamma = y^*(b)S(b)y(b) = S[y(b), a]. \tag{3.6}$$

It follows from (3.3) and (3.6) that (3.2) is true, and the theorem is proved.

IV. **The singularity condition.** We can now prove the following necessary condition.

THEOREM 4.1. *If $[a, \infty)$ affords a minimum limit to J, then*

$$\liminf_{x=\infty} S[y(x), a] \geqq 0 \tag{4.1}$$

for each A-admissible curve for which

$$\liminf_{x=\infty} J(y)\,\Big|_a^x < \infty. \tag{4.2}$$

Having established Theorem 3.1, the proof of Theorem 4.1 proceeds in the same way as the proof due to Morse and Leighton [6] for $n = 1$.

The condition of Theorem 4.1 that (4.1) holds for each A-admissible curve for which (4.2) holds will be called the *singularity condition* belonging to $[a, \infty)$.

V. **Sufficient conditions.** We come to the following theorem.

THEOREM 5.1. *If $(a, \infty]$ does not contain the conjugate point of $x = \infty$ and if the singularity condition belonging to $[a, \infty)$ is satisfied, then $[a, \infty)$ affords J a minimum limit among A-admissible curves.*

Using the extension of the Hilbert integral found in Chapter IV and using (3.1) the proof follows in the same fashion as that found in [6] where $n = 1$.

VI. **Principal quadratic functionals.** Functionals (1.1) in which $Q(x)=0$ will be called *principal* quadratic functionals. In this case, equation (1.1) becomes

$$f_0(x,y,y') = y^{*\prime}(x)R(x)y'(x) + y^*(x)P(x)y(x), \tag{6.1}$$

and then equation (1.2) becomes

$$J_0(y)\,\big|_a^b = \int_a^b f_0(x,y,y')\,dx. \tag{6.2}$$

These functionals were studied systematically first by Leighton [3], and later by Leighton and Martin [4].

We shall first prove a necessary condition.

THEOREM 6.1. *If the interval $[a,\infty)$ affords an A-minimum limit to J_0 then for any i, $1 \leqq i \leqq n$, the functional*

$$J_i(z) = \int_a^x [r_{ii}(x)z'^2(x) - P_{ii}(x)z^2(x)]\,dx \quad (i \text{ not summed})$$

is afforded an A-minimum limit by $[a,\infty)$.

We see that $J_i(z) = J_0(y)$, where $y=(y_1,\cdots,y_n)$ is defined so that $y_j=0$ if $j\neq i$ and $y_i=z$ and the theorem follows immediately.

We are now prepared to prove a basic theorem.

THEOREM 6.2. *If the interval $[a,\infty)$ affords an A-minimum limit to J_0 and if $P(x)$ is positive definite for large x, the matrix $\int_a^x P(x)dx$ is bounded.*

Suppose the contrary. Since $P(x)$ is positive definite for large x, there then exists a diagonal element, $\int_a^x p_{ii}(x)dx$ (i not summed), which is unbounded; that is,

$$\lim \int_a^x p_{ii}(x)dx = \infty \qquad (i \text{ not summed}). \tag{6.3}$$

But by Theorem 6.1, the functional

$$J_i = \int_a^x [r_{ii}(x)z'^2(x) - p_{ii}(x)z^2(x)]\,dx$$

is afforded an A-minimum limit by $[a,\infty)$. It follows from a result due to Leighton and Martin [4, p. 104] that $\int_a^x p_{ii}dx$ is bounded. From this contradiction, we conclude that the theorem is true.

In order to give sufficient conditions we require the following lemmas.

LEMMA 6.1. *If $A(x) = \| a_{ij}(x) \|$ is a positive definite matrix for large x and $\int_a^x A(x)dx$ is bounded, then*

$$\lim \int_a^x \max_{i,j} |a_{ij}(x)| \, dx < \infty.$$

One can prove the lemma easily by showing that $\max_{i,j}|a_{ij}(x)|$ is equal to some diagonal element of $A(x)$ when $A(x)$ is positive definite.

LEMMA 6.2. *If $U(x)$ is a bounded solution matrix of* (EE), *and if* $\lim \int_a^x \max_{i,j}|p_{ij}(x)| \, dx < \infty$, *then $R(x)U'(x)$ is bounded.*

Let $R(x) = \| r_{ij}(x) \|$, $U(x) = \| u_{ij}(x) \|$, and $P(x) = \| p_{ij}(x) \|$. Then, from (EE), we see that

$$r_{ik}(x)u'_{kj}(x) = r_{ik}(a)u'_{kj}(a) - \int_a^x p_{ik}(t)u_{kj}(t)dt$$

for any $i,j = 1,2,\cdots,n$. Setting $u(x) = \max_{i,j}|u_{\ j}(x)|$ and $p(x) = \max_{i,j}|p_{ij}(x)|$, we have that

$$|r_{ik}(x)u'_{kj}(x)| \leqq |r_{ik}(a)u'_{kj}(a)| + n^2 \int_a^x p(t)u(t)dt.$$

Since $u(x)$ is positive and bounded and since $\int_a^x p(t)dt$ is bounded, we see that $\int_a^x p(t)u(t)dt$ is bounded. The lemma now follows from the last inequality.

LEMMA 6.3. *If $Q(x) = \| q_{ij}(x) \|$ is a positive definite matrix for $a \leqq x < \infty$, and $\int_a^x Q(t)dt$ is bounded, and if $A(x) = \| a_{ij}(x) \|$ is a bounded matrix, then $\int_a^x A^*(t)Q(t)A(t)dt$ is bounded.*

Let $q(x) = \max_{i,j}|q_{ij}(x)|$ and $a(x) = \max_{i,j}|a_{ij}(x)|$. Then,

$$\left| \int_a^x a_{ki}(t)q_{kh}(t)a_{hj}(t)\,dt \right| \leqq n^2 \int_a^x q(x)a^2(x)\,dx. \tag{6.4}$$

Since $\int_a^x q(t)dt$ is bounded from Lemma 6.1 and $a^2(x)$ is also bounded, the right-hand side of (6.4) is bounded, and the lemma follows.

LEMMA 6.4. *If the interval $[a, \infty]$ does not contain a point conjugate to $x = \infty$, then the matrix solution $V(x)$ of* (EE) *such that*

$$V(a) = 0, \quad V'(a) = I, \tag{6.5}$$

is antiprincipal.

Since $V(x)$ is a prepared solution of (EE) which is nonsingular on $[a, \infty)$, there exists an antiprincipal solution $U(x)$ that is nonsingular on $[b, \infty)$ for some b. Furthermore, we can write the principal solution $W(x)$ in the form

$$W(x) = U(x) \int_x^\infty (U^*RU)^{-1}dx \qquad (x > b).$$

It can be readily seen by using (6.5) that $V(x)$ can be written as

(6.6) $$V(x) = U(x)W^*(a)R(a) - W(x)U^*(a)R(a),$$

or

(6.7) $$V(x) = U(x)\left[W^*(a) - \int_x^\infty (U^*RU)^{-1}dx\, U^*(a)\right]R(a) \quad (x \geqq b).$$

Set

$$A(x) = \left[W^*(a) - \int_x^\infty (U^*RU)^{-1}dx\, U^*(a)\right]R(a);$$

then $\lim A(x) = W^*(a)R(a)$, and $\det[W^*(a)R(a)] \neq 0$; hence, $A^{-1}(x)$ is nonsingular for large x, and $\lim A^{-1}(x) = R^{-1}(a)W^{*-1}(a)$. In particular, $A^{-1}(x)$ is bounded. But

$$\int^x (V^*RV)^{-1}dx = \int^x A^{-1}(U^*RU)^{-1}A^{*-1}dx$$

is bounded by Lemma 6.3. The theorem follows, then, from the definition of an antiprincipal solution matrix.

THEOREM 6.3. *If the interval* $[a, \infty]$ *does not contain a point conjugate to* $x = \infty$, *if* $\int_a^x P(x)dx$ *is bounded, and if* $V(x)$ *is a solution matrix of* (EE) *such that* $V(a) = 0$, $V'(a) = I$, *and* $R(x)V'(x)$ *is bounded, then* J *is afforded an A-minimum limit by* $[a, \infty)$.

If $S(x) = R(x)V'(x)V^{-1}(x)$, then

$$S(x) = S(b) - \int_b^x S(t)R^{-1}(t)S(t)dt - \int_b^x P(t)dt \qquad (b > a),$$

and using the symmetry of $S(x)$ we have

(6.8) $$\begin{aligned} S(x) = &- \int_b^x R(t)V'(t)[V^*(t)R(t)V(t)]^{-1}V^{*\prime}(t)R(t)\,dt \\ &- \int_b^x P(t)dt + S(b). \end{aligned}$$

By Lemma 6.4, the integral $\int_b^x [V^*(t)R(t)V(t)]^{-1}dt$ is bounded. Therefore, by Lemma 6.3, the first integral in (6.8) is bounded, and hence $S(x)$ is bounded. It follows that the singularity condition is satisfied. Theorem 6.3 now follows from Theorem 5.1.

One should notice that Theorem 6.3 does not require that the matrix $P(x)$ be positive definite for large x.

We state without proof a theorem due to Morse and Leighton when $n = 1$. The proof in n dimensions is precisely analogous.

THEOREM 5. *If $[b, \infty)$ affords J an A-minimum limit for any $b > a$, then J has an A-minimum limit on $[a, \infty)$.*

We come to an important theorem.

THEOREM 6.4. *If the interval $(a, \infty]$ does not contain a point conjugate to $x = \infty$, if $P(x)$ is positive definite for large x, and if $\int_a^x P(x)dx$ and $\int_a^x R^{-1}(x)dx$ are bounded, then J_0 possesses an A-minimum limit.*

We shall first show that the hypothesis of the theorem implies that any solution matrix $U(x) = \| u_{ij}(x) \|$ of (EE) is bounded.

This is easy to see by transforming (2.1) with of course $Q(x) = 0$, into the $2n$ system of first-order equations

$$u' = A(x)u, \tag{6.9}$$

where $A(x)$ is the $2n$ by $2n$ matrix

$$\begin{pmatrix} 0 & R^{-1}(x) \\ P(x) & 0 \end{pmatrix}$$

and $u^*(x) = (y_1, \cdots, y_n, r_{1k}y_k', \cdots, r_{nk}y_k')$. Since $\int_a^x P(x)dx$ and $\int_a^x R^{-1}(x)dx$ are bounded, then in view of Lemma 6.1, $\int_a^x \max_{i,j} | a_{ij}(x) | dx$ is bounded where $A(x) = \| a_{ij}(x) \|$. It is well known that if $\int_a^x \max_{i,j} | a_{ij}(x) | dx$ is bounded, then all solutions of (6.9) are bounded and hence any solution matrix $U(x)$ of (EE) is bounded. From Lemma 6.2 we have that $R(x)U'(x)$ is bounded, and from Theorem 6.3 J_0 is afforded an A-minimum limit by $[b, \infty)$ for any $b > a$. Theorem 6.4 then follows from Theorem 5.

We actually have proved, in view of Lemma 6.1, a stronger result that Theorem 6.4. We may state this result as follows.

THEOREM 6.5. *If the interval $(a, \infty]$ does not contain a point conjugate to $x = \infty$, $\lim \int_a^x \max_{i,j} | p_{ij}(x) | \, dx < \infty$, and if the matrices $\int_a^x P(x)dx$ and $\int_a^x R^{-1}(x)dx$ are both bounded, then J_0 possesses an A-minimum limit.*

The proof is the same as that of Theorem 6.4 and is valid since Theorem 6.3 and Lemma 6.2 do not require that $P(x)$ be positive definite.

Suppose now that the interval $(a, \infty]$ does not contain a point conjugate to $x = \infty$, $P(x)$ is positive definite for large x, $\int_a^x P(x)dx$ is bounded, and that all characteristic roots of $\int_a^x R^{-1}(x)dx$ are unbounded. In this case, Reid [7] has pointed out that using the proof of a theorem due to Sternburg [8], one can prove that there exists a solution matrix $U(x)$ of (EE) such that

$$\lim R(x)U'(x)U^{-1}(x) = 0.$$

Using this same proof one can show that if $S(x)$ is the singularity matrix, then $\lim S(x) = 0$. The singularity condition is then obviously satisfied, and hence under these conditions Theorem 5.1 leads to the conclusion that J_0 has an A-minimum limit. We summarize these remarks in the following theorem.

THEOREM 6.6. *If the interval* $(a, \infty]$ *does not contain a point conjugate to* $x = \infty$, *if* $P(x)$ *is positive definite for large* x, *if* $\int_a^x P(x)dx$ *is bounded, and if all characteristic roots of* $\int_a^x R^{-1}(x)dx$ *are unbounded, then* J_0 *possesses an A-minimum limit.*

We shall indicate at the end of the next section how this theorem can be proved in another way.

VII. **An oscillation theorem.** In this section we shall consider the differential equation

$$[R(x)y']' + P(x)y = 0, \tag{7.1}$$

where $R(x)$ and $P(x)$ are continuous positive definite (real) matrices on $[a, \infty)$, and y is a (column) vector with n components. We will state sufficient conditions that (7.1) be oscillatory. We say that (7.1) is oscillatory if the conjugate point to $x = \infty$ is infinite.

Equation (7.1) can be written in matrix form as

$$[R(x)Y'(x)]' + P(x)Y(x) = 0, \tag{7.2}$$

where the solution matrix $Y(x)$ has for its columns the solution vectors of (7.1). We shall say that (7.2) is oscillatory if there exists a prepared (cf. p. 61) solution matrix $Y(x)$ of (7.2), where $\det Y(x) \not\equiv 0$, such that $\det Y(x)$ has arbitrarily large zeros. It is known (cf. [8, p. 314]) that (7.1) is oscillatory if and only if (7.2) is oscillatory.

We continue by defining $z(x, x_0)$ as the solution matrix of (7.2) such that

$$z(x_0, x_0) = I \text{ and } z_x(x_0, x_0) = 0. \tag{7.3}$$

We then define the first zero, if it exists, of $\det z(x, x_0)$ less than $x = x_0$ to be the focal point of the hyperplane $x = x_0$. If $\det z(x, x_0) \neq 0$ for $x < x_0$, we say that the focal point of the hyperplane $x = x_0$ does not exist. The kth focal point of the hyperplane $x = x_0$ will be defined, if it exists, to be the kth zero of $\det z(x, x_0)$ less than $x = x_0$.

We shall now give a representation of $z(x, x_0)$ that is more useful for our purposes. To this end, let $V(x)$ and $W(x)$ be the two solution matrices of (7.2) that satisfy the conditions

$$V(b) = W'(b) = 0,\ R(b)V'(b) = W(b) = I\ (b \neq x_0). \tag{7.4}$$

Recall that any two solution matrices of (7.2) must satisfy (2.3) for some constant matrix C. Using (2.3) and (7.4), we see that

$$V^{*\prime}RW - V^{*}RW' = I, \tag{7.5}$$

and that

$$V^{*\prime}RV - V^{*}RV' = W^{*\prime}RW - W^{*}RW' = 0. \tag{7.6}$$

Using (7.5) and (7.6), we can write

$$\begin{pmatrix} V^{*\prime}R & -V^{*} \\ -W^{*\prime}R & W^{*} \end{pmatrix} \begin{pmatrix} W & V \\ RW' & RV' \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{7.7}$$

where the matrices are the obvious $(2n)$-square block matrices. We then rewrite (7.7) as

$$\begin{pmatrix} W & V \\ RW' & RV' \end{pmatrix} \begin{pmatrix} V^{*\prime}R & -V^{*} \\ -W^{*\prime}R & W^{*} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{7.8}$$

In particular, (7.8) yields

$$WV^{*\prime}R - VW^{*\prime}R = I, \tag{7.9}$$

and

$$W'V^{*\prime} - V'W^{*\prime} = 0. \tag{7.10}$$

We finally have from (7.3), (7.9), and (7.10) that

$$z(x, x_0) = W(x)V^{*\prime}(x_0)R(x_0) - V(x)W^{*\prime}(x_0)R(x_0). \tag{7.11}$$

We continue with two lemmas (cf. [3, p. 257]).

LEMMA 7.1. *If the kth focal point α of the hyperplane $x = x_0$ exists, and if $P(x_0)$ is positive definite, then $\alpha(x)$ is a strictly increasing function in some neighborhood of $x = x_0$.*

To prove the lemma, let $V(x)$ and $W(x)$ be the two solution matrices of (7.2) satisfying (7.4) where $b \neq x_0$. An integration by parts indicates that

$$\int_a^x (V^{*\prime}RV' - V^{*}PV)dx = V^{*}RV' \Big|_a^x. \tag{7.12}$$

Since the left-hand member of (7.12) possesses a derivative, the right-hand member does also, and

$$\frac{d}{dx}(V^{*}RV') = V^{*\prime}RV' - V^{*}PV. \tag{7.13}$$

It follows from (7.9) that

$$V^{-1}W - W^{*\prime}V^{*\prime -1} = (V^{*\prime}RV)^{-1};$$

or, in view of (7.10) and (7.6), that

$$V^{-1}W - V'^{-1}W' = (V^*RV')^{-1}. \tag{7.14}$$

Differentiating (7.14) and using (7.6), (7.9), (7.10), and (7.13), one can readily verify that

$$\frac{d}{dx}(V'^{-1}W') = -(V'^{-1}R^{-1})P(V'^{-1}R^{-1})^*. \tag{7.15}$$

It follows from (7.15) that $V'^{-1}W'$ is of class C' except in the zeros of $\det V'(x)$. By hypothesis, the first zero α of $\det z(x, x_0)$ less than $x = x_0$ exists. Thus, if $V(x)$ is such that $\det V'(x_0) \neq 0$ and $\det V(\alpha) \neq 0$, we have from (7.11) that

$$\det[V^{-1}(\alpha)W(\alpha) - W^{*\prime}(x_0)V^{*\prime -1}(x_0)] = 0.$$

In view of (7.10), $W^{*\prime}V^{*\prime -1}$ is symmetric; accordingly, we have

$$\det[V^{-1}(\alpha)W(\alpha) - V'^{-1}(x_0)W'(x_0)] = 0. \tag{7.16}$$

It follows from (7.16) that there exists a constant vector $c \neq 0$ such that

$$[V^{-1}(\alpha)W(\alpha) - V'^{-1}(x_0)W'(x_0)]c = 0, \tag{7.17}$$

and

$$c^*[V^{-1}(\alpha)W(\alpha) - V'^{-1}(x_0)W'(x_0)]c = 0. \tag{7.18}$$

We wish to solve equation (7.18) for α using the implicit-function theorem. Note that it has already been established that the left-hand member of (7.18) is of class C'. If we set the left-hand member of (7.18) equal to $f(\alpha, x_0)$ and use (7.17), we have that

$$\frac{\partial f}{\partial \alpha}(\alpha, x_0) = c^*\left[\frac{d}{d\alpha} V^{-1}(\alpha)W(\alpha)\right]c = -c^*(V^*(\alpha)R(\alpha)V(\alpha))^{-1}c; \tag{7.19}$$

accordingly, $\partial f(\alpha, x_0)/\partial\alpha$ is strictly negative since $(V^*(\alpha)R(\alpha)V(\alpha))^{-1}$ is positive definite, and $c \neq 0$. Therefore, the implicit-function theorem applies to (7.18) and affirms the existence of $d\alpha/dx_0$; consequently,

$$\begin{aligned} c^*\left[\frac{d}{d\alpha} V^{-1}(\alpha)W(\alpha)\right]c\,\frac{d\alpha}{dx_0} &= c^*\left[\frac{d}{dx_0} V'^{-1}(x_0)W'(x_0)\right]c \\ &= c^*(V'^{-1}(x_0)R^{-1}(x_0))P(x_0)\,(V'^{-1}(x_0)R^{-1}(x_0))\,c. \end{aligned}$$

The last equality above follows from (7.15). Thus,

$$\frac{d\alpha}{dx_0} = \frac{c^*(V'^{-1}(x_0)R^{-1}(x_0))P(x_0)\,(V'^{-1}(x_0)R^{-1}(x_0))^*c}{c^*(V^*(\alpha)R(\alpha)V(\alpha))^{-1}c}. \tag{7.20}$$

Since $P(x_0)$ and $R(\alpha)$ are positive definite and $c \neq 0$, it follows from (7.20) that $d\alpha/dx_0 > 0$ and that $\alpha(x_0)$ is strictly increasing in a neighborhood of $x = x_0$. To complete the proof of the lemma, we must consider the possibility that $\det V'(x_0) = 0$ or $\det V(\alpha) = 0$. We shall choose the point b which in (7.4) defines $V(x)$ in such a manner that this does not occur. Assume that α is the first focal point of $x = x_0$ and take b to be any point on the interval (α, x_0). First notice that $\det z(x, x_0)$ does not vanish on (α, x_0), and in particular, $\det z(b, x_0) \neq 0$. Now $\det V'(x_0) \neq 0$ since otherwise there would exist a constant vector $c \neq 0$ such that $V'(x_0)c = 0$, that is, the vector solution $y(x) = V(x)c$ is such that $y'(x_0) = 0$ and therefore by the uniqueness theorem $y(x) = z(x, x_0)c_1$ for some constant vector $c_1 \neq 0$. But

$$z(b, x_0)c = y(b) = V(b)c = 0;$$

accordingly, $\det z(b, x_0) = 0$ which is not the case. Also, $\det V(\alpha) \neq 0$. Otherwise we could choose a point b^* on the interval (b, x_0) and then the conjugate point of b^*, $c(b^*)$, would be on the interval (α, b^*). This would imply that the determinant of the solution matrix $V_1(x)$, defined by $V_1(b^*) = 0$ and $V_1{}'(b^*) = I$, would have $(n+1)$ zeros on $[c(b^*), b^*]$ which contradicts the fact that $\det z(x_0, x)$ has no zeros there. Furthermore, if α is the kth focal point of $x = x_0$, let x_0^* be the $(k-1)$st focal point, then we can choose b on the interval (α, x_0^*) and proceed as before. These last remarks conclude the proof of the lemma.

LEMMA 7.2. *If the interval $(a, \infty]$ does not contain the point conjugate to $x = \infty$, if the matrix $P(x)$ is positive definite for large x, and if $V(x)$ is the solution matrix of* (7.3) *such that $V(a) = 0$ and $V'(a) = I$, then* $\det V'(x)$ *does not vanish for large x.*

Suppose the lemma is false and that the zeros of $\det V'(x)$ are $x = x_i$ $(i = 1, 2, \cdots)$, where the sequence $\{x_i\}_{i=1}$ is strictly increasing and $\lim x_i = \infty$. There exists a nonzero constant vector c such that $V'(x_1)c = 0$. By definition of the matrix solution $z(x, x_1)$ given by (7.3) and by the uniqueness of solutions, we must have that $V(x)c = z(x, x_1)c_1$ for some nonzero constant vector c_1. We then have $z(a, x_1)c_1 = 0$; thus, $\det z(a, x_1) = 0$. If $\alpha_k(x_1)$ is the kth focal point of $x = x_1$, then we must have $\alpha_1(x_1) \geqq a$. But from Lemma 7.1, we see that $\alpha_k(x)$ is a strictly increasing function on $[x_1, \infty)$ for any k, and hence that $a \leqq \alpha_1(x_1) < \alpha_1(x_2)$. Therefore, $\det z(x, x_2)$ has at least one zero on (a, x_2). But we can also show in a similar manner as before that $\det z(a, x_2) = 0$. It follows that $\alpha_2(x_2) \geqq a$. Proceeding in this manner, we see that $\det z(x, x_{n+2})$ has at least $(n+1)$ zeros counting multiplicity, on (a, x_{n+2}). But this contradicts Theorem 4, since $\det V(x)$ does not vanish on (a, ∞) by Theorem 2.1. From this contradiction we infer the truth of the lemma.

THEOREM 7.1. *The system* (7.1) *of n equations is oscillatory if the limits of r*

characteristic roots of the matrix $\int_a^x R^{-1}(x)dx$ *and the limits of* s *characteristic roots of the matrix* $\int_a^x P(x)dx$ *exist and are positively infinite and* $r+s>n$.

Suppose that the theorem is false and assume that (7.1) is not oscillatory; that is, there is no point conjugate to $x=\infty$ on some interval (α,∞), where α is chosen so that $\alpha \geqq a$. From Lemma 7.2, it follows that the solution matrix $V(x)$ of (EE), such that $V(\alpha)=0$ and $V'(\alpha)=I$, satisfies the condition

$$\det V'(x) \neq 0 \qquad (x \geqq b), \tag{7.21}$$

where $b>\alpha$. Set $S(x)=R(x)V'(x)V^{-1}(x)$. Then from (EE), it is easy to see that

$$S(x)=S(b)-\int_b^x S(x)R^{-1}(x)S(x)dx-\int_b^x P(x)dx, \tag{7.22}$$

and

$$S^{-1}(x)=S^{-1}(b)+\int_b^x S^{-1}(x)P(x)S^{-1}(x)dx+\int_b^x R^{-1}(x)dx, \tag{7.23}$$

where we recall from Theorem 2.1 that $\det V(x)\neq 0$ on (α,∞). Accordingly, $S(x)$ is defined on $[b,\infty)$ and (7.21) is nonsingular on $[b,\infty)$.

It is easy to see from the preparedness (cf. (2.4)) of $V(x)$ that $S(x)$ is symmetric, so that the matrices $\int_b^x S(x)R^{-1}(x)S(x)dx$ and $\int_b^x S^{-1}(x)P(x)S^{-1}(x)$ are positive definite. Using this fact and (7.22) and (7.23), it can readily be shown with the aid of the Courant-Fischer min-max theorem that $S(x)$ has s characteristic roots whose limits are negatively infinite and $S^{-1}(x)$ has r characteristic roots whose limits are positively infinite. But this contradicts the fact that $r+s>n$. From this contradiction we infer the truth of the theorem.

We shall now give an example to indicate that the condition in Theorem 7.1 that $r+s>n$ cannot be relaxed. We take $n=2$, $r_{12}=p_{12}=0$, $r_{11}=1$, $r_{22}=e^{2x}$, $p_{11}=1/x^2\ln x$, $p_{22}=e^{2x}$. This system consists of two independent equations. One vector solution is $y^*=(\ln x, e^{-x})$; and it is easy to see that this system is not oscillatory. It is also easy to verify that, for this system, $r=s=1$. So that $r+s=n$, and (7.1) is not oscillatory.

In the case when all the characteristic roots of the matrix $\int^x R^{-1}(x)dx$ have limits that are positively infinite, Reid has shown, using the proof of a theorem due to Sternberg [8], that (7.1) is oscillatory if $\int^x P(x)dx$ is unbounded. This is the case in the terminology of Theorem 7.1 when $r=n$ and $s>1$. As mentioned at the end of the last section, this proof can be utilized to show that the singularity matrix $S(x)$ satisfies:

$$\lim S(x)=0,$$

if the hypothesis of Theorem 6.6 holds. However, we have shown in the proof of Theorem 7.1, that if $r=n$ and (7.1) is not oscillatory, then $\lim S(x)=0$. Thus, we have indicated an alternate proof of Theorem 6.6.

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