

PIERCING POINTS OF HOMEOMORPHISMS OF DIFFERENTIABLE MANIFOLDS

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1. **Introduction.** In this paper we investigate the problem of relating homeomorphisms of euclidean n -space E^n (resp. a differentiable manifold M^n) onto itself to diffeomorphisms of E^n (resp. of M^n). Throughout this paper, "diffeomorphism" shall mean C^p -diffeomorphism, where $p > 0$. A discussion of these problems is facilitated by introducing the following equivalence relation \sim on the set $H(E^n)$ of homeomorphisms of E^n onto itself. If $F, G \in H(E^n)$, we say $F \sim G$ if there exist homeomorphisms $H_0 = F, H_1, \dots, H_m = G$, where each $H_i \in H(E^n)$, and nonempty open sets U_1, U_2, \dots, U_m of E^n such that $H_i|U_i = H_{i-1}|U_i, i = 1, 2, \dots, m$. One asks, for example, whether a given homeomorphism $F \in H(E^n)$ is equivalent under \sim to a diffeomorphism.

Fundamental in the study of this type of question is the notion of *stable* homeomorphisms of E^n onto itself. Recall that a homeomorphism $H \in H(E^n)$ is called stable if there exist homeomorphisms H_1, \dots, H_m , where each $H_i \in H(E^n)$ and nonempty open sets U_1, \dots, U_m of E^n such that $H = H_1 H_2 \dots H_m$, and $H_i|U_i = 1, i = 1, 2, \dots, m$. All orientation-preserving diffeomorphisms of E^n onto itself are stable. It is readily seen that if $F \sim G$, and G is stable, then so is F . It also can be proved (cf. Theorem 5.4 of [1]) that if F and G are any two stable homeomorphisms, then $F \sim G$. It follows easily from these latter two statements that $F \sim G$ if and only if $G^{-1}F$ is a stable homeomorphism of E^n . Finally, the annulus conjecture is equivalent to the conjecture that all orientation-preserving homeomorphisms of E^n onto itself are stable. This latter conjecture is known to be true for $n = 1, 2, 3$.

In an effort to relate homeomorphisms to diffeomorphisms, we define (cf. §3) the notion of a piercing point of a homeomorphism. In §§3-7 we develop some basic properties relating to piercing points. A proof is given in §8 of a result announced by the author in [2]. Theorem 1 of §9 relates the notion of piercing point to stability. The author thanks William Huebsch for many helpful conversations.

2. **Notation.** Let the points of E^n be written $x = (x^1, \dots, x^n)$, and provide E^n with the usual euclidean norm and metric

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$$\|x\| = \left[\sum_{i=1}^n (x^i)^2 \right]^{1/2}, \quad d(x, y) = \|x - y\|.$$

For c a fixed point of E^n , and $r > 0$ a constant, we denote the $(n-1)$ -sphere about c , of radius r , by

$$S^{n-1}(c, r) = \{x \in E^n \mid d(x, c) = r\}.$$

We often delete the superscript $n-1$ when there is no danger of confusion. By a topological $(n-1)$ -sphere M in E^n we mean the image in E^n of $S^{n-1}(c, r)$ under some homeomorphism h . We say that h defines M . If M is a topological $(n-1)$ -sphere in E^n , we denote the bounded component of $E^n - M$ by JM , and the closure of JM in E^n by JM .

A topological $(n-1)$ -sphere M in E^n will be called elementary if some (hence every, as is readily proved) homeomorphism h defining M is extendable as a homeomorphism into E^n of an open neighborhood N of $S^{n-1}(c, r)$ relative to E^n . This is equivalent (cf. [3]) to requiring that M is locally flat. A homeomorphism h of an elementary $(n-1)$ -sphere M in E^n which is extendable over a neighborhood of M as a homeomorphism will itself be termed elementary.

If M is a topological $(n-1)$ -sphere in E^n , and $f: JM \rightarrow E^n$ is a homeomorphism of JM into E^n , then

$$(\alpha) \quad f(JM) = Jf(M).$$

A proof of (α) can be found, for example, in [4].

3. Piercing points. We assume in what follows that $n \geq 2$.

DEFINITION 1. Let $f: U \rightarrow E^n$ be a homeomorphism of U into E^n , where U is an open subset of E^n . A point $x \in U$ is called a *piercing point* of f if there exists a C^p -imbedding ($p > 0$) $\sigma: [-1, 1] \rightarrow U$, a diffeomorphism $H \in H(E^n)$, and an $(n-1)$ -hyperplane P in E^n such that

- (i) $\sigma(0) = x$,
- (ii) $Hf\sigma([-1, 1]) \cap P = Hf\sigma(0)$,
- (iii) $Hf\sigma(-1)$ and $Hf\sigma(1)$ lie in opposite components of $E^n - P$.

One verifies that every point of a diffeomorphism $H \in H(E^n)$ is a piercing point of H . On the other hand, we will prove (§8) that there exist homeomorphisms $F \in H(E^n)$ having a dense set of nonpiercing points.

PROPOSITION 1. Let U, V be open subsets of E^n , let $f: U \rightarrow E^n$ be a homeomorphism of U into E^n , and let $g: V \rightarrow E^n$ be a diffeomorphism of V into E^n . Then a point $x \in U \cap f^{-1}(V)$ [resp. $x \in U \cap g(V)$] is a piercing point of f if, and only if, x is a piercing point of gf [resp. $g^{-1}(x)$ is a piercing point of fg].

Proof. In each situation considered, it can be assumed without loss of generality that g is a diffeomorphism of E^n onto itself (cf. [5, pp. 28-29]). Suppose that x is a piercing point of f . Then, corresponding to f and x , there exist σ, H ,

and P as in Definition 1. Then σ, Hg^{-1} , and P [resp. $g^{-1}\sigma, H, P$] can be used to verify that x is a piercing point of gf [resp. $g^{-1}(x)$ is a piercing point of fg]. Conversely, suppose that x is a piercing point of H [resp. g^{-1} is a piercing point of fg]. Then σ, Hg , and P [resp. $g\sigma, H, P$] can be used to verify that x is a piercing point of f .

With the aid of Proposition 1, the notion of piercing point may be extended, in the natural way, to homeomorphisms from one nonbounded differentiable n -manifold into another. More precisely, we make the following definition.

DEFINITION 1'. Let M_1^n, M_2^n be nonbound differentiable n -manifolds having differentiable structures D_1, D_2 , respectively, both of class at least $p > 0$. Let $f: U \rightarrow M_2^n$ be a homeomorphism of U into M_2^n , where U is an open subset of M_1^n . Then x is called a piercing point of f if there exist coordinate systems $(U_1, h_1) \in D_1, (U_2, h_2) \in D_2$ at $x, f(x)$, respectively, such that $h_1(x)$ is a piercing point (in the sense of Definition 1) of the homeomorphism

$$h_2 f h_1^{-1} : h_1(U \cap U_1 \cap f^{-1}(U_2)) \rightarrow E^n.$$

It is easily verified, using Proposition 1 and its proof, that this definition is independent of the choice of coordinate systems at x and $f(x)$.

4. Example 1. We now construct homeomorphisms $f, g \in H(E^2)$ such that $\mathbf{0} = (0, 0)$ is a piercing point of $f, f(\mathbf{0}) = \mathbf{0}$ is a piercing point of g , but $\mathbf{0}$ is not a piercing point of the composition gf . Now it is easily seen that a point $x \in U \subset E^2$ is a piercing point of a homeomorphism $h: U \rightarrow E^2$ if, and only if, the point $h(x)$ is a piercing point of $h^{-1}: h(U) \rightarrow E^2$. Hence if f, g are as above, we see that the homeomorphisms $gf, f = g^{-1}(gf)$, and $f^{-1} = (gf)^{-1}g$, show that all the conclusions of Proposition 1 can fail when the hypothesis is weakened to merely requiring, for example, that $g(x)$ be a piercing point of the homeomorphism g .

Let $x = (0, 0) = \mathbf{0} \in E^2$, and set $S = S^1(\mathbf{0}, 1)$. Denote the line segments having one end point at $\mathbf{0}$ and the other end point at $(0, 1), (\sqrt{2}/2, \sqrt{2}/2), (-\sqrt{2}/2, \sqrt{2}/2), (-\sqrt{3}/2, -\frac{1}{2}), (-\frac{1}{2}, -\sqrt{3}/2), (\frac{1}{2}, -\sqrt{3}/2), (\sqrt{3}/2, -\frac{1}{2})$, by L_0, \dots, L_6 , respectively. Let

$$M_1 = \{y = (y^1, y^2) \in S \mid 0 \leq y^1 \leq \sqrt{2}/2, -\sqrt{2}/2 \leq y^2 \leq 1\} \cup L_0 \cup L_1,$$

$$M_2 = \{y = (y^1, y^2) \in S \mid -\sqrt{2}/2 \leq y^1 \leq 0, \sqrt{2}/2 \leq y^2 \leq 1\} \cup L_0 \cup L_2,$$

and

$$M_3 = \{y = (y^1, y^2) \in S \mid -1 \leq y^1 \leq 1, -1 \leq y^2 \leq \sqrt{2}/2\} \cup L_1 \cup L_2.$$

Let

$$\{y_i\} = \{1/i(\sqrt{2}/2, \sqrt{2}/2)\}, \quad \{z_i\} = \{1/i(-\sqrt{2}/2, \sqrt{2}/2)\}, \quad i = 1, 2, \dots.$$

We now define f on M_1 as follows. Set $f(w) = w$ for $w = (w^1, w^2) \in L_0$, and

$$f(w) = (w^1 \cos(5\pi\sqrt{2}/12w^1) + w^2 \sin(5\pi\sqrt{2}/12w^1), \\ -w^1 \sin(5\pi\sqrt{2}/12w^1) + w^2 \cos(5\pi\sqrt{2}/12w^1))$$

for $w \in M_1 \cap S$. We complete the definition of f on M_1 as follows.

Set $f(y_{2i-1}) = 1/i(\sqrt{3}/2, -1/2)$, and $f(y_{2i}) = 1/i(-1/2, -\sqrt{3}/2)$, $i = 1, 2, \dots$. Then let f map the segment between y_i and y_{i+1} in the manner indicated in Figure 1 below.

Define f on M_2 by letting

$$f((w^1, w^2)) = (-f^1(-w^1, w^2), f^2(-w^1, w^2)) \text{ for } w \in L_0 \cup (M_2 \cap S) \cup \bigcup_{i=1}^{\infty} z_i.$$

Complete the definition of f on M_2 by mapping the segment between z_i and z_{i+1} in the manner indicated in Figure 1.

To define f on M_3 , note that f is defined on $(M_3 - S) \cup \{\sqrt{2}/2, \sqrt{2}/2\} \cup \{-\sqrt{2}/2, \sqrt{2}/2\}$. We complete the definition of f on M_3 by mapping $M_3 - (L_1 \cup L_2)$ homeomorphically onto $S - \{f(M_1 \cup M_2)\}$.

We now have defined f on $M_1 \cup M_2 \cup M_3$ consistently as a homeomorphism. Since every homeomorphism of a topological 1-sphere M in E^2 admits an extension as a homeomorphism over JM (cf. [6]), we may extend $f|_{M_i}$ to a homeomorphism of JM_i into E^2 , $i = 1, 2, 3$. Actually, it is clear that $f|_{M_i}$ is elementary, and hence we could use the Schoenflies extension theorem (cf. [7] or [8]) to get the extension of $f|_{M_i}$ over JM_i . This latter method of obtaining the extension is employed when modifying the example to dimensions greater than 2. Now since $fJM_i = Jf(M_i)$ (cf. (α) of §2), we obtain a homeomorphism f of JS into E^n . For convenience, we apply the Schoenflies extension theorem again to assume that $f \in H(E^2)$. Note that $\mathbf{0}$ is a piercing point of f .

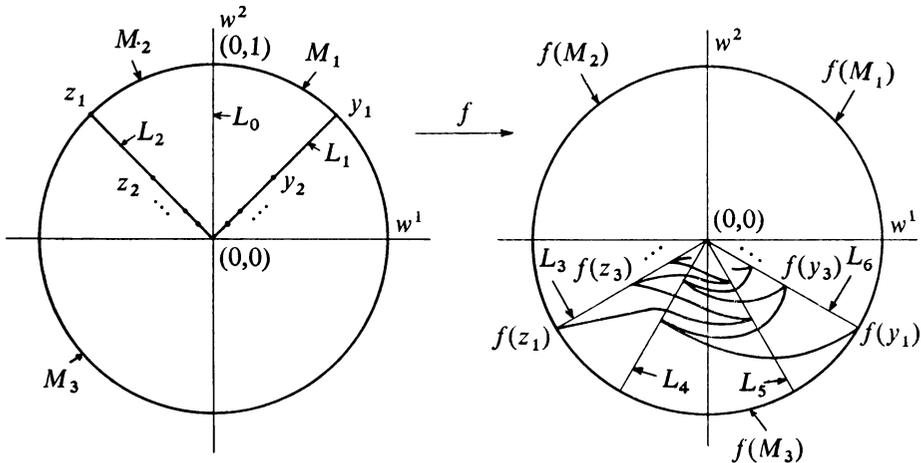


FIGURE 1

We now consider a homeomorphism $g \in H(E^2)$ with the following properties (cf. Figure 2). Let L_3, L_4, L_5 , and L_6 be as in Figure 1. Denote the line segments having one end point at $\mathbf{0}$ and the other end point at $(-\frac{1}{2}, \sqrt{3}/2)$, $(-\sqrt{3}/2, \frac{1}{2})$, $(\sqrt{3}/2, \frac{1}{2})$, $(\frac{1}{2}, \sqrt{3}/2)$, by L_7, L_8, L_9, L_{10} , respectively. We suppose $g(L_i) = L_{i+4}$,

$i = 3, 4, 5, 6$. We further suppose that g is the identity on the w^2 -axis. A homeomorphism $g \in H(E^2)$ with these properties is easily constructed. Note that $\mathbf{0}$ is a piercing point of g .

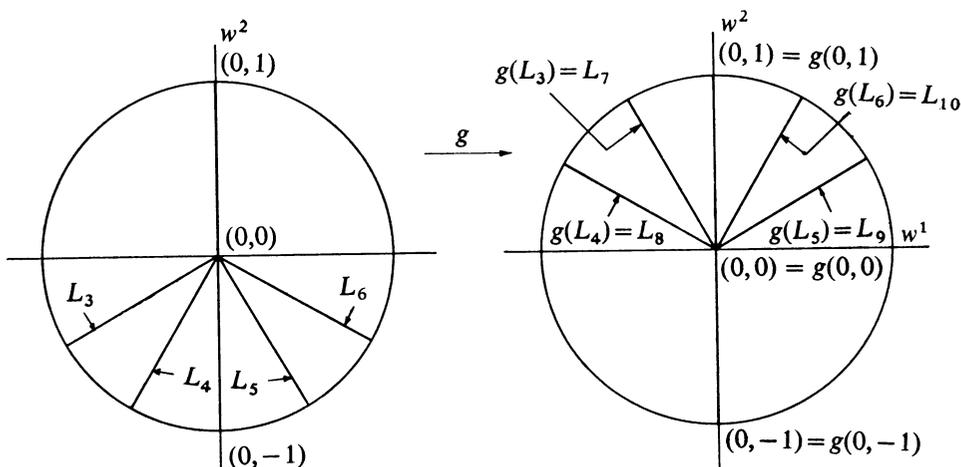


FIGURE 2

It is readily seen that $\mathbf{0}$ is not a piercing point of gf .

REMARKS. An analogous construction can be carried out for any $n \geq 2$. Also, one verifies with the aid of Theorem 4.1 of [5] (only really needed for f), that f and g can be constructed so as to be diffeomorphisms on $E^n - \mathbf{0}$.

5. **One-sided piercing points.** Note that the homeomorphism gf of Example 1, while not having $\mathbf{0}$ as a piercing point, nevertheless leaves the segment from $(0, 1)$ to $\mathbf{0}$ pointwise invariant. Hence $\mathbf{0}$ could be called a “one-sided” piercing point of gf . We make this concept precise in the following definition.

DEFINITION 2. Let $f: U \rightarrow E^n$ be a homeomorphism of U into E^n , where U is an open subset of E^n . A point $x \in U$ is called a *one-sided* piercing point of f if there exists C^p -imbedding ($p > 0$) $\sigma: [0, 1] \rightarrow U$, a diffeomorphism $H \in H(E^n)$ and an $(n - 1)$ -hyperplane P in E^n such that

- (i) $\sigma(1) = x$,
- (ii) $Hf\sigma([0, 1]) \cap P = Hf\sigma(1)$.

A point $x \in U$ which is *not* a one-sided piercing point will be called a *spiral point* of f .

We see, then, that a spiral point $x \in U$ of f has the following characteristic property: given any C^p -imbedding ($p > 0$) $\sigma: [0, 1] \rightarrow U$, any diffeomorphism $H \in H(E^n)$, and any $(n - 1)$ -hyperplane P in E^n , there exists a sequence of points $t_i \in [0, 1]$ converging to 1 and such that $Hf\sigma(t_i) \in P$. Clearly, a spiral point of f is a nonpiercing point of f , but the converse does not hold as the homeomorphism gf of Example 1 shows.

PROPOSITION 2. Let $f: U \rightarrow E^n$ be a homeomorphism of U into E^n , where U is

an open subset of E^n . Then the set of one-sided piercing points of f is dense in U . In fact, if $\sigma: [0, 1] \rightarrow U$ is any C^p -imbedding, ($p > 0$), then there exists a $t \in (0, 1)$ such that $\sigma(t)$ is a one-sided piercing point of f .

Proof. Let $\sigma: [0, 1] \rightarrow U$ be a C^p -imbedding, and let P be any $(n - 1)$ -hyperplane in E^n such that $f\sigma(0)$ and $f\sigma(1)$ lie in opposite components of $E^n - P$. Since $f\sigma([0, 1]) \cap P \neq \emptyset$, there exists a (unique) $t \in (0, 1)$ such that $f\sigma[0, t] \cap P = \sigma(t)$. Hence $\sigma(t)$ is a one-sided piercing point of f .

REMARK. The notion of "one-sided" piercing point extends, in the natural way (cf. Definition 1'), to differentiable manifolds. Proposition 2 holds with this notion so extended.

6. An alternative definition. Suppose we altered Definition 1 by requiring that H be the identity diffeomorphism of E^n onto itself. The question arises as to whether every piercing point in the old sense would also be one in this new more restrictive sense. Example 2 below answers this latter question in the negative.

EXAMPLE 2. Let $S = S^1(0, 1)$, and let $f: E^2 \rightarrow E^2$ be a homeomorphism of E^2 onto itself having the following properties. First $f(x) = x$ for $x \in C \cup L$, where L is the radius segment of S joining $0 = (0, 0)$ to $(0, 1)$. Secondly, the image under f of every radius segment of S makes an angle of 0° with L at 0 (cf. Figure 3). Such homeomorphism clearly exists (in fact, f may be required to be a diffeomorphism on $E^2 - 0$). Note that 0 is *not* a piercing point of f if H is required to be the identity map of E^n .

We now construct a diffeomorphism $H \in H(E^n)$ which will show that 0 is a piercing point (in the sense of Definition 1) of f .

We can assume f is so constructed that there exists a C^∞ -imbedding $\tau: (L \cup L') \rightarrow E^2$ such that $\tau(L) = f(L)$ and $\tau|_{L'} = 1$, where L, L' are the radius segments of S which go through $(\sqrt{2}/2, \sqrt{2}/2), (0, -1)$, respectively. Let H be a C^∞ -diffeomorphism of E^2 onto itself such that $H|(f(L) \cup L') = \tau^{-1}$. Let $\sigma: [-1, 1] \rightarrow E^2$ be the linear imbedding defined by $\sigma(-1) = (0, -\frac{1}{2}), \sigma(1) = (0, \frac{1}{2})$. Finally, let P be the x^2 axis. One verifies that σ, H , and P satisfy the conditions of Definition 1 with respect to f and 0 .

REMARK. Example 2 also shows that Proposition 1 would not be satisfied if our definition of piercing point would have been the more restrictive one. This example can be modified to yield the corresponding results for $n \geq 2$.

7. Property Q. We now introduce a property which concerns the existence of piercing points of a uniform type.

DEFINITION 3. Let $f: U \rightarrow E^n$ be a homeomorphism of U into E^n , where U is an open set in E^n . We say that f has property Q at a point $x_0 \in U$ if there exists a diffeomorphism L of E^n onto itself such that the following three conditions are satisfied:

- (i) $L^{-1}(x_0)$ is a piercing point of fL , i.e. there exists σ, H , and P satisfying (i), (ii), and (iii) of Definition 1 relative to fL and $L^{-1}(x_0)$,

- (ii) σ is linear, i.e. $\sigma([-1, 1])$ is a straight line segment,
- (iii) there exists an open set W in P such that for all $x \in L^{-1}f^{-1}H^{-1}(W)$, H, P , and σ_x defined by $\sigma_x(t) = \sigma(t) + x - \sigma(0)$ satisfy conditions (i), (ii), and (iii) of Definition 1 with respect to fL and x .

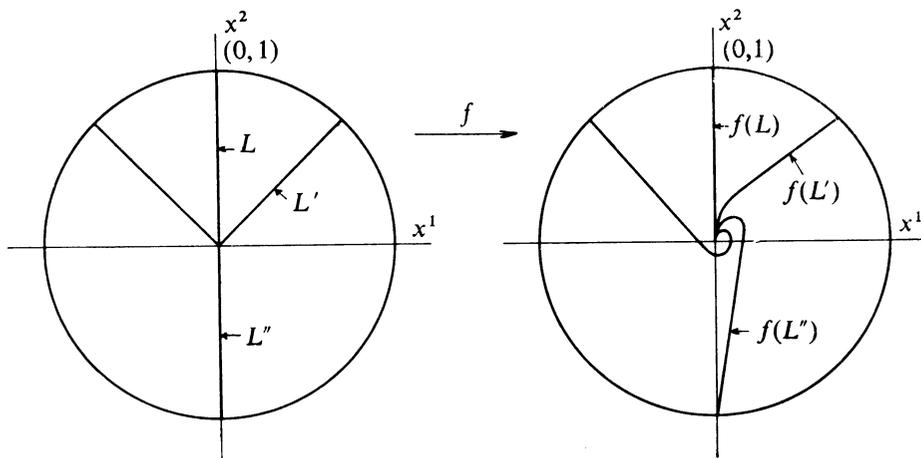


FIGURE 3

REMARKS. Again, one verifies that if $f: U \rightarrow E^n$ is a C^p -imbedding, ($p > 0$), then f has property Q at each point of its domain. Also, it is readily seen that the definition of "property Q " can be extended in the natural way (cf. Definition 1') to differentiable manifolds. One verifies that if M_1^n, M_2^n are nonbounded differentiable n -manifolds, and if $f: U \rightarrow M_2^n$ is a C^p -imbedding of U into M_2^n , where U is an open subset of M_1^n , then f has property Q at each point of U .

8. **A homeomorphism having a dense set of nonpiercing points.** We now construct a homeomorphism F of E^n onto itself having a dense set of nonpiercing points. For $c = (c^1, c^2, \dots, c^n)$, $x = (x^1, x^2, \dots, x^n)$, and $r > 0$, we define the homeomorphism $F_{c,r,i}$, $i = 2, \dots, n$, of E^n onto itself, as follows:

$$F_{c,r,i}(x) = x, \quad [x \in CJS(c,r) \cup JS(c,r/2)]$$

$$F_{c,r,i}(x) = ((x^1 - c^1)\cos\alpha(x) - (x^i - c^i)\sin\alpha(x) + c^1, x^2, \dots, x^{i-1},$$

$$(x^1 - c^1)\sin\alpha(x) + (x^i - c^i)\cos\alpha(x) + c^i, x^{i+1}, \dots, x^n)$$

$$[x \in JS(c,r) - JS(c,r/2)] \text{ where } \alpha(x) = 4\pi \left(\frac{r - \|x - c\|}{r} \right).$$

In the above formula, and throughout the following, CX denotes the complement in E^n of X . We then define the homeomorphism $F_{c,r}$ of E^n onto itself by setting $F_{c,r} = F_{c,r,2}F_{c,r,3} \dots F_{c,r,n}$. We then choose a constant $\delta > 0$ with the following property (β).

(β) If $S(x_1, r_1), \dots, S(x_m, r_m)$ are $(n - 1)$ -spheres in E^n such that $JS(x_i, r_i) \cap JS(x_j, r_j) = \emptyset, i \neq j, 1 \leq i, j \leq n$, and if $H = F_{x_n, r_n} F_{x_{n-1}, r_{n-1}} \dots F_{x_1, r_1}$ (note that H is independent of the order of the factors), then $d(H(x), H(y)) \geq \delta d(x, y)$ for all $x, y \in E^n$.

Such a δ clearly exists. We now define, inductively, a sequence of homeomorphisms $\{F_i\}$ of E^n into itself, and set $F = \lim_{i \rightarrow \infty} F_i$.

Let X be a countable dense subset in E^n of distinct points $x_i, i = 1, 2, \dots$. Set $F_0 = 1$. Select a positive constant $r_{11} < \frac{1}{2}$ and such that $S(x_1, r_{11}) \cap X = \emptyset$. Set $F_1 = F_{x_1, r_{11}} = F_{11}$. Now consider $F_1(x_2)$. By our choice of $S(x_1, r_{11})$, and since $F_1|_{S(x_1, r_{11})} = 1$, we have $F_1(x_2) \notin S(x_1, r_{11})$. We have two cases.

Case 1. $F_1(x_2) \in JS(x_1, r_{11})$. Then select positive constants r_{12}, r_{22} such that:

$$(2.1) \quad r_{12} < \frac{1}{2} r_{11},$$

$$(2.2) \quad \max(r_{12}, r_{22}) < \delta^2 / 2^2,$$

$$(2.3) \quad JS(F_1(x_2), r_{22}) \subset JS(x_1, r_{11}),$$

$$(2.4) \quad JS(F_1(x_1), r_{12}) \cap JS(F_1(x_2), r_{22}) = \emptyset,$$

$$(2.5) \quad \{S(F_1(x_1), r_{12}) \cup S(F_1(x_2), r_{22}) \cup S(F_1(x_1), r_{12}/2) \cup S(F_1(x_2), r_{22}/2)\} \\ \cap \{X \cup F_1(X)\} = \emptyset.$$

Then set $F_{12} = F_{F_1(x_1), r_{12}}, F_{22} = F_{F_1(x_2), r_{22}}$, and

$$(2.6) \quad F_2 = F_{22} F_{12} F_{11} = F_{22} F_{12} F_1.$$

Case 2. $F_1(x_2) \in CJS(x_1, r_{11})$. Then select positive constants r_{12}, r_{22} such that (2.1), (2.2), (2.4) and (2.5) hold, together with the following relation analogous to (2.3).

$$(2.3)' \quad JS(F_1(x_2), r_{22}) \subset CJS(x_1, r_{11}).$$

Then construct F_2 as in (2.6). Note that the following relations are satisfied.

$$(2.7) \quad F_2(x_1) = F_1(x_1) = x_1, \quad F_2(x_2) = F_1(x_2).$$

$$(2.8) \quad F_1(y) \in JS(F_1(x_i), r_{ij}) \Rightarrow F_2(y) \in JS(F_1(x_i), r_{ij}), \quad i = 1, 2; i \leq j \leq 2.$$

$$(2.9): \quad F_1(y) \in CJS(F_1(x_i), r_{ij}) \Rightarrow F_2(y) \in CJS(F_1(x_i), r_{ij}), \quad i = 1, 2; i \leq j \leq 2.$$

$$(2.10) \quad d(F_2(x), F_2(y)) = d(F_{22} F_{12} F_1(x), F_{22} F_{12} F_1(y)) \geq \delta d(F_1(x), F_1(y)) \\ \geq \delta^2 d(x, y) \quad [x, y \in E^n].$$

Suppose, inductively, that positive constants $r_{ij}, i = 1, 2, \dots, k - 1, i \leq j \leq k - 1$ have been chosen, together with homeomorphisms $F_0 = 1, F_1, \dots, F_{k-1}$ of E^n onto itself, such that the following conditions are satisfied. First, for $1 \leq m \leq k - 1, F_m = F_{mm} F_{m-1m} \dots F_{1m} F_{m-1}$, where $F_{ij} = F_{F_{j-1}(x_i), r_{ij}}, i = 1, 2, \dots, k - 1, i \leq j \leq k - 1$, and, moreover:

$$(k-1.1) \quad r_{ij} < \frac{1}{2}r_{ij-1} < \dots < \frac{1}{2}r_{ii},$$

$$(k-1.2) \quad \max(r_{ij}) < \delta^i/2^j,$$

$$(k-1.3) \quad JS(F_{j-1}(x_i), r_{ij}) \cap JS(F_{j-1}(x_m), r_{mj}) = \emptyset \quad [l \neq m, 1 \leq j \leq k-1, l, m \leq j],$$

$$(k-1.4) \quad \left\{ \bigcup_{i=1, \dots, k-1; i \leq j \leq k-1} S(F_{j-1}(x_i), r_{ij}) \cup S(F_{j-1}(x_i), r_{ij}/2) \right\} \\ \cap \{X \cup F_1(X) \cup \dots \cup F_{k-1}(X)\} = \emptyset,$$

$$(k-1.5) \quad F_{j-1}(x_j) \in JS(F_{j-1}(x_l), r_{lp}) \Rightarrow JS(F_{j-1}(x_j), r_{jj}) \subset JS(F_{j-1}(x_l), r_{lp}) \\ [j = 1, \dots, k-1; l \leq p \leq j-1],$$

$$(k-1.6) \quad F_{j-1}(x_j) \in CJS(F_{j-1}(x_l), r_{lp}) \Rightarrow JS(F_{j-1}(x_j), r_{jj}) \subset CJS(F_{j-1}(x_l), r_{lp}) \\ [j = 1, \dots, k-1; 1 \leq l \leq p \leq j-1].$$

Note by (k-1.4) that (k-1.5) and (k-1.6) cover the possible locations of $F_{j-1}(x_j)$. Note also that the following relations are necessarily satisfied:

$$(k-1.7) \quad F_j(x_l) = F_{l-1}(x_l) \quad [j = 1, \dots, k-1, 1 \leq l \leq j],$$

$$(k-1.8) \quad F_{j-1}(y) \in JS(F_{j-1}(x_l), r_{lp}) \Rightarrow F_q(y) \in JS(F_{j-1}(x_l), r_{lp}) \\ [j = 1, \dots, k-1, 1 \leq l \leq p \leq j-1 \leq q \leq k-1, y \in E^n],$$

$$(k-1.9) \quad F_{j-1}(y) \in CJS(F_{j-1}(x_l), r_{lp}) \Rightarrow F_q(y) \in CJS(F_{j-1}(x_l), r_{lp}) \\ [j = 1, \dots, k-1, 1 \leq l \leq p \leq j-1 \leq q \leq k-1, y \in E^n],$$

$$(k-10) \quad d(F_j(x), F_j(y)) \geq \delta^j d(x, y) \quad [x, y \in E^n, \quad j = 1, \dots, k-1],$$

$$(k-11) \quad d(F_j(x), F_{j-1}(x)) < \delta^j/2^j < 1/2^j \quad [j = 1, \dots, k-1].$$

Clearly, we may choose positive constants r_{ik} , $i = 1, \dots, k$, and define $F_k = F_{kk}F_{k-1k} \dots F_{1k}F_{k-1}$ so that the relations (k.1)–(k.11) analogous to (k-1.1)–(k-1.11) hold. Hereafter, we understand (m.i) to mean the relation in stage m of our construction analogous to (k-1.1) above. Set $F = \lim_{k \rightarrow \infty} F_k$. Then using (k.11), F is a continuous mapping of E^n onto itself. Hence to show that F is a homeomorphism, it suffices to show that F is biunique. To verify the biuniqueness of F , let x, y be distinct points of E^n . We have the following two cases.

Case 1. There exists a sequence $k_1 < k_2 < \dots$ such that, for example, $F_{k_i}(x) \neq F_{k_{i-1}}(x)$.

Then $F_{k_i}(x) \in JS(F_{k_{i-1}}(x_{l_i}), r_{l_i k_{i-1}})$ for some $l_i \leq k_{i-1}$. Using (k.5), (k.6) and (k.8), we see that $F(x) \in JS(F_{k_{i-1}}(x_{l_i}), r_{l_i k_{i-1}})$, $i = 1, 2, \dots$. Set $\zeta = d(x, y)$, and choose $p = k_j$ when j is so large that $1/2^{p-1} < \zeta/3$. Now $F(x) \in JS(F_{p-1}(x_{l_j}), r_{l_j p-1})$, and by (k.10), we have

$$(\xi) \quad d(F_{p-1}(x), F_{p-1}(y)) \geq \delta^{p-1} d(x, y) = \delta^{p-1} \zeta.$$

Now by (k.2), (β), and our choice of p , we have

$$(\eta) \quad r_{i,jp-1} < \delta^{p-1}/2^{p-1} < \delta^{p-1} \zeta/3 < d(F_{p-1}(x), F_{p-1}(y))/2.$$

Then $F_{p-1}(y) \in CJS(F_{p-1}(x_{i_j}), r_{i,jp-1})$, and using (k.9), $F(y) \notin JS(F_{p-1}(x_{i_j}), r_{i,jp-1})$. Hence $F(x) \neq F(y)$. A similar proof holds when there exists a sequence $m_1 < m_2 < \dots$ such that $F_{m_i}(y) \neq F_{m_{i-1}}(y)$.

Case 2. There exist integers $M(x), N(y)$ such that $F_l(x) = F_{M(x)}(x), l \geq M(x)$, and $F_p(y) = F_{N(y)}(y), p \geq N(y)$.

Then $F(y) = F_{N(y)}(y)$ and $F(x) = F_{M(x)}(x)$. Suppose $M(x) = N(y)$. Then since $F_{M(x)} = F_{N(y)}$ is a homeomorphism, we have $F(x) = F_{M(x)}(x) \neq F_{N(y)}(y) = F(y)$. Now if $M(x) < N(y)$, then $F(x) = F_{M(x)}(x) = F_{N(y)}(x)$, and since $F_{N(y)}$ is a homeomorphism, we have $F(x) = F_{N(y)}(x) \neq F_{N(y)}(y) = F(y)$. A similar proof holds when $N(y) < M(x)$.

This completes the proof of the biuniqueness of F , and therefore F is a homeomorphism of E^n onto itself.

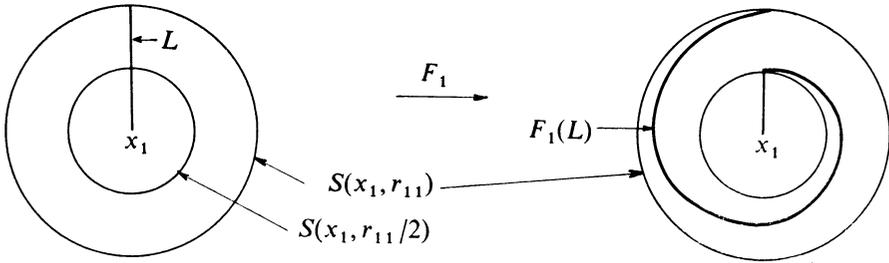
CLAIM. X consists of nonpiercing points of F . We show first that x_1 is a nonpiercing point of F . Figure 4 illustrates the situation when $n = 2$. The situation for general n is analogous.

Clearly, x_1 is a nonpiercing point of F , in fact, x_1 is a spiral point. Now consider x_j , where $j > 1$. Construct a new sequence of homeomorphisms $\{G_{i,j}\}$, $i = 1, 2, \dots$, where $G_{i,j}$ is obtained from F_i by deleting all factors in F_i which are of the form F_{kl} , where $k < j$ (with $G_{1,j} = G_{2,j} = \dots = G_{j-1,j} = 1$). Setting $G_j = \lim_{i \rightarrow \infty} G_{i,j}$, one verifies, in a manner analogous to that concerning F and x_1 , that G_j is a homeomorphism of E^n onto itself having $F_{j-1}(x_j)$ as a nonpiercing (spiral) point. Setting $V^j = F_{j-1}^{-1}(S(F_{j-1}(x_j), r_{jj}))$, we see that V^j is a neighborhood of x_j , and $F|V^j = G_j F_{j-1}|V$. Moreover, using (j.4) and the fact that $F_{c,r}|(E^n - \{S(c,r) \cup S(c,r/2)\})$ is a C^∞ -diffeomorphism, there exists an open neighborhood U^j of x_j in V^j such that $F_{j-1}|U^j$ is a C^∞ -diffeomorphism. It follows from Proposition 1 that x_j is a nonpiercing point of $G_j F_{j-1}$, and hence x_j is a nonpiercing point of F . This completes the proof of the claim.

REMARKS. Now $F|S(x_1, r_{11}) = 1$, and hence we see that F has property Q at every point of $S(x_1, r_{11})$. Also, F is stable, since it is equivalent under \sim to the stable homeomorphism $\hat{F} \in H(E^n)$ defined by $\hat{F}|CJS(x_1, r_{11}) = F|CJS(x_1, r_{11})$, and $\hat{F}|JS(x_1, r_{11}) = 1$. Now if T is any triangulation of E^n , we see that F could have been constructed to have a dense set of nonpiercing points, and, moreover, reduce to the identity on the $(n-1)$ -skeleton T^{n-1} of T . We merely take our countable dense set X , and all the spheres entering into our construction, to be disjoint from T^{n-1} . In particular, if P is an $(n-1)$ -hyperplane in E^n , then F could have been required to reduce to the identity on P (and hence have property Q at each point of P).

THEOREM 1. If M^n is any nonbounded differentiable n -manifold, then there exists a homeomorphism F of M^n onto itself having a dense set of nonpiercing points (cf. Definition 1').

Stage 1.



Stage 2.
(enlarged)

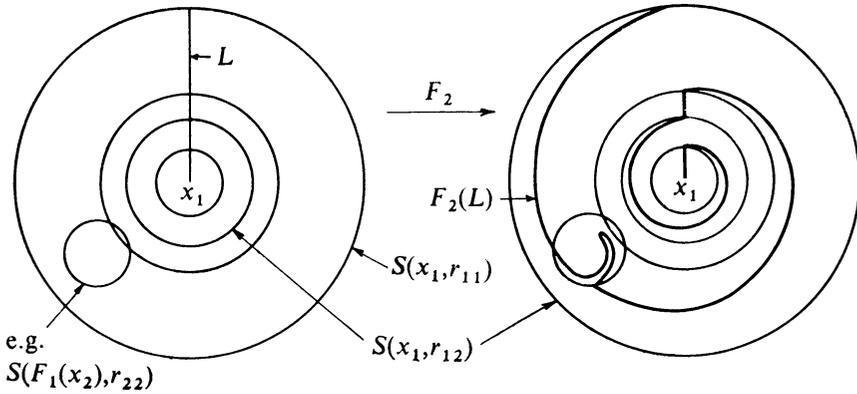


FIGURE 4

Proof. We take a countable covering (U_i, h_i) of M^n by coordinate systems. Let $f_1: h_1(U_1) \rightarrow h_1(U_1)$ be a homeomorphism of $h_1(U_1)$ onto itself having a dense set of nonpiercing points, and such that $d(x, f_1(x)) \rightarrow 0$ as x approaches the boundary of $h_1(U_1)$. Then let $F_1: M^n \rightarrow M^n$ be the homeomorphism of M^n onto itself defined by $F_1|U_1 = h_1^{-1}f_1h_1$, and $F_1|(M^n - U_1) = 1$. Note that there is a dense subset of U_1 consisting of nonpiercing points of F_1 . Let $f_2: h_2(U_2 - \bar{U}_1) \rightarrow h_2(U_2 - \bar{U}_1)$ be a homeomorphism of $h_2(U_2 - \bar{U}_1)$ onto itself having a dense set of nonpiercing points, and such that $d(x, f_2(x)) \rightarrow 0$ as x approaches the boundary of $h_2(U_2 - \bar{U}_1)$. Then let $F_2: M^n \rightarrow M^n$ be the homeomorphism of M^n onto itself defined by $F_2|U_1 = F_1|U_1$, $F_2|(U_2 - \bar{U}_1) = h_2^{-1}f_2h_2$, and $F_2|(M^n - (U_1 \cup U_2)) = 1$. Inductively, we construct homeomorphisms $F_i: M^n \rightarrow M^n$ of M^n onto itself such that $F_i|(U_1 \cup \dots \cup U_{i-1}) = F_{i-1}|(U_1 \cup \dots \cup U_{i-1})$, $F_i|(M^n - (U_1 \cup \dots \cup U_i)) = 1$, and there exists a dense subset of $U_1 \cup \dots \cup U_i$ consisting of nonpiercing points of F_i . Then set $F = \lim_{i \rightarrow \infty} F_i$. It is readily seen that F is a homeomorphism of M^n onto itself having a dense set (in M^n) of nonpiercing points.

9. Applications. We now prove a theorem which relates the notion of piercing point to stability of homeomorphisms.

THEOREM 2. *For any integer n , if every homeomorphism $H \in H(E^k)$, $2 \leq k \leq n$, is such that $H \sim G$, where G has property Q at some point, then all orientation preserving homeomorphisms of E^n onto itself are stable.*

We first prove the following lemma.

LEMMA 1. *Suppose, for some n , that all orientation-preserving homeomorphisms of E^{n-1} onto itself are stable. If $F \in H(E^n)$ is orientation-preserving, and if there exist $(n - 1)$ -hyperplanes P, P' in E^n , and an open set U in P' , such that $f(U) \subset P$, then F is stable.*

Proof of Lemma 1. Choose a point $x \in U$, and let $JS^{n-2}(x, r) = B$ be a closed $(n - 1)$ -ball in $P' \cap U$. Let M be an elementary topological $(n - 1)$ -sphere in E^n such that $M \cap P' = B$. Since we have assumed that all orientation-preserving homeomorphisms of E^{n-1} onto itself are stable, and since $F(B) \subset P$, we can modify $F|B$ (cf. Theorem 5.4 of [1]) to obtain a homeomorphism $\tilde{F}: B \rightarrow E^n$ such that $\tilde{F}|S^{n-2}(x, r) = F|S^{n-2}(x, r)$, $\tilde{F}(B) \subset P$, and $\tilde{F}|JS^{n-2}(x, s)$ is a C^p -imbedding ($p > 0$) for some $s < r$. Note that $\tilde{F}(B) = F(B)$ (cf. (α) of §2). We then obtain a homeomorphism $\hat{F}: M \rightarrow E^n$ by setting

$$\begin{aligned} \hat{F}(x) &= F(x), & [x \in M - JS^{n-2}(x, r)], \\ \hat{F}(x) &= \tilde{F}(x), & [x \in JS^{n-2}(x, r)]. \end{aligned}$$

Since $\hat{F}(M) = F(M)$, \hat{F} is elementary. Hence, using the Schoenflies extension theorem, \hat{F} admits extension to a homeomorphism of E^n onto itself, which we still denote by \hat{F} . Moreover, using the tubular neighborhood theorem, $\hat{F}|JS^{n-2}(x, s)$ may be extended over an open neighborhood of $JS^{n-2}(x, s)$ in E^n as a diffeomorphism. Hence, we can assume (cf. [9]) that $\hat{F}|V$ is a diffeomorphism for some open neighborhood V of x in E^n . Therefore \hat{F} is stable. Since

$$\hat{F}|(M - B) = F|(M - B),$$

we can assume (cf. [9]), moreover, that $\hat{F} \sim F$. Hence F is stable.

REMARK. One verifies that the hypotheses of Lemma 1 can be weakened by allowing P and P' to be diffeomorphs of E^{n-1} in E^n .

Proof of Theorem 2. Using induction, the above remark, and the fact that all orientation-preserving homeomorphisms of E^n onto itself are stable for $n = 1, 2, 3$, it suffices to prove the following proposition (γ).

(γ) *If the homeomorphism $G \in H(E^n)$ has property Q at some point, then $G \sim F$, where $F(U) \subset P_1$ for an open set U in P_2 , and P_1, P_2 are diffeomorphs of E^{n-1} in E^n .*

To verify (γ), suppose G has property Q at a point $x_0 \in E^n$. Hence there exist σ, H, P, W , and σ_x in Definition 3 relative to G and x_0 . To simplify our discussion, we note that for the purposes of verifying (γ), we can assume without loss of generality that L is the identity diffeomorphism of E . Indeed, GL has property Q

at $L^{-1}(x_0)$ using σ , identity, H, P, W , and it is clear that L can be taken as orientation-preserving, which implies that $GL \sim G$.

Now if V is a sufficiently small neighborhood of $HG(x_0)$ in W , the straight line segments $\sigma_x([-1, 1])$, as x varies throughout V , are mutually disjoint. To see this, note first that Definition 1 and condition (iii) of Definition 3 (with $L = 1$) imply that $\sigma_x([-1, 1]) \cap G^{-1}H^{-1}(W) = x$ for all $x \in W$. Since the segments $\sigma_x([-1, 1])$ are all translates of one another, we see that if $V \subset W$ is small enough, the $\sigma_x([-1, 1]) \cap \sigma_y([-1, 1]) \neq \emptyset$ for $x, y \in V$ implies that $x \in \sigma_y([-1, 1])$, and hence $x = y$. Actually, it can be proved that the segments $\sigma_x([-1, 1])$ are mutually disjoint for all $x \in W$, but we won't need this fact.

Let $B = JS^{n-2}(HG(x_0), r)$ be an $(n - 1)$ -ball in P such that $B \subset V$. Let P^* be the $(n - 1)$ -hyperplane in E^n going through $x_0 = \sigma(0)$ and such that the segment $\sigma([-1, 1])$ is normal to P^* at $\sigma(0)$. We also suppose r chosen so small that the $(n - 1)$ -hyperplanes P', P'' which are parallel to P^* and go through $\sigma(\frac{1}{2}), \sigma(-\frac{1}{2})$, respectively, have the following property: for each $x \in G^{-1}H^{-1}(B)$, the segments $\sigma_x([-1, 1])$ intersect P', P'' in (continuously varying) points $\sigma_x(t'_x), \sigma_x(t''_x)$, respectively, where $-1 < t''_x < 0 < t'_x < 1$. Note that $t''_{x_0} = -\frac{1}{2}, t'_{x_0} = \frac{1}{2}$. Hence the segments $\sigma_x([t''_x, t'_x])$, as x varies throughout $G^{-1}H^{-1}(B)$, "fiber" the neighborhood

$$N = \{ \bigcup \sigma_x([t''_x, t'_x]) \mid x \in G^{-1}H^{-1}(B) \}.$$

Let $B' = JS^{n-1}(HG(x_0), r/2)$, and let $B^* = \{ \bigcup \sigma_x([t''_x, t'_x]) \mid x \in G^{-1}H^{-1}(B') \}$. It is clear that

$$\text{BdB}^* = \{ \bigcup \sigma_x([t''_x, t'_x]) \mid x \in G^{-1}H^{-1}(S^{n-2}(HG(x_0), r/2)) \} \cup \{ B^* \cap (P' \cup P'') \}$$

is an elementary topological $(n - 1)$ -sphere in E^n . Setting $M = \text{BdB}^*$, we consider homeomorphism $\Phi: M \rightarrow E^n$ which maps the segment $\sigma_x([t''_x, t'_x])$ homeomorphically onto the segment $\sigma_x([t''_x, 0])$, and reduces to the identity $B^* \cap P''$. Then Φ is an elementary homeomorphism such that $\Phi(M) = M'$, where

$$M' = \{ \bigcup \sigma_x([t''_x, 0]) \mid x \in G^{-1}H^{-1}(S^{n-2}(HG(x_0), r/2)) \} \cup \{ B^* \cap P'' \}.$$

Set $F = G\Phi$. Then F is an elementary homeomorphism of M into E^n , and hence may be extended to a homeomorphism of E^n onto itself, which we still denote by F . Since $F|_{(B^* \cap P'')} = G|_{(B^* \cap P'')}$, we can assume (cf. [9]) that $F \sim G$. Moreover, setting $U = \{ \bigcup \sigma_x(t_x) \mid x \in G^{-1}H^{-1}(JS^{n-2}(HG(x_0), r/2)) \}$, we see that U is an open set in P' , and $F(U) = G\Phi(U) = G(G^{-1}H^{-1}(JS^{n-2}(HG(x_0), r/2))) = H^{-1}(JS^{n-2}(HG(x_0), r/2)) \subset H^{-1}(P)$. Then setting $P_1 = H^{-1}(P)$, $P_2 = P'$, we see that Proposition (γ) is verified, which completes the proof of Theorem 2.

We now show that any homeomorphism $F \in H(E^n)$ is equivalent under \sim to a homeomorphism having piercing points.

Since all orientation-preserving homeomorphisms of E^n onto itself are stable

for $n = 1, 2, 3$, it follows that if $F \in H(E^n)$, $n = 1, 2, 3$, then $F \sim H$ where H has an n -cell of piercing points. The following theorem extends this latter result to a similar (but weaker) result in the higher dimensions.

THEOREM 3. *Let $F \in H(E^n)$, where $n \geq 4$. Then $F \sim H$ where H has a k -cell of piercing points, $k \leq 2n/3 - 1$.*

Proof. Let K be any k -cell in E^n . Then $F(K)$ is a flat k -cell in E^n . Moreover, $F(K)$ is stably flat by a theorem of P. Roy (cf. [10]), and $F^{-1}|_{F(K)}: F(K) \rightarrow K$ admits an extension to a stable homeomorphism G of E^n onto itself. Since G is stable, we can assume that $G|_U = 1$, where U is some nonempty open set in E^n . Set $H = GF$. Then $H|_K = 1$, and hence K consists entirely of piercing points of H . Since $H|_{F^{-1}(U)} = F|_{F^{-1}(U)}$, we see that $F \sim H$ and the theorem is proved.

REMARK. Theorem 3 can be strengthened by requiring that $H|(E^n - V) = F|(E^n - V)$, where V is any nonempty open set in E^n . Hence Theorem 3 can be extended to an analogous result about differentiable manifolds.

We conclude by noting a relationship between our notion of piercing point and a notion of "piercing" which has been discussed in the literature. A topological $(n - 1)$ -sphere M in E^n is said to be pierced by a straight line segment yz at a point $x_0 \in M$ if $yz \cap M = x_0$, and if y and z lie in opposite components of $E^n - M$. An example was given by Fort (cf. [11]) of a wild sphere which can be pierced at each point by a straight line segment. On the other hand, our example of §8 shows that there exist *elementary* spheres M in E^n which can *not* be pierced, even by diffeomorphs of straight line segments, at a dense set of points of M . To verify this latter statement, observe first that if $G \in H(E^n)$, and if $M = G(S^{n-1}(c, r))$ can be pierced at a point x_0 by a diffeomorph of a straight line segment, then x_0 is a piercing point of G^{-1} . We now note that the homeomorphism $F \in H(E^n)$ constructed in §8 is such that $F(x)$ is a nonpiercing (spiral) point of F^{-1} , for all $x \in X$. Hence if we choose X so that $X \cap S(c, r)$ is dense in $S(c, r)$, then the elementary sphere $M = F(S(c, r))$ can *not* be pierced at the points of the dense subset $F(X \cap S(c, r))$ of M .

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