TOPOLOGICAL PROPERTIES OF THE HILBERT CUBE AND THE INFINITE PRODUCT OF OPEN INTERVALS

BY
R. D. ANDERSON

1. For each $i>0$, let I_i denote the closed interval $0 \leqq x \leqq 1$ and let ${}^\circ I_i$ denote the open interval $0<x<1$. Let $I^\infty = \prod_{i>0} I_i$ and ${}^\circ I^\infty = \prod_{i>0} {}^\circ I_i$. I^∞ is the Hilbert cube or parallelotope sometimes denoted by Q. ${}^\circ I^\infty$ is homeomorphic to the space sometimes called s, the countable infinite product of lines.

The principal theorems of this paper are found in §§5, 7, 8, and 9. In §5 it is shown as a special case of a somewhat more general theorem that for any countable set G of compact subsets of ${}^\circ I^\infty$, ${}^\circ I^\infty \backslash G^*$ is homeomorphic to ${}^\circ I^\infty$ (where G^* denotes the union of the elements of G)(1).

In §7, it is shown that a great many homeomorphisms of closed subsets of I^∞ into I^∞ can be extended to homeomorphisms of I^∞ onto itself. The conditions are in terms of the way in which the sets are coordinatewise imbedded in I^∞. A corollary is the known fact (Keller [6], Klee [7], and Fort [5]) that if X is a countable closed subset of I^∞, then every homeomorphism of X into I^∞ can be extended to a homeomorphism of I^∞ onto itself. In a further paper based on the results and methods of this paper, the author will give a topological characterization in terms of imbeddings of those closed subsets X of I^∞ for which homeomorphisms of X into $W_1 = \{p \mid p \in I^\infty$ and the first coordinate of p is zero$\}$ can be extended to homeomorphisms of X onto itself. In his recent dissertation, Raymond Wong has settled a question of Blankinship [4] by showing that there do exist two Cantor Sets in I^∞ such that no homeomorphism of one onto the other can be extended to a homeomorphism of I^∞ onto itself.

In §8 the results of §7 are used to give conditions under which the union of two Hilbert cubes can be seen to be homeomorphic to I^∞.

In §9 it is proved that many countable infinite products not obviously homeomorphic to ${}^\circ I^\infty$ are, in fact, homeomorphic to ${}^\circ I^\infty$. A theorem (Theorem 9.5) equivalent to the following is proved: "For $i=1, 2, \ldots$, let C_i be a closed n_i-cell

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(1) Klee [8] shows that an arbitrary normed linear space can lose a single compact set without changing its topological character. The author [1] has the same conclusion (not only for compact sets, but for certain noncompact sets which are appropriately imbedded) for a class of topological linear spaces which does not appear to include all normed spaces and does not include s but does include certain nonnormable spaces. In [2] Bessaga and Klee establish that the removal of a single point from any of a very general class of topological linear spaces including s does not change the topological character of the space.

$(0 < n_i < \infty)$ and let Y_i be a subset of C_i which contains the interior of C_i. Then the product of the Y_i's is homeomorphic with $°I^\infty$ if and only if each Y_i is a G_δ set and $Y_i \neq C_i$ for infinitely many i."

The arguments of this paper use purely set-theoretic topological methods.

A REMARK CONCERNING STABLE HOMEOMORPHISMS. It will be shown in a further paper that every homeomorphism of I^∞ onto itself is stable in the sense of Brown-Gluck, i.e., every homeomorphism is the finite product of homeomorphisms each the identity on some nonempty open set. However, it follows easily (as was pointed out to the author by Raymond Wong) that all the homeomorphisms of I^∞ onto itself used in this paper can be specified to be stable. Specifically, the results of §§3–7 on existence of homeomorphisms, can be strengthened by adding the condition that the homeomorphisms asserted to exist also be stable.

2. **Definitions, notation, and preliminary lemmas.** Let Z denote the set of positive integers. For $\alpha \subset Z$, let I_α denote $\prod_{j\in\alpha} I_j$ and let $°I_\alpha$ denote $\prod_{j\in\alpha} °I_j$. For $\alpha \subset \beta \subset Z$, let τ_α denote the projection of I^∞ (or of I_β where appropriate) onto I_α. For α the set whose only element is i, τ_α may be written as τ_i.

For α a nonnull proper subset of Z, α' will denote $Z \backslash \alpha$.

The collection $\{\alpha_i\}_{i>0}$ is said to be a *partition* of Z provided

(1) for each $i, j > 0$, $\alpha_i \cap \alpha_j = \varnothing$ if and only if $i \neq j$ and

(2) $\bigcup_{i>0} \alpha_i = Z$. A collection satisfying condition (1) is called a *subpartition.* A partition (or subpartition) is said to be *simple* if each element is finite.

For any infinite $\alpha \subset Z$ we consider $I_\alpha = \prod_{i\in\alpha} I_i$ to be endowed with metric ρ_α defined by

$$\rho_\alpha(x, y) = \sqrt{\sum_{i\in\alpha} \frac{1}{2^i}(x_i - y_i)^2}$$

where $x = \{x_i\}$ and $y = \{y_i\}$, $x_i, y_i \in I_i$. For $\alpha = Z$ we let ρ be ρ_α. As $°I_\alpha \subset I_\alpha$, the above metric is inherited by $°I_\alpha$. For finite α, we consider ρ_α to be the ordinary Euclidean metric with each $I_j, j \in \alpha$, being of length 1. The set $°I_\alpha$ is dense in I_α and is called the *pseudo-interior* of I_α. Let $B(I_\alpha) = I_\alpha \backslash °I_\alpha$ and let $B(I_\alpha)$ be called the *pseudo-boundary* of I_α. For α finite, $B(I_\alpha)$ is the boundary of the finite-dimensional cell I_α. For α infinite, $B(I_\alpha)$ is dense in I_α^∞.

REMARK. $B(I^\infty)$ and $°I^\infty$ are clearly not homeomorphic to each other. Note that both $B(I^\infty)$ and $°I^\infty$ are dense in I^∞ and $B(I^\infty)$ is an F_σ set. Since I^∞ is compact and hence of the second category, $°I^\infty$ cannot also be an F_σ set.

By a $\beta^*(I_\alpha)$-homeomorphism we shall mean a homeomorphism of I_α onto itself which carries $B(I_\alpha)$ *onto* $B(I_\alpha)$. By a $\beta(I_\alpha)$*-homeomorphism* we shall mean a homeomorphism of I_α onto itself carrying $B(I_\alpha)$ *into* $B(I_\alpha)$. For $\alpha = Z$, we simplify the notation to β^*- and β-homeomorphisms.

Let $G(I^\infty)$ denote the group of all homeomorphisms of I^∞ onto itself. For $K \subset I^\infty$ and $f \in G(I^\infty)$, let $f|K$ be the homeomorphism f restricted to the domain K. For $M \subset I^\infty$, f is said to be supported on M if $f|(I^\infty \backslash M)$ is the identity.

Many of the homeomorphisms with which we shall be concerned may be described in something like the following manner. Let $\alpha=\{1, 2\}$. Let h be a homeomorphism of I_α onto itself such that (1) $h|B(I_\alpha)$ is the identity, (2) for some $p \in I_\alpha$, $\tau_1(p) \neq \tau_1(h(p))$ and (3) for each $p \in I_\alpha$, $\tau_2(p)=\tau_2(h(p))$. Let I^∞ be regarded as $I_\alpha \times I_{\alpha'}$ and for $q \in I^\infty$, let $q=(\tau_\alpha(q), \tau_{\alpha'}(q))$. Then define $f(q)=(h(\tau_\alpha(q)), \tau_{\alpha'}(q))$. (Incidentally, f is a β^*-homeomorphism.) Associated with f are three subsets of Z which in the definition to follow we shall label $\alpha(f)$, $\beta(f)$, and $\gamma(f)$. In our example $\alpha(f)=\{1\}$, $\beta(f)=\alpha'$ and $\gamma(f)=\{2\}$. In effect, $\alpha(f)$ represents the set of directions in which f acts, $\beta(f)$ represents the set of directions which can be ignored or factored out when defining f and $\gamma(f)$ represents the other directions.

For $f \in G(I^\infty)$, let $\alpha(f)$ denote the set of all elements i of Z such that for some $p \in I^\infty$, $\tau_i(p) \neq \tau_i(f(p))$. Clearly $\alpha(f)=\varnothing$ if and only if f is the identity. For $j \in Z$, f is said to be independent of j provided that $j \notin \alpha(f)$ and for any $p, q \in I^\infty$ for which $\tau_{Z\setminus\{j\}}(p)=\tau_{Z\setminus\{j\}}(q)$, $\tau_{Z\setminus\{j\}}(f(p))=\tau_{Z\setminus\{j\}}(f(q))$. Let $\beta(f)$ denote the set of all $j \in Z$ for which f is independent of j. Let $\gamma(f)=Z\setminus(\alpha(f) \cup \beta(f))$.

For any $\alpha \subset Z$ and any $i \in \alpha$, let $W_i(I_\alpha)$ be the set of all points of I_α with i-coordinate 0 and let $^\circ W_i(I_\alpha)$ be the set of all points of $W_i(I_\alpha)$ whose other coordinates are properly between 0 and 1. For $\alpha=Z$ we let $W_i(I^\infty)$ and $^\circ W_i(I^\infty)$ be denoted by W_i and $^\circ W_i$ respectively.

Let, for each i, $f_i \in G(I^\infty)$. We shall be concerned with the infinite product (composition) of the $\{f_i\}_{i>0}$ in the form $\ldots\ldots f_3 f_2 f_1$ in cases where such composition is, in fact, a homeomorphism. To this end we give the following definition. The (*formal*) *left product* $L\prod_{i>0} f_i$ of $\{f_i\}$ is the transformation μ of I^∞ into the space C of closed subsets of I^∞ defined by $\mu(p)=\lim_{i>0}\sup\{f_i \cdots f_2 f_1(p)\}$. For a double sequence $\{f_{ij}\}_{i,j>0}$ we define $L\prod_{i,j>0} f_{ij}$ by ordering $\{f_{ij}\}$ as a simple sequence, $i, j<i', j'$ if $i+j<i'+j'$ or if $i+j=i'+j'$ and $i<i'$. We are interested in conditions under which (1) μ is a transformation of I^∞ into I^∞ as a subset of C, (2) μ is onto I^∞, (3) μ is continuous, and (4) μ is 1-1.

LEMMA 2.1. *If $\{\alpha(f_i)\}_{i>0}$ is a subpartition, then $\mu=L\prod_{i>0} f_i$ is a continuous transformation of I^∞ onto I^∞.*

Proof. Consider $p \in I^\infty$ and $k \in Z$. If $k \notin \bigcup_{i>0} \alpha(f_i)$ then $\tau_k(\mu(p))=\tau_k(p)$ and if, for some i, $k \in \alpha(f_i)$, then $\tau_k(\mu(p))=\tau_k(f_i(p))$. Hence μ carries I^∞ into I^∞. Also since, for each i, f_i is continuous and τ_k is continuous then μ is continuous in each coordinate and hence is continuous. Finally we verify that μ carries I^∞ onto I^∞. We suppose it does not. Since $\mu(I^\infty)$ is closed there is some j such that for $\alpha=\{1, 2, \ldots, j\}$, $\tau_\alpha(\mu(I^\infty)) \neq I_\alpha$. But there is a k such that for $k'>k$, $\alpha \cap \alpha(f_{k'})=\varnothing$. Hence $\tau_\alpha(\mu(I^\infty))=\tau_\alpha(f_k \cdots f_2 \cdot f_1(I^\infty))=I_\alpha$, a contradiction.

REMARK. Under the hypotheses of Lemma 2.1 μ need not be 1-1. For example, if $\alpha(f_i)=i$ and $\gamma(f_i)=i+1$, for each i, then for some $p \neq q$ we can define $\{f_i\}$ so as successively to make $\tau_k(\mu(p))=\tau_k(\mu(q))$ for all k.

There are many conditions other than those of Lemma 2.1 under which μ is a mapping and even a homeomorphism. In the next lemma, we observe some other conditions for μ to be a mapping, such conditions being designed to fit applications in §§5 and 6.

LEMMA 2.2. *Suppose that for each $i, f_i \in G(I^\infty)$ and for each $i, j > 0$, $\alpha(f_i) \cap \alpha(f_j)$ is the single element $t \in Z$. And suppose that $\{\tau_t(f_n \cdots f_1(p))\}_{n>0}$ converges for each $p \in I^\infty$ and the mapping thus defined is continuous from I^∞ onto I_t. Then $\mu = L\prod_{i>0} f_i$ is a continuous transformation of I^∞ onto I^∞.*

Proof. The proof of this lemma is like that for Lemma 2.1 except that the properties of μ with respect to the tth direction are implied by the special hypotheses.

LEMMA 2.3. *If $\{\alpha(f_i)\}_{i>0}$ is a subpartition and if*

$$\Big(\bigcup_{i>0} \gamma(f_i)\Big) \cap \Big(\bigcup_{i>0} \alpha(f_i)\Big) = \varnothing,$$

then $\mu = L\prod_{i>0} f_i$ is a homeomorphism onto I^∞.

Proof. Using Lemma 2.1, it suffices to show that $\mu = 1\text{-}1$. Suppose $p \neq q$ and let k be an element of Z such that $\tau_k(p) \neq \tau_k(q)$. Then if

$$k \notin \bigcup_{i>0} \alpha(f_i), \quad \tau_k(\mu(p)) = \tau_k(p) \neq \tau_k(q) = \tau_k(\mu(q)).$$

If for every $k \notin \bigcup_{i>0} \alpha(f_i)$, $\tau_k(p) = \tau_k(q)$ then for some i and some j, $i \in \alpha(f_j)$ and $\tau_i(p) \neq \tau_i(q)$. Since $f_j \in G(I^\infty)$, and $\gamma(f_j) \subset Z \backslash \bigcup_{i>0} \alpha(f_i)$, for some $i' \in \alpha(f_j)$,

$$\tau_{i'}(\mu(p)) = \tau_{i'}(f_j(p)) \neq \tau_{i'}(f_j(q)) = \tau_{i'}(\mu(q))$$

as was to be shown.

In the following lemma we give an easy metric condition on $\{f_i\}_{i>0}$, using uniform continuity of $(f_j \cdots f_2 \cdot f_1)^{-1}$ so that $L\prod_{i>0} f_i$ must be a homeomorphism.

LEMMA 2.4. *Let $\{f_i\}_{i>0}$ be a collection of elements of $G(I^\infty)$ such that $\{\alpha(f_i)\}_{i>0}$ is a subpartition. Let $\{\varepsilon_i\}_{i>0}$ be a sequence of positive numbers with, for each i, $\varepsilon_{i+1} < \frac{1}{2}\varepsilon_i$. For each i, let δ_i be a positive number such that $\rho(p, q) < \varepsilon_i/2$ whenever $p, q \in I^\infty$ with $\rho(f_i \cdots f_1(p), f_i \cdots f_1(q)) < \delta_i$. Suppose, for each i, the distance between f_{i+1} and the identity is less than* $\min(\delta_i, \delta_{i-1}/2, \delta_{i-2}/2^2, \ldots, \delta_1/2^{i-1})$. *Then $\mu = L\prod_{i>0} f_i$ is a homeomorphism.*

Proof. By Lemma 2.1, μ is a mapping. By the conditions of the theorem, if two points are at a distance $> \varepsilon_n$ from each other, then $f_n \cdots f_2 f_1$ keeps the points separated by a distance of δ_n and the introduction of the additional factors cannot bring them together.

We next introduce four lemmas giving conditions under which homeomorphisms can be asserted to be β or β^*.

LEMMA 2.5. For $f \in G(I^\infty)$ *with $\alpha(f)$ finite, f is β^*.*

Proof. Let $x \in I^\infty$ be expressed as (p, q) where $p \in I_{\alpha(f)}$ and $q \in I_{\beta(f)\cup\gamma(f)}$. For each $q \in I_{\beta(f)\cup\gamma(f)}$, $f[(p, q)] = (p', q)$ and since $I_{\alpha(f)}$ is a finite dimensional cube, p' is on the boundary of $I_{\alpha(f)}$ if and only if p is on the boundary of $I_{\alpha(f)}$. But then $f[(p, q)] \in B(I^\infty)$ if and only if $(p, q) \in B(I^\infty)$ and thus f is β^*.

LEMMA 2.6. *Any finite product of β (or β^*) homeomorphisms is β (or β^*).*

Proof. Obvious.

LEMMA 2.7. *If, for each $i>0$, $h_i \in G(I^\infty)$, if $h = L\prod_{i>0} h_i \in G(I^\infty)$, if $\{\alpha(h_i)\}_{i>0}$ is a simple subpartition, and if, for each $i>0$, h_i is β (or β^*), then h is β (or β^*).*

Proof. Suppose $p \in B(I^\infty)$. Then for some j, $\tau_j(p) = 0, 1$. If $j \notin \bigcup_{i>0} \alpha(h_i)$, then $\tau_j(h(p)) = \tau_j(p)$. If $j \in \alpha(h_i)$ then since h_i is β (or β^*) and $I_\alpha(h_i)$ is a finite cube, there exists $k \in \alpha(h_i)$ such that $\tau_k(h_i(p)) = 0, 1$ and hence $\tau_k(h(p)) = \tau_k(h_i(p)) = 0, 1$. Therefore h is β.

Suppose each h_i is β^*. Then, as above, h is β. Let $q \in {}^\circ I^\infty$. For each $j \notin \bigcup_{i>0} \alpha(h_i)$, $\tau_j(h(q)) = \tau_j(q)$ and hence cannot be 0 or 1. For each $j \in \alpha(h_i)$, $\tau_j(h_i(q)) = \tau_j(h(q))$ and $\tau_j(h_i(q)) \neq 0, 1$ since h_i is β^*. Therefore h is β^*.

LEMMA 2.8. *If, for each $i>0$, $h_i \in G(I^\infty)$, if $h = L\prod_{i>0} h_i \in G(I^\infty)$, if $\{\alpha(h_i)\}_{i>0}$ is a subpartition, if $\bigcup_{i>0} \alpha(h_i) \cap \bigcup_{i>0} \gamma(h_i) = \varnothing$, and if for each $i>0$, h_i is β (or β^*), then h is β (or β^*).*

Proof. Suppose $p \in B(I^\infty)$. Then, for some j, $\tau_j(p) = 0, 1$. If $j \notin \bigcup_{i>0} \alpha(h_i)$, then $\tau_j(h(p)) = \tau_j(p)$. If $j \in \alpha(h_i)$, then since h_i is β (or β^*), either (1) there exists a $j' \in \gamma(h_i)$ for which $\tau_{j'}(p) = 0, 1$ or (2) there exists a $k \in \alpha(h_i)$ such that $\tau_k(h_i(p)) = 0, 1$ and hence $\tau_k(h(p)) = \tau_k(h_i(p)) = 0, 1$. Therefore h is β. The argument that h is β^* if each h_i is β^* is like that of Lemma 2.7.

LEMMA 2.9. *Let $\{\alpha_i\}_{i>0}$ be a collection of subsets of Z such that for each i, α_i is infinite. Let, for each $i>0$, $\{\alpha_{ij}\}_{j>0}$ be a simple subpartition the union of whose elements is contained in α_i. Then there exists a subpartition $\{\beta_i\}_{i>0}$ such that* (1) *for each $i>0$, $\beta_i \subset \alpha_i$ and* (2) *for each $i>0$, β_i is the union of infinitely many sets α_{ij}.*

Proof. This lemma is a standard set-theoretic proposition. It may be proved by an inductive constructive argument. Let $\{\gamma_i\}$ be any partition of Z with, for each i, γ_i infinite. For each k, let b_k be inductively selected so that, if $k \in \gamma_i$, then b_k is the α_{ij} with least index j for which $b_k \cap b_s = \varnothing$ for each $s<k$. Let β_i be the union of all elements b_k for which $b_k = \alpha_{ij}$ for some j and for which $k \in \gamma_i$.

3. Straightening weakly thin compact sets.

DEFINITIONS. A set $K \subset I^\infty$ is said to be *weakly thin* provided that (1) K is closed and (2) there exists a simple subpartition $\{\alpha_i\}_{i>0}$ such that, for each i, $\tau_{\alpha_i}(K) \neq I_{\alpha_i}$. A set $K \subset I^\infty$ is said to be *thin* provided that (1) K is closed and (2) there exists a simple subpartition $\{\alpha_i\}_{i>0}$ such that, for each i, α_i consists of a single integer and $\tau_{\alpha_i}(K) \subset {}^\circ I_{\alpha_i}$. We say that K is *weakly thin* (or *thin* as appropriate) *with respect to* $\{\alpha_i\}$.

REMARK. Examples of closed sets which are not weakly thin are closed sets which contain nonempty open sets or closed sets whose complements in I^∞ are not homotopically trivial. For finite α, let $p \in {}^\circ I_\alpha$. Then $\tau_\alpha^{-1}(p)$ is not weakly thin nor is its complement homotopically trivial.

LEMMA 3.1. *Let α be a subset of Z. Let $\{K_i\}_{i>0}$ be a collection of closed sets each weakly thin with respect to a subpartition whose elements are disjoint from α. There exists a β^*-homeomorphism g such that $\alpha(g) \subset \alpha'$ and, for each $i>0$, $g(K_i)$ is thin with respect to a subset of α'.*

Proof. By Lemma 2.9 there exists a subpartition $\{\beta_i\}_{i>0}$ such that $\bigcup_{i>0} \beta_i \cap \alpha = \varnothing$ and for each i, there exists a simple subpartition $\{\beta_{ij}\}_{j>0}$ with respect to which K_i is weakly thin and with for each j, $\beta_{ij} \subset \beta_i$. If for two disjoint finite sets $\gamma_1, \gamma_2 \subset Z$, and for any $i>0$, $\tau_{\gamma_1}(K_i) \neq I_{\gamma_1}$ and $\tau_{\gamma_2}(K_i) \neq I_{\gamma_2}$, then $\tau_{\gamma_1 \cup \gamma_2}(K_i) \neq I_{\gamma_1 \cup \gamma_2}$ and $\tau_{\gamma_1 \cup \gamma_2}(K_i) \not\supset B(I_{\gamma_1 \cup \gamma_2})$. Thus for each i and each j we may let $\gamma_{ij} = \beta_{i\,2j-1} \cup \beta_{i\,2j}$ and, for each i, $\{\gamma_{ij}\}_{j>0}$ is a simple subpartition with respect to which K_i is weakly thin, with for each j, $\gamma_{ij} \subset \beta_i$ and with $\tau_{\gamma_{ij}}(K_i) \not\supset B(I_{\gamma_{ij}})$. For each i and each j, let g_{ij} be a β^*-homeomorphism such that $\alpha(g_{ij}) = \gamma_{ij}$, $\gamma(g_{ij}) = \varnothing$, and for some $n_j \in \gamma_{ij}$, $\tau_{n_j}(g_{ij}(K_i)) \subset {}^\circ I_{n_j}$. By Lemma 2.3, $L \prod_{i,j>0} g_{ij}$ is an element of $G(I^\infty)$ and by Lemmas 2.5 and either 2.7 or 2.8, $g = L \prod_{i,j>0} g_{ij}$ is β^*. By construction, for each i, since $G(K_i)$ is compact, $g(K_i)$ is thin and the lemma is proved.

LEMMA 3.2. *Let $i \in Z$, let t be a number, $0<t<1$, and let $\alpha \subset Z$ with $i \notin \alpha$. Let K be a closed subset of I^∞ such that (1) $0 < \tau_i(K) < 1$ and (2) for $p, q \in K$ with $p \neq q$, $\tau_\alpha(p) \neq \tau_\alpha(q)$. Then there exists a β^*-homeomorphism f such that $\alpha(f)$ is the set whose only element is i, $\gamma(f) \subset \alpha$ and $\tau_i(f(K)) = t \in I_i$.*

Proof. This lemma is a simple consequence of the Tietze Extension Theorem and is similar to lemmas in the literature. Let K' be the set of points of $\tau_i^{-1}(t)$ for which $\tau_\alpha(K') = \tau_\alpha(K)$. Let g be the map of K' into ${}^\circ I_i$ defined as follows: for $p \in K'$, let q be the point of K for which $\tau_\alpha(p) = \tau_\alpha(q)$; then $g(p) = \tau_i(q)$. Since K is compact, $\tau_i(K)$ is compact and there exist t_1, t_2 such that $0<t_1<t<t_2<1$ and such that $\tau_i(K) \subset [t_1, t_2] \subset I_i$. Let g^* be the map from $\tau_{\alpha\cup\{i\}}(K')$ onto $\tau_{\alpha\cup\{i\}}(K)$ induced by g; for $p \in K'$, $g^*(\tau_{\alpha\cup\{i\}}(p)) = \tau_{\alpha\cup\{i\}} g(p)$.

Let g^* be extended to a map g' of $\tau_\alpha \tau_i^{-1}(t)$ into $[t_1, t_2]$. Then for each $p \in \tau_\alpha \tau_i^{-1}(t)$ let the interval $[0, g'(p)]$ be mapped linearly onto $[0, p]$ and let $[g'(p), 1]$ be mapped linearly onto $[p, 1]$, all intervals being orthogonal to $\tau_\alpha \tau_i^{-1}(t)$. The resultant map $\tilde{f}$ of $I_{\alpha\cup\{i\}}$ into $I_{\alpha\cup\{i\}}$ is a homeomorphism. Let $I^\infty = I_{\alpha\cup\{i\}} \times U_{(\alpha\cup\{i\})'}$ and for $p = (p_1, p_2)$ coordinates of these factors let $f(p) = (\tilde{f}(p_1), p_2)$. Then f is the desired homeomorphism as is easy to verify.

A closed set $K \subset I^\infty$ is said to be *straight* with respect to the set $\{n_i\}_{i>0}$ of integers provided that, for any i, $\tau_{n_i}(K)$ is a single point of ${}^\circ I_{n_i}$.

LEMMA 3.3. *Let K be a thin set and let $\{n_j\}_{j>0}$ be a set of integers such that, for each j, $\tau_{n_j}(K) \subset {}^\circ I_{n_j}$. For any infinite subset $\{m_s\}_{s>0}$ of $\{n_j\}_{j>0}$ there exists a*

β^*-homeomorphism f such that $f(K)$ is straight with respect to some infinite subset of $\{m_s\}$ and $\alpha(f)\subset\{m_s\}$.

Proof. If, for infinitely many s, $\tau_{m_s}(K)$ is a single point, f can be taken as the identity. Alternatively, let $\{m_{s_r}\}_{r>0}$ be the infinite subset of $\{m_s\}$ for which $\tau_{m_{s_r}}(K)$ is nondegenerate. Let $\{\alpha_i\}_{i>0}$ be a subpartition with $\bigcup_{i>0}\alpha_i=\{m_{s_r}\}_{r>0}$ and with each α_i infinite. We shall select f so that $f(K)$ is straight with respect to α_1 and so that $\alpha(f)=\{m_{s_r}\}_{r>0}$.

Let $\alpha_1=\{t_i\}$ and, for each i, let $\alpha_{i+1}=\{v_{ij}\}_{j>0}$. Let $\delta_{ij}=\{t_i, v_{ij}\}$. Then $I_{\delta_{ij}}$ is a square and since K is compact, $\tau_{\delta_{ij}}(K)$ is a compact subset of ${}^\circ I_{\delta_{ij}}$. Let σ_{ij} be a homeomorphism of $I_{\delta_{ij}}$ onto itself such that (1) σ_{ij} is the identity on $B(I_{\delta_{ij}})$, (2) for each $p\in I_{\delta_{ij}}$, $\tau_{t_i}(p)=\tau_{t_i}(\sigma_{ij}(p))$, and (3) for $p,q\in\tau_{\delta_{ij}}(K)$ with $|\tau_{t_i}(p)-\tau_{t_i}(q)|>1/2^i$, $|\tau_{v_{ij}}(\sigma_{ij}(p))-\tau_{v_{ij}}(\sigma_{ij}(q))|\neq 0$. We may simply distort ${}^\circ I_{\delta_{ij}}$ in the v_{ij} direction to produce such a σ_{ij}. Letting, for each $i,j>0$, $I^\infty=I_{\delta_{ij}}\times I_{\delta'_{ij}}$ and for $q\in I^\infty$ letting $q=(\tau_{\delta_{ij}}(q),\ \tau_{\delta'_{ij}}(q))$, we define $\phi_{ij}(q)=(\sigma_{ij}(\tau_{\delta_{ij}}(q)),\ \tau_{\delta'_{ij}}(q))$. Then ϕ_{ij} is a β^*-homeomorphism with $\alpha(\phi_{ij})$ the set consisting of v_{ij} itself and with $\gamma(\phi_{ij})$ the set consisting of t_i itself.

By Lemmas 2.3, 2.5 and 2.7, $\phi=L\prod_{i,j>0}\phi_{ij}$ is a β^*-homeomorphism with $\alpha(\phi)=\bigcup_{i>0}\alpha_{i+1}$ and $\gamma(\phi)=\alpha_1$. Also, for any $p,q\in K$ and any $i>0$, if $\tau_{t_i}(\phi(p))\neq\tau_{t_i}(\phi(q))$, then for some $j\in\alpha_{i+1}$, $\tau_j(\phi(p))\neq\tau_j(\phi(q))$.

Now by Lemma 3.2, for each $i>0$, there exists a β^*-homeomorphism f_i with $\alpha(f_i)$ the set consisting of t_i itself, $\gamma(f_i)\subset\alpha_{i+1}$, and $\tau_{t_i}(f_i\phi(K))=\frac{1}{2}\in I_{t_i}$. By Lemmas 2.3 and 2.7, we may let $f^*=L\prod_{i>0}f_i$ and $f=f^*\phi$ is the desired homeomorphism.

Lemma 3.4. *Let α be a subset of Z. Let $\{K_i\}_{i>0}$ be any collection of closed sets each thin with respect to a subset of α'. Then there exists a β^*-homeomorphism h such that $\alpha(h)\subset\alpha'$ and for each i, $h(K_i)$ is straight with respect to some infinite subset of α'.*

Proof. By Lemma 2.9, we may let $\{\beta_i\}_{i>0}$ be a subpartition such that, for each i, $\beta_i\cap\alpha=\varnothing$, β_i is infinite, and K_i is thin with respect to the collection of all single element subsets of β_i. Let, for each i, h_i be a β^*-homeomorphism as in Lemma 3.3 with $\alpha(h_i)\cup\gamma(h_i)\subset\beta_i$. Then by Lemmas 2.3 and 2.8 $L\prod_{i>0}h_i$ is the desired homeomorphism h.

Theorem 3.5. *Let α be a subset of Z. Let $\{K_i\}_{i>0}$ be any collection of closed sets such that each is weakly thin with respect to a subpartition whose elements are disjoint from α. Then there exists a β^*-homeomorphism h such that $\alpha(h)\subset\alpha'$ and, for each i, $h(K_i)$ is straight with respect to some infinite subset of α'.*

Proof. Theorem 3.5 is an immediate consequence of Lemmas 3.1 and 3.4.

4. **Extending homeomorphisms—Klee's method.** The proof of the following two theorems is by a method due to Klee [7]. The setting is a bit different from Klee's.

THEOREM 4.1. *Let α be a subset of Z such that α and α' are nonnull. Let K and K' be closed sets in $^\circ I^\infty$ such that, for each $i \in \alpha$, $\tau_i(K')$ is a single point and, for each $j \in \alpha'$, $\tau_j(K)$ is a single point. Then any homeomorphism f of K onto K' can be extended to a β^*-homeomorphism F.*

Proof. We consider I^∞ as $I_\alpha \times I_{\alpha'}$ and let K_f be the graph of f, i.e.,

$$K_f = \{(\tau_\alpha(p), \tau_{\alpha'}(f(p))) \mid p \in K\}.$$

Let g be the homeomorphism of K onto K_f defined by $g(p) = (\tau_\alpha(p), \tau_{\alpha'}(f(p)))$ for $p \in K$ and let g' be the homeomorphism of K' onto K_f defined by

$$g'(p) = (\tau_\alpha(f^{-1}(p)), \tau_{\alpha'}(p))$$

for $p \in K'$.

We shall define a β^*-homeomorphism h such that $h|K = g|K$ and another β^*-homeomorphism h' such that $h'|K' = g'|K'$. Then $(h')^{-1}h$ is the desired f.

It suffices simply to define h since h' may be defined analogously. For each $i \in \alpha'$, Lemma 3.2 asserts the existence of a β^*-homeomorphism h_i with $\alpha(h_i)$ the set whose only element is i and with $\gamma(h_i) \subset \alpha$ such that for $p \in K$, $h_i(p)$ is the point whose ith coordinate is $\tau_i(g(p))$.

Let $h = L \prod_{i>0} h_i$ and by Lemmas 2.3 and 2.7, h is the desired homeomorphism.

THEOREM 4.2. *Let K and K' be subsets of $^\circ I^\infty$ closed in I^∞ and let f be any homeomorphism of K onto K'. Then f can be extended to a β^*-homeomorphism.*

Proof. Since K and K' are closed subsets of I^∞ they are compact and as they are subsets of $^\circ I^\infty$ they are (weakly) thin. By Theorem 3.5 there is a β^*-homeomorphism h such that $h(K)$ and $h(K')$ are each straight with respect to some infinite set of integers. By Lemma 2.9 there exist two disjoint infinite sets α, $\beta \in Z$ such that K is straight with respect to α and K' with respect to β. Let g be a β^*-homeomorphism with I_β mapped onto $I_{\alpha'}$ and $I_{\beta'}$ mapped onto I_α. Then gf, i.e., f followed by $g|K'$, is a homeomorphism of K onto $g(K')$. Since K is straight with respect to α and $g(K')$ is straight with respect to α', we may invoke Theorem 4.1 to assert the existence of a β^*-homeomorphism θ which extends gf. Then $g^{-1}\theta|K = f|K$ and as g^{-1} and θ are both β^*-homeomorphisms, $g^{-1}\theta$ is the desired extension.

Let β and β' be disjoint closed nonnull sets whose union is the set of all positive integers greater than 1.

The following two propositions are immediate consequences of Theorems 4.1 and 4.2 respectively, obtained by identifying $\prod_{i>1} I_i$ coordinatewise with I^∞.

COROLLARY 4.3. *Let K and K' be closed sets in $^\circ W_1$ such that, for each $i \in \beta$, $\tau_i(K')$ is a single point and, for each $j \in \beta'$, $\tau_j(K)$ is a single point. Then any homeomorphism f of K onto K' can be extended to a β^*-homeomorphism F such that $1 \in \beta(F)$.*

THEOREM 4.4. *Let K and K' be closed sets in $^\circ W_1$. Then any homeomorphism f of K onto K' can be extended to a β^*-homeomorphism F such that $1 \in \beta(F)$.*

5. **Pushing weakly thin sets to** $B(I^\infty)$. In this section we begin by proving the key lemma establishing a certain β-homeomorphism.

LEMMA 5.1. *Let* $\alpha=\{n_i\}_{i>0}$ *be any infinite set of positive integers with, for each* i, $n_i<n_{i+1}$ *and with* α' *being nonnull. Let* $K\subset I^\infty$ *be any closed set which is straight with respect to* α. *There exists a* β*-homeomorphism* h *such that* (1) $\alpha(h)\subset\alpha$, (2) *for* $p\in {}^\circ I^\infty$, $h(p)$ *is an element of* $B(I^\infty)$ *if and only if* $p\in K$, (3) *for* $p\in K$, $\tau_{n_1}(h(p))=0$ *and for* $k\neq n_1$, $\tau_k(p)=\tau_k(h(p))$ *and* (4) *for any* $p\in B(I^\infty)$ *and any* i *for which* $\tau_i(p)=0, 1$, *there is a* $k\leqq i$ *for which* $\tau_k(h(p))=0, 1$.

Proof. We shall exhibit h as an infinite left product of finitely elementary homeomorphisms h_i where, for each i, $\alpha(h_i)$ consists of n_1 and n_{i+1} and $\gamma(h_i)$ is null. Each h_i is to move K (or really $h_{i-1}h_{i-2}\cdots h_1(K)$) a step closer to $W_{n_1}=\tau_{n_1}^{-1}(0)$. Without loss of generality we may assume that, for each i, $\tau_{n_i}(K)$ is the single point $1/2^i\in I_{n_i}$ (for we could have earlier recoordinatized each factor of I^∞ to achieve this). Let ω be the point of I_α whose n_ith coordinate is $1/2^i$.

Let q_1, q_2, q_3, q_4, and q_5 denote the points of $C_i=I_{n_1}\times I_{n_{i+1}}$ whose coordinates are $(0, 0)$, $(0, 1/2^{i+1})$, $(1/2^i, 1/2^{i+1})$, $(1/2^i, 0)$, and $((1/2^{i+1})\cdot(1/2^{i+1}))$ respectively. Let M_i denote the closed rectangular region in C_i whose vertices are q_1, q_2, q_3, and q_4. To define h_i, for $i>1$, we use four sets:

S_i is the infinite product $I_{n_{i+2}}\times I_{n_{i+3}}\times\cdots$;

T_i is a neighborhood of $(1/2^2, 1/2^3,\ldots, 1/2^i)$ in $I_{n_2}\times I_{n_3}\times\cdots\times I_{n_i}$;

U_i is a neighborhood of M_i in C_i; and

V_i is a neighborhood of $\tau_{\alpha'}(K)$ in $\tau_{\alpha'}(I^\infty)$.

In the case of h_1, as T_1 is vacuous, we use only S_1, U_1, and V_1.

The homeomorphism h_i is to be supported on $S_i\times T_i\times U_i\times V_i$ (or on $S_1\times U_1\times V_1$ in the case of h_1). The sizes of the neighborhoods are to be selected (as specified later) in terms of $h_{i-1}\cdot\cdots\cdot h_1$ so that the infinite left product h of the $\{h_i\}$ will be defined and will move exactly K to W_{n_1}. In fact h will move K onto the projection of K in W_{n_1} and will move exactly $K\cap {}^\circ I^\infty$ from ${}^\circ I^\infty$. Also for any $p\notin K$, $h(p)$ will be $h_nh_{n-1}\cdots h_1(p)$ for all sufficiently large n and for any p not in the projection of K on W_{n_1}, $h^{-1}(p)$ will be $(h_nh_{n-1}\cdots h_1)^{-1}(p)$ for all sufficiently large n.

Let λ_i be a homeomorphism of C_i onto itself such that λ_i is supported on U_i, is isotopic to the identity with support on U_i, carries the interval $[q_3, q_4]$ onto the interval $[q_5, q_2]$ and, for each $p\in C_i$, $\tau_{n_1}\lambda_i(p)\leqq\tau_{n_1}(p)$. Let Λ_i^t be an isotopy of λ_i to the identity e with $\Lambda_i^0=\lambda_i$ and $\Lambda_i^1=e$, with, for each t, Λ_i^t supported on U_i, and with, for each t and each $p\in C_i$, $\tau_{n_1}\Lambda_i^t(p)\leqq\tau_{n_1}(p)$. This last condition helps imply condition (4) of the lemma.

Let $Y_i=\overline{V}_i\times\overline{T}_i$ and, for $y\in Y_i$ let $y=(y_V, y_T)$ with $y_V\in\overline{V}_i$ and $y_T\in\overline{T}_i$. By the Urysohn Lemma we may let ϕ_i be a continuous map of Y_i onto $[0, 1]$ such that (1) $\phi_i(y_V, y_T)=0$ if and only if $y_V\in\tau_{\alpha'}(K)$ and $y_T=(1/2^2, 1/2^3,\ldots, 1/2^i)$ and (2) $\phi_i(y_V, y_T)=1$ if $y_V\in\overline{V}_i\backslash V_i$ or $y_T\in\overline{T}_i\backslash T_i$.

For any $p\in S_i\times\overline{U}_i\times Y_i$ let p be expressed as (p_1, p_2, p_3) with p_1, p_2, p_3 points of the factors above. For any t, $0 \leqq t \leqq 1$, any $p_1 \in S_i$, any $p_2 \in \bar{U}_i$ and any $p_3 \in \phi_i^{-1}(t)$, let $h_i(p) = (p_1, \Lambda_i^t(p_2), p_3)$. Clearly under the given conditions since ϕ_i is continuous and Λ_i^t is an isotopy, h_i so defined is a homeomorphism of $S_i \times \bar{U}_i \times Y_i$ onto itself and h_i is the identity on the boundary of $S_i \times \bar{U}_i \times Y_i$ in I^∞. We let h_i be defined as the identity outside of $S_i \times \bar{U}_i \times Y_i$ and, thus, this extended h_i is by Lemma 2.5, a β^*-homeomorphism.

We define h_1 as above but with $Y_1 = \bar{V}_1$. Since λ_i carries the interval $[q_3, q_4]$ to the interval $[q_5, q_2]$ in C_i, then, for any j, $\tau_{n_1}(h_j \cdots h_2 h_1(K))$ is the point $1/2^{j+1}$ of I_{n_1}, and $\tau_{n_j}(h_j \cdots h_2 h_1(K)) = \tau_{n_j}(K)$ is the point $1/2^j$ of I_{n_j}.

We now ask for conditions on T_i, U_i, and V_i so that h will be the desired homeomorphism.

Let U_1 be the $(1/10)$-neighborhood of M_1 in C_1 and let $V_1 = \tau_{\alpha'}(I^\infty)$. Suppose we have given $h_1, \ldots, h_{i-1}$. Let δ_i be a positive number $< 1/10^i$ such that for any two points $p, q \in I^\infty$ with $\rho(h_{i-1} \cdot \cdots \cdot h_1(p), h_{i-1} \cdot \cdots \cdot h_1(q)) > \delta_i$, then $\rho(p, q) < 1/10^i$.

Let V_i be the $(\delta_i/3)$-neighborhood of $\tau_{\alpha'}(K)$ in $\tau_{\alpha'}(I^\infty)$. Let T_i be the $(\delta_i/3)$-neighborhood of $(1/2^2, \ldots, 1/2^i)$ in $I_{n_2} \times I_{n_3} \times \cdots \times I_{n_i}$. Let U_i be the $(\delta_i/3)$-neighborhood of M_i in C_i.

We note that the metric ρ on infinite products is a scaled down version of the metric d on finite products. Thus a δ-neighborhood of a point in a finite product corresponds to an open subset of the δ-neighborhood of the point in the finite product regarded as a projection of the infinite product.

We consider the difference in action between $\tilde{h}_n = h_n h_{n-1} \cdot \cdots \cdot h_1$ and $\tilde{h}_{n-1} = h_{n-1} \cdot \cdots \cdot h_1$. Since h_n is supported on $S_n \times T_n \times U_n \times V_n$, then $\tilde{h}_n$ differs from $\tilde{h}_{n-1}$ in domain $\tilde{h}_n^{-1}(S_n \times T_n \times U_n \times V_n)$ and in range $S_n \times T_n \times U_n \times V_n$. In effect, this range is a small neighborhood of the projection of K on W_{n_1} whereas the domain is a small neighborhood of K itself. Furthermore the set which is the projection of $\tilde{h}_n(K)$ on W_{n_1} is the projection of $\tilde{h}_{n-1}(K)$ on W_{n_1} and such a set is moved by h_n (through the action of λ_n) to a nearby set on W_{n_1} and then is eventually left alone. Since $\tau_{\alpha'}$ carries K homeomorphically onto $\tau_{\alpha'}(K)$ and no h_n affects any α' coordinate, then no two points of K are brought close together. From these considerations and Lemma 2.2 it follows that $h = L\prod_{i>0} h_i$ exists and is the desired homeomorphism.

LEMMA 5.2. *Let α be a subset of Z and let $\{K_i\}_{i>0}$ and $\{M_i\}_{i>0}$ be any collection of closed sets in I^∞ each straight with respect to an infinite set of integers disjoint from α. Then there exists a β-homeomorphism h such that*

(1) *for $p \in ({}^\circ I^\infty \backslash \bigcup_{i>0} K_i)$, $h(p) \in {}^\circ I^\infty$,*

(2) *for any i, there is a j_i such that $h(K_i) \subset W_{j_i}$,*

(3) *for each i, $h(M_i)$ is straight with respect to some infinite set of integers, and*

(4) *$\alpha(h) \subset \alpha'$.*

Proof. By Lemma 2.9, there exists a subpartition $\{\alpha_i\}$ of infinite sets such that for each i, $\alpha \cap \alpha_i = \varnothing$, K_i is straight with respect to α_{2i-1} and M_i is straight with respect to α_{2i}. We shall define a set $\{h_i\}_{i>0}$ of β-homeomorphisms inductively by repeated use of Lemma 5.1. Thus for some $j_i \in \alpha_{2i-1}$, h_i is to move K_i to W_{j_i} with $\alpha(h_i) \subset \alpha_{2i-1}$.

Since $(h_{i-1}\cdots h_i)^{-1}$ is a homeomorphism of a compact set we let δ_i be a positive number $< 1/10^i$ such that if $p, q \in I^\infty$ and $\rho(h_{i-1}\cdots h_1(p), h_{i-1}\cdots h_1(q)) < \delta_i$, then $\rho(p, q) < 1/10^i$. By use of the product metric on $I_{\alpha_{2i+2j-1}}$ we may require that for each $j \geqq 0$, h_{i+j} move no point more than $(1/2^{j+2})\delta_i$. Since for $p, q \in I^\infty$, $\rho(p, q) > 0$, it follows as in Lemma 2.4 that $h = L\prod_{i>0} h_i$ is an element of $G(I^\infty)$. By condition (4) of Lemma 5.1, it follows that h is a β-homeomorphism.

THEOREM 5.3. *Let α be a collection of integers. Let $\{K_i\}_{i>0}$ be any collection of closed sets such that each is weakly thin with respect to a subpartition whose elements are disjoint from α. Then there exists a β-homeomorphism h such that*

(1) *for $p \in ({}^\circ I^\infty \backslash \bigcup_{i>0} K_i)$, $h(p) \in {}^\circ I^\infty$,*

(2) *for any $i > 0$, there is a j_i such that $h(K_i) \subset W_{j_i}$ and*

(3) $\alpha(h) \subset \alpha'$.

Proof. This theorem follows immediately from Theorem 3.5 and Lemma 5.2.

COROLLARY 5.4. *Let $\{K_i\}_{i>0}$ be any collection of weakly thin subsets of I^∞. Then there exists a β-homeomorphism h such that for $p \in {}^\circ I^\infty$, $h(p) \in B(I^\infty)$ if and only if $p \in \bigcup_{i>0} K_i$.*

COROLLARY 5.5. *Let ${}^\circ I^\infty$ be regarded as the product of lines and as space and let $\{K_i\}_{i>0}$ be any collection of closed sets in ${}^\circ I^\infty$ such that for each i, K_i is bounded above (or below) in infinitely many directions. Then ${}^\circ I^\infty \backslash \bigcup_{i>0} K_i$ is homeomorphic to ${}^\circ I^\infty$.*

Proof. We imbed ${}^\circ I^\infty$ in I^∞ in the natural way and for each i, the closure of K_i is weakly thin. Hence Corollary 5.5 follows from Theorem 5.3.

We may regard this corollary as a theorem giving conditions under which a subset of ${}^\circ I^\infty$ is homeomorphic to ${}^\circ I^\infty$. As a special case of Corollary 5.5 we may assert:

COROLLARY 5.6. *Suppose M is a subset of ${}^\circ I^\infty$ and M is the countable union of compact sets. Then ${}^\circ I^\infty \backslash M$ is homeomorphic to ${}^\circ I^\infty$.*

Our last corollary applies the theorem to a more general setting.

COROLLARY 5.7. *For any separable metric space X and for any countable collection $\{K_i\}_{i>0}$ of compact subsets of $X \times {}^\circ I^\infty$, $(X \times {}^\circ I^\infty) \backslash \bigcup_{i>0} K_i$ is homeomorphic to $X \times {}^\circ I^\infty$.*

Proof. Let $I^\infty = I^\infty_\alpha \times I^\infty_{\alpha'}$ where α and α' are each infinite. Let f be an imbedding of $X \times {}^\circ I^\infty$ into I^∞ defined by imbedding X in ${}^\circ I^\infty_{\alpha'}$ and ${}^\circ I^\infty$ in I^∞_α the latter imbedding carrying ${}^\circ I^\infty$ onto ${}^\circ I^\infty_{\alpha'}$. Now $\{f(K_i)\}_{i>0}$ is a collection of closed sets as in Theorem 5.3 with α, above, regarded as α of the theorem. The homeomorphism h of the theorem carries $f[(X\times{}^\circ I^\infty)\backslash\bigcup_{i>0}K_i]$ onto $f(X\times{}^\circ I^\infty)$. Thus $f^{-1}hf$ (with suitable restrictions of f and h understood) is the desired homeomorphism.

Finally we state a theorem concerning the pushing of compact sets in $B(I^\infty)$ into ${}^\circ I^\infty$.

THEOREM 5.7. *Let M be any compact subset of ${}^\circ W_1$ and let α be any infinite subset of Z with α' infinite and $1\in\alpha'$. Then there exists a β-homeomorphism f such that (1) for $p\in B(I^\infty)$, $f^{-1}(p)\in{}^\circ I^\infty$ if and only if $p\in M$ and (2) $f^{-1}(M)$ is straight with respect to α.*

Proof. Let $\beta\subset Z$ be such that $\beta\supset(\alpha\cup\{1\})$, $\beta\backslash\alpha$ is infinite and β' is infinite. Let K be a subset of ${}^\circ I^\infty$ such that K is homeomorphic to M and K is straight with respect to β. By Lemma 5.1 let h be a β-homeomorphism such that (1) $\alpha(h)\subset\beta\backslash\alpha$, (2) for $p\in I^\infty$, $h(p)$ is an element of $B(I^\infty)$ if and only if $p\in K$ and (3) for $p\in K$, $\tau_1(h(p))=0$ and for $k\in\beta\backslash(\alpha\cup\{1\})$, $\tau_k(p)=\tau_k(h(p))$. Hence $h(K)\subset{}^\circ W_1$. By Theorem 4.4, we may let g be a β^*-homeomorphism carrying $h(K)$ onto M such that $1\in\beta(g)$. Then gh is the desired homeomorphism.

6. The contraction theorem.

THEOREM 6.1. *There exists a β^*-homeomorphism, h, such that for $p\in I^\infty$ with $\tau_1(p)=0$, $\tau_1(h(p))=0$ and, for each $j>1$, $0<\tau_j(h(p))<1$.*

Proof. We shall inductively construct a sequence $\{h_i\}_{i>0}$ of finitely elementary homeomorphisms such that $\alpha(h_i)=\{I_1, I_{i+1}\}$, $\gamma(h_i)\subset\{2, 3, \ldots, i\}$ and the left product of the h_i's exists as the desired homeomorphism h. As before, $W_1=\tau_1^{-1}(0)$ and ${}^\circ W_1=\{0\}\times{}^\circ I_2\times{}^\circ I_3\times\cdots$. Let, for $i>1$, $W_1(i)$ denote the set of all points p of W_1 for which, for each $1<j\leqq i$, $0<\tau_j(p)<1$.

Let U_1 be the (1/10)-neighborhood of W_1 in I^∞. Let h_1 be defined so that (1) h_1 is supported on U_1, (2) $h_1(W_1)\subset W_1(2)$, (3) for each $j>2, j\in\beta(h_1)$ and (4) for each $p\in I^\infty$, $\tau_1(p)\geqq\tau_1(h_1(p))$. Inductively, let U_i be the $(1/10^i)$-neighborhood of $h_{i-1}\cdots h_1(W_1)$. Let h_i be defined so that (1) h_i is supported on U_i, (2)

$$h_i[h_{i-1}\cdots h_i(W)]\subset W_1(i+1),$$

(3) for each $j>i+1, j\in\beta(h_i)$, (4) $\alpha(h_i)=\{1, i+1\}$, (5) for each $p\in I^\infty$ with $\tau_1(p)>1/10^i$, $h_ih_{i-1}\cdots h_1(p)=h_{i-1}\cdots h_1(p)$, and (6) for each $p\in I^\infty$, $\tau_1(p)\geqq\tau_1(h_i(p))$. Note that condition (5) may be achieved since $h_{i-1}\cdots h_2h_1$ is uniformly continuous and h_i could be supported on a small neighborhood of $h_{i-1}\cdots h_2h_1(W_1)$ inside U_i.

We wish to verify that, in fact, the left product of $\{h_i\}_{i>0}$ is a β^*-homeomorphism satisfying the conditions of the theorem. Since with $t=1$, the conditions of Lemma 2.2 with respect to τ_t are clearly satisfied using particularly conditions (1), (2), and (5) above, and since conditions (3) and (4) imply that the other hypotheses of Lemma 2.2 are satisfied, then the left product of $\{h_i\}_{i>0}$ is a mapping h of I^∞ onto itself. To verify that h is a homeomorphism it suffices to note that no two points

p and q can be mapped to the same point by h. From conditions (5) and (6) it follows that for some n and all $m>n$

$$\tau_1(h_n\cdots h_1(p)) = \tau_1(h_m\cdots h_1(p)) = \tau_1(h(p))$$

and

$$\tau_1(h_n\cdots h_1(q)) = \tau_1(h_m\cdots h_1(q)) = \tau_1(h(q)).$$

But then, as in Lemma 2.3, if $p\neq q$, there is some $k>0$ such that $\tau_k(h(p))\neq\tau_k(h(q))$ for otherwise, for some j, $h_j\cdots h_1$ could not be a homeomorphism. Conditions (2) and (4) (principally) imply that $h(W_1)\subset {}^\circ W_1$. To see that h is a β^*-homeomorphism we observe using conditions (3), (4), and (5), that no point of ${}^\circ I^\infty$ is moved to W_1, or, in fact, to $B(I^\infty)$, and that no point of $B(I^\infty)$ is moved to ${}^\circ I^\infty$ since, for $p\in B(I^\infty)$, if $\tau_j(p)=0$, 1, then either $\tau_j(h(p))=0$, 1 or $\tau_1(h(p))=0$ as implied by condition (6) in the presence of the other conditions.

We now reformulate the contraction theorem in a more general setting for use in §9. Let $\alpha=\{n_i\}$ be any infinite subset of Z.

For each i, let f_i be an order-preserving homeomorphism on I_{n_i} onto I_i. For $p=\{p_{n_i}\}\in I_\alpha$, let $f(p)=f(\{p_{n_i}\})=\{f_i(p_{n_i})\}$ and f so defined is a homeomorphism of I_α onto I^∞. Each β- or β^*-homeomorphism g of I^∞ produces a $\beta(I_\alpha)$ or $\beta^*(I_\alpha)$-homeomorphism of I_α, namely $f^{-1}gf$. The set $W_1\in I^\infty$ is $f(W_1(I_\alpha))$ and ${}^\circ W_1=f({}^\circ W_1(I_\alpha))$.

These considerations lead us immediately to the desired corollary.

Corollary 6.2. *Let $\alpha=\{n_i\}_{i>0}$ be any infinite subset of Z. There exists a $\beta^*(I_\alpha)$-homeomorphism, h, such that, for $p\in I_\alpha$ with $\tau_{n_1}(p)=0$, $\tau_{n_1}(h(p))=0$ and, for each $j>1$, $0<\tau_{n_j}(h(p))<1$.*

Note. We can let any element of α be listed as n_1.

Remark 6.3. The lemmas, theorems, and definitions of the preceding sections can be similarly reformulated to refer to I_α instead of I^∞.

7. **The principal extension theorem.**

Definition. A closed set K is said to be *normally imbedded* in $B(I^\infty)$ if for some finite subset $\alpha\subset Z$, $\tau_\alpha(K)$ is a subset of the boundary of $\tau_\alpha(I^\infty)$.

Definition. A closed set K is said to be *calm* in I^∞ if K is the finite union of closed sets each of which is either weakly thin in I^∞ or normally imbedded in $B(I^\infty)$.

Theorem 7.1. *Let M be a closed subset of I^∞ and let f be a homeomorphism of M into I^∞. If $M\cup f(M)$ is calm, then f can be extended to an element of $G(I^\infty)$.*

Proof. Since $M\cup f(M)$ is calm, let δ be a finite subset of Z such that $M\cup f(M)$ is the union of closed sets N_1 and N_2 with the property that $\tau_\delta(N_1)$ is a subset of the boundary of $\tau_\delta(I^\infty)$ and N_2 is the union of finitely many closed sets each weakly thin with respect to a subpartition whose elements are in $Z\backslash\delta$. By Theorem 5.3 there exist a β-homeomorphism h and a finite set $\alpha\subset Z$ such that $\tau_\alpha(h(M\cup f(M)))$

is a closed subset of $B(\tau_\alpha(I^\infty))$. Let g be a β^*-homeomorphism such that $\alpha(g)=\alpha \cup \{1\}$ and $g(h(M \cup f(M))) \subset W_1$. Clearly g can be produced from a homeomorphism of the finite cell $I_{\alpha\cup\{1\}}$.

By Theorem 6.1 there exists a β^*-homeomorphism ϕ such that $\phi gh(M \cup F(M))$ is a (compact) subset of $^\circ W_1$. Hence by Theorem 4.4 there exists a β^*-homeomorphism η extending $\phi ghfh^{-1}g^{-1}\phi^{-1}$ (with h, g, and ϕ cut down appropriately) from the set $\phi gh(M)$ onto $\phi ghf(M)$. But then $h^{-1}g^{-1}\phi^{-1}\eta\phi gh$ extends f as was to be shown.

Theorem 7.1 gives a rather strong homogeneity property of the Hilbert cube: namely, for many closed subsets of I^∞ any homeomorphism from one to the other may be extended to an element of $G(I^\infty)$. (For ordinary 1-point homogeneity, M and $f(M)$ may be regarded as single points.) In a paper entitled *On topological infinite deficiency*, a definitive theorem will be given.

8. **Unions of two Hilbert cubes.** In this section we use the result of §7 to get conditions under which the union of two Hilbert cubes is homeomorphic to a Hilbert cube.

Let A and A' be disjoint metric spaces and let K and K' be closed subsets of A and A' respectively. Suppose there exists a homeomorphism f of K onto K'. Let $A \cup_f A'$ denote the space whose points are (1) the points of $A\backslash K$, (2) the points of $A'\backslash K'$, and (3) the pairs $(k, f(k))$ for $k \in K$. Let g and g' be the canonical 1-1 transformations of A and A' respectively into $A \cup_f A'$. A set U in $A \cup_f A'$ is open if and only if $g^{-1}(U)$ and $g'^{-1}(U)$ are open in A and A' respectively.

THEOREM 8.1. *Suppose A, A', K, and K' are all homeomorphic to I^∞ and suppose K and K' are calm subsets of A and A' respectively. Then for any homeomorphism h of K onto K', $A \cup_f A'$ is homeomorphic to I^∞.*

Proof. This theorem follows readily from Theorem 7.1. Let g and g' be as in the explanation of $A \cup_f A'$. Let f be a homeomorphism of $g(A)$ onto itself carrying the set of points of $g(A)$ with first coordinate 0 onto $g(K)$. Let f' be a similar homeomorphism of $g(A')$ onto itself. We give the desired coordinatization of $A \cup_f A'$ as a Hilbert cube. For any $x \in g(A)$ let every coordinate of x after the first be the similar coordinate of $f^{-1}(x)$ and let the first coordinate of x be $\frac{1}{2}(1-x_1)$ where x_1 is the first coordinate of $f^{-1}(x)$. For any $x \in g'(A')$, let the first coordinate of x be $\frac{1}{2}+\frac{1}{2}(x_1)$ where x_1 is the first coordinate of $f'^{-1}(x)$ and let every other coordinate be that already assigned to the point of $g'(K')$ whose other coordinates under f'^{-1} agree with those of $f'^{-1}(x)$.

CONJECTURE. *If the intersection of two Hilbert cubes is a Hilbert cube then their union is a Hilbert cube.*

In a paper to be published separately the author has shown that the product of any dendron and I^∞ is homeomorphic to I^∞. It is easy to see that the above conjecture could not be true unless, for example, the product of a triod and I^∞ were homeomorphic to I^∞. But the conjecture is, in fact, much stronger than this latter statement.

QUESTION. *If the union of two Hilbert cubes is a Hilbert cube must their intersection be a Hilbert cube?* It seems likely that the answer is in the negative but the author does not know how to prove it. Clearly such intersection must be an infinite-dimensional Cantorian manifold. If there exists (as seems likely) an infinite-dimensional Cantorian manifold which is an absolute retract and is not homeomorphic to I^∞, then it might be possible to slice a Hilbert cube into two Hilbert cubes by such a set.

9. **Products homeomorphic to $^\circ(I)^\infty$.** In this section we apply the procedures of the preceding sections to show that many infinite products not obviously homeomorphic to $^\circ I^\infty$ are, in fact, homeomorphic to this set.

DEFINITIONS. A set Y is said to be a *near n-cell*, $n>0$, if Y is a subset of a closed n-cell V such that Y contains Int V. We say that V *carries* Y. A near n-cell Y for which $V\backslash Y\neq\varnothing$ is called *proper*. A near n-cell Y for which $Y\backslash\text{Int } V$ is a G_δ subset of $B(V)$ is called a G_δ near n-cell.

The following lemma is almost obvious and is given without proof.

LEMMA 9.1. *If Y_1 and Y_2 are G_δ near n_i-cells, $i=1, 2$, and Y_1 is proper, then $Y_1\times Y_2$ is a proper G_δ near (n_1+n_2)-cell.*

LEMMA 9.2. *Let Y be a proper near n-cell. Then for any set $\alpha=\{j_1,\dots,j_n\}$ of distinct positive integers, there exists a homeomorphism ϕ of Y into I_α such that* (i) *I_α carries $\phi(Y)$*, (ii) *$\tau_{j_1}(\phi(Y)\backslash{}^\circ I_\alpha)$ is the point* $0\in I_{j_1}$, *and* (iii) *for each k, $1<k\leqq n$, $0<\tau_{j_k}(\phi(Y))<1$.*

Proof. Let V be an n-cell which carries Y and let p be a point of $V\backslash Y$. Let σ be a map of I_α onto V such that for each $q\in V$ with $q\neq p$, $\sigma^{-1}(q)$ is a single point of I_α and such that $\sigma^{-1}(p)$ is the closure of the set of all points of the boundary of I_α with j_1 coordinate >0. Then $\sigma^{-1}|Y$ is the desired homeomorphism ϕ.

We call I_{j_1} of the above lemma the end-factor and j_1 the end-index of I_α.

THEOREM 9.3. *Let, for each $i>0$, Y_i be a G_δ near n_i-cell and let, for infinitely many i, Y_i be proper. Then $Y=\prod_{i>0} Y_i$ is homeomorphic to $^\circ I^\infty$.*

Proof. By Lemma 9.1 we may take products of pairs of Y_i's so that each product is a proper near m_k-cell for some $m_k\geqq 2$. Without loss of generality, we assume the Y_i's themselves to be proper and each n_i to be $\geqq 2$. By Lemma 9.2 we may regard I^∞ as $V_1\times V_2\cdots$ where, for each j, V_j is a finite product $I_{j_1},\dots,I_{j_{n_i}}$ and V_j carries Y_j in the manner of Lemma 9.2. Let T be the collection of all end-indices of the various V_j's.

For each j, let X_j be the set of all points of V_j with end-factor coordinate $\neq 1$ and other coordinates neither 0 nor 1. Thus we have $^\circ V_j\subset Y_j\subset X_j\subset V_j$. Letting $X=\prod_{i>0} X_i$ we also have $^\circ I^\infty\subset Y\subset X\subset I^\infty$.

Our strategy will be first to exhibit a homeomorphism f of I^∞ onto itself such that $f(X)={}^\circ I^\infty$. Then we find a β-homeomorphism h deleting a countable union of straight sets (relatively closed in ${}^\circ I^\infty$) from ${}^\circ I^\infty$ so that hf carries Y onto ${}^\circ I^\infty$, i.e., we use h to eliminate the points of $f(X\backslash Y)$ from ${}^\circ I^\infty$.

Let $\{\alpha_j\}$ be a partition of Z such that for each $j>0$, α_j is infinite and $\alpha_j \cap T$ is a single element.

Let j^* denote the element of T in α_j. Let W'_{j^*} denote the set of all points of I_{α_j} with j^* coordinate 0. Let ${}^\circ W'_{j^*}$ denote the subset of W'_{j^*} consisting of all points of W'_{j^*} each of whose other coordinates is neither 0 nor 1.

We note that $X=\prod_{j>0}({}^\circ I_{\alpha_j} \cup {}^\circ W'_{j^*})$ since ${}^\circ I_{\alpha_j} \cup {}^\circ W'_{j^*}$ is the product of an appropriate half-open interval by infinitely many open intervals. Thus $\prod_{j>0}({}^\circ I_{\alpha_j} \cup {}^\circ W'_{j^*})$ can be refactored to produce X.

By Corollary 6.2, there exists a $\beta^*(I_{\alpha_j})$-homeomorphism η_j such that $\eta_j(W'_{j^*})\subset {}^\circ W'_{j^*}$. By Theorem 5.7 and Remark 6.3 there exists a $\beta(I_{\alpha_j})$-homeomorphism ϕ_j such that $\phi_j^{-1}\eta_j(W'_{j^*})$ is a straight set in ${}^\circ I_{\alpha_j}$ and

$$\phi_j({}^\circ I_{\alpha_j}) = {}^\circ I_{\alpha_j} \cup \eta_j(W'_{j^*}).$$

For each j, consider $I^\infty = I_{\alpha_j}\times I_{\alpha'_j}$. For each j, let $\tilde{\eta}_j$ and $\tilde{\phi}_j$ be defined in terms of the above factorization as follows: for $p\in I^\infty$, $\tilde{\eta}_j(p)=(\eta_j(\tau_{\alpha_j}(p)),\ \tau_{\alpha'_j}(p))$ and $\tilde{\phi}_j(p)=(\phi_j(\tau_{\alpha_j}(p)),\ \tau_{\alpha'_j}(p))$. By Lemmas 2.3 and 2.8, $\eta=L\prod_{j>0}\eta_j$ and $\phi=L\prod_{j>0}\phi_j$ are β^*- and β-homeomorphisms respectively.

The homeomorphism $\phi^{-1}\eta$ does not serve as the f of our strategy since, for each j, $\phi_j^{-1}\eta_j$ carries W'_{j^*} instead of ${}^\circ W'_{j^*}$ into ${}^\circ I^\infty$. For each j and each $k\in\alpha_j$ with $k\neq j^*$, let $C_k(j)$ denote the (closed) set of points of W'_{j^*} with k-coordinate 0 or 1. Then ${}^\circ W'_{j^*}=W'_{j^*}\backslash(\bigcup_{k\in\alpha_j;\,k\neq j^*}C_k(j))$. Also since $\phi_j^{-1}\eta_j(W'_{j^*})$ is straight in I_{α_j} then so is $\phi_j^{-1}\eta_j(C_k(j))$ for each k. Therefore by Lemma 5.2, there is a $\beta(I_{\alpha_j})$-homeomorphism g_j such that g_j carries ${}^\circ I_{\alpha_j}\backslash\bigcup_k \phi_j^{-1}\eta_j(C_k(j))$ onto ${}^\circ I_{\alpha_j}$, and $g_j\phi_j^{-1}\eta_j(W'_{j^*})$ is straight in I_{α_j}. Let $\tilde{g}_j$ be defined with respect to g_j as were $\tilde{\eta}_j$ and $\tilde{\phi}_j$ with respect to η_j and ϕ_j. Let $g=L\prod_{j<0}\tilde{g}_j$ and g, by Lemmas 2.3 and 2.8 is a β-homeomorphism. Hence $g\phi^{-1}\eta$ is the desired homeomorphism f of our strategy and $g\phi^{-1}\eta(\tau_t^{-1}(0))$ is straight for each $t\in T$.

For any i, we consider V_i, X_i, and Y_i. By definition of near n-cell and the construction of Lemma 9.2, for each i, there exists a countable collection $R_\lambda(i)$, $\lambda>0$, of closed subsets of the closure of $X_i\backslash{}^\circ V_i$ such that $X_i\backslash\bigcup_{\lambda>0}R_\lambda(i)=Y_i$. For j^* the end-index of V_i and $j^*\in\alpha_j$, the closure of $X_i\backslash{}^\circ V_i$ crossed the factors of I^∞ not factors of V_i, is precisely the set W'_{j^*} crossed the factors of I^∞ not in α_j and is precisely the set $\tau_{j^*}^{-1}(0)$. Let $\mathcal{M}$ be the collection of all sets $R_\lambda(i)$ crossed with the factors of I^∞ not factors of V_i. Since for $t\in T$, $g\phi^{-1}\eta(\tau_t^{-1}(0))$ is straight in I^∞, then so is each set $g\phi^{-1}\eta(M)$ for $M\in\mathcal{M}$. The collection of all such sets $g\phi^{-1}\eta(M)$ is a countable collection of straight sets and by Lemma 5.2 there is a β-homeomorphism h of I^∞ onto itself such that for $p\in{}^\circ I^\infty$, $h(p)\in B(I^\infty)$ if and only if p is an element of some such set $g\phi^{-1}\eta(M)$. But $X\backslash\bigcup_{M\in\mathcal{M}}M$ must be the set Y since in order for a point p to be in $X \backslash Y$, it is necessary and sufficient that there be an i such that the projection of p on V_i is in $X_i \backslash Y_i$.

Thus Theorem 9.3 is proved.

REMARK. The question as to whether a given set M, ${}^\circ I^\infty \subset M \subset I^\infty$, is homeomorphic to ${}^\circ I^\infty$ is partially answered by the results of §5: the images of ${}^\circ I^\infty$ under β-homeomorphisms are such sets. Also §5 explicitly gives us conditions under which certain subsets of ${}^\circ I^\infty$ are homeomorphic to ${}^\circ I^\infty$.

COROLLARY 9.4. *In order that the countable infinite product of intervals (each open, closed, or half-open) be homeomorphic to* ${}^\circ I^\infty$ *it is necessary and sufficient that infinitely many of the factors be open or half-open.*

Proof. The sufficiency follows explicitly from Theorem 9.3. The necessity follows from the fact that ${}^\circ I^\infty$ is not locally compact whereas such a product with only finitely many open or half-open factors would be locally compact.

COROLLARY 9.5. $I^\infty \times {}^\circ I^\infty$ *is homeomorphic to* ${}^\circ I^\infty$.

In [3] Bessaga and Klee show that Corollary 9.4 above is a corollary of one of their theorems dealing with infinite (not necessarily countable) products.

Sierpiński [9] has shown that the absolute G_δ's are the spaces which are topologically complete. Thus the G_δ requirement of the definition of near n-cell cannot be weakened at all if Theorem 9.3 is to be true. In fact, from Sierpiński's result and Theorem 9.3 we get our final theorem.

THEOREM 9.5. *In order that a countable infinite product of near n-cells be homeomorphic to* ${}^\circ I^\infty$ *it is necessary and sufficient that* (1) *each factor be a* G_δ *near n-cell and* (2) *infinitely many of the factors be proper near n-cells.*

REFERENCES

1. R. D. Anderson, *On a theorem of Klee*, Proc. Amer. Math. Soc. (to appear).

2. Czeslaw Bessaga and Victor Klee, *Two topological properties of topological linear spaces*, Israel J. Math. **2** (1964), 211–220.

3. ———, *Every non-normable Frechet space is homeomorphic with all its closed convex bodies*, Math. Ann. (to appear).

4. W. A. Blankinship, *Generalization of a construction of Antoine*, Ann. of Math. **53** (1951), 276–297.

5. M. K. Fort, *Homogeneity of infinite products of manifolds with boundary*, Pacific J. Math. **12** (1962), 879–884.

6. Ott-Heinrich Keller, *Die Hcmiomorphie der kompakten konvexen Mengen im Hilbertschen Raum*, Math. Ann. **105** (1931), 748–758.

7. V. L. Klee, *Some topological properties of convex sets*, Trans. Amer. Math. Soc. **78** (1955), 30–45.

8. ———, *A note on topological properties of normed linear spaces*, Proc. Amer. Math. Soc. **7** (1956), 673–674.

9. W. Sierpiński, *Sur les ensembles complets d'un espace (D)*, Fund. Math. **11** (1928), 203–205.

LOUISIANA STATE UNIVERSITY,
BATON ROUGE, LOUISIANA