# UNSTABLE HOMOTOPY OF $O(n)$

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**Introduction.** Let $\Pi_m^p(X)$ denote the $p$-primary component of $\Pi_m(X)$. In this paper $\Pi_m^p(O(n))$ is determined, together with its image under $J$, for all $n$, all odd primes $p$, and all $m<2p(p-1)-2$.

Since $\Pi_m^p(O(n))=\Pi_m^p(O(n-1))\oplus\Pi_m^p(S^{n-1})$ when $n$ is even, it is only necessary to discuss odd values of $n$. This is done by studying the homotopy exact couple of the fiberings $O(2j-1)\to O(2j+1)\to V_{2j+1,2}$ using an isomorphism $\Pi_m^p(V_{2j+1,2})\cong \Pi_m^p(S^{4j-1})$. The associated spectral sequence converges to the known stable homotopy groups of $O$, and calculation of the differentials yields the desired results. Only one differential presents any trouble.

Let $\theta^{m+n,m}$ be the group of isotopy classes of homotopy $m$-spheres embedded in $R^{m+n}$. Associating with each element of $\theta^{m+n,m}$, $m\geqq 5$ and $n\geqq 3$, its normal bundle we have a homomorphism $\theta^{m+n,m}\to \Pi_{m-1}(SO(n))$; denote its image by $N(m, n)$ and the $p$-component of $N(m, n)$ by $N^p(m, n)$. $N^p(m+1, n)$ is determined for all $n$, all odd primes $p$, and $m<2(p-1)-2$. These follow easily from the results on $\Pi_m^p(O(n))$.

The results are taken from my Ph.D. thesis (1965), University of Chicago, and I thank Professors M. G. Barratt and A. Liulevicius for their guidance. Some unpublished results of Barratt on the homotopy of spheres are extensively used in the paper. I am indebted to the referee whose suggestions very much improved the presentation.

*Statement of results.* Let $p$ be an odd prime, and let $i$: $O(n)\to O(n+1)$ be the inclusion.

LEMMA A. $\Pi_m^p(O(2k))\cong\Pi_m^p(O(2k-1))\oplus\Pi_m^p(S^{2k-1})$.

This is easy; the groups are embedded by obvious maps of $O(2k-1)$, $S^{2k-1}$ to $O(2k)$.

Hereafter $n$ will be odd and $m<2p(p-1)-2$, unless otherwise stated.

THEOREM B. (i) $\Pi_{4q}^p(O(n))=0$ *for all* $q$

(ii) $\Pi_{4q-1}^p(O(n))=0$ *for all* $q$, *all* $n<2q+1$.

(iii) $\Pi_{4q-1}^p(O(n))\cong\Pi_{4q-1}^p(O)=Z$ *for all* $q$, *all* $n\geqq 2q+1$.

The image of $J$ in the last case has been determined by Adams.

THEOREM C. $\Pi_{4q+1}^p(O(n))=0$ *except that, for each* $r$, $k$ *such that* $r\geqq 2k>0$, $\Pi_m^p(O(n))=Z_p$ *if* $m=2rp-2(r-2k)-3$, *and* $2k+1\leqq n\leqq 2kp-p+1$. *These groups are annihilated by* $J$, *and mapped epimorphically by* $(i)^2$ *when* $n\geqq 2k+1$.

Received by the editors September 8, 1965.

Now suppose $m \equiv 2 \pmod 4$. It is convenient to define $t = t_m$ by $2tp \leqq m < 2(t+1)p$, and to write $\lambda_m$ for the least positive integer congruent to $\frac{1}{2}m+2 \pmod{(p-1)}$.

THEOREM D. (i) $\Pi^p_m(O(n)) = 0$ *for* $n < \lambda_m$ *and* $n \geqq \frac{1}{2}m+2$.

(ii) $\Pi^p_m(O(n)) = Z_{p^{s+1}}$ *when* $\lambda_m + s(p-1) \leqq n < \lambda_m + (s+1)(p-1)$ *and* $n < \frac{1}{2}m+2$ *except that the group is only* $Z_{p^s}$ *if* $s \geqq \frac{1}{2}m - (t+1)(p-1) \geqq 0$.

(iii) *These groups are mapped monomorphically by* $(i)^2$ *when* $n < \frac{1}{2}m$, *and annihilated by* $J$ *when* $p \neq 3$.

REMARK. In (ii), the exception only arises when $2(t+1)(p-1) \leqq m < 2(t+1)p$. As $n$ increases, the group builds up from $Z_p$ when $n = \lambda_m$ to $Z_{p^t}$, the exponent of $p$ increasing by 1 whenever $n$ increases by $(p-1)$, *except* for the only value of $n$ congruent to $p+2 \pmod{2p}$ and to $\frac{1}{2}m+2 \pmod{(p-1)}$.

For completeness we state the following Proposition E without proof.

PROPOSITION E. *The nonzero groups* $\Pi^3_m(O(2j+1))$ *and their images under* $J$ *are given for* $m \leqq 10$ *by the table.*

| | $m=3, j \geqq 1$ | $m=6, j=1$ | $m=7, j \geqq 2$ | $m=9, j=1$ | $m=10, j=1,2$ |
|---|---|---|---|---|---|
| $\Pi^3_m(O(2j+1))$ | $Z$ | $Z_3$ | $Z$ | $Z_3$ | $Z_3$ |
| $\operatorname{Im} J$ | $Z_3$ | $Z_3$ | $Z_3$ | 0 | $Z_3$ |
| generator | $\alpha_1$ | $\alpha_1\alpha_1$ | $\alpha_2$ | — | $\alpha_1\alpha_2$ |

This may be well known. The calculations use the following data from [2].

PROPOSITION F. *The group* $\Pi^p_m(S^{2j-1})$ *is zero for* $m-2j+1 < 2p(p-1)-2$ *except that*

(i) *it is* $Z_p$, *generated by the stable generator* $\alpha_t$, *for all* $j \geqq 2$ *when* $m-2j+1 = 2t(p-1)-1$;

(ii) *it is* $Z_p$, *generated by*

$$\theta^j_r = \{\alpha_1, \ldots, \alpha_1, \alpha_{r-j+1}\}$$

(*where* $\theta^2_r = \alpha_1\alpha_{r-1}$, *and* $\theta^j_r$ *is a j-fold Toda bracket when* $j > 2$), *when* $m-2j+1 = 2t(p-1)-2$. *Here* $\theta^j_r$ *is annihilated by double suspension.*

PROPOSITION G. *Let* $\zeta\colon \Pi^p_m(S^{2k-1}) \to \Pi^p_m(O(2k))$ *be the embedding. Then* $J(\varphi(\alpha)) = \pm[\iota, \iota]E^{2k}\alpha$. *This is well known.*

The following result is true for *all* $m \geqq 2p$ and congruent to $2 \pmod 4$; let $m = 4k-2$ and $n$ be odd.

THEOREM H. $\Pi^p_{4k-2}(O(n)) = Z_{p^t}$ *for* $2k+p+2 \leqq n < 2k$ *where* $p^t$ *is the highest power of* $p$ *dividing* $(2k-1)!$. *These groups are isomorphic under* $(i)^2 = \bar{\imath}$.

The following three theorems are true for all $n$, even or odd, $p \neq 3$, and $m < 2p(p-1)-2$.

THEOREM I. *$N^p(m+1, n)=0$ if $m\equiv 3$ or 0 (mod 4).*

THEOREM J. *$N^p(4q+2, n)=0$ except that for each $r$, $k$ such that $r\geqq 2k>0$, $N^p(m+1, n)=Z_p$ if $m=2rp-2(r-2k)-3$ and $2k+1<n\leqq 2k-p+1$.*

Let $\lambda_m$ and $t=t_m$ be the same as in Theorem D.

THEOREM K. (i) *$N^p(m+1, n)=0$ for $n\leqq\lambda_m$ and $n\geqq\frac{1}{2}m+2$.*

(ii) *$N^p(m+1, n)=Z_{p^{s+1}}$ when $\lambda_m+s(p-1)<n\leqq\lambda_m+(s+1)(p-1)$ and $n<\frac{1}{2}m+2$ except that the group is only $Z_{p^s}$ if $s\geqq\frac{1}{2}m-(t+1)(p-1)\geqq 0$.*

It is immediate from Proposition E that $N^3(m+1, n)=0$ for $m\leqq 10$ except that $N^3(10, 4)=Z_3$.

1. **Preliminaries.** The fibering $O(n)\to O(n+1)\to S^n$ gives rise to the exact sequence

$$(1.1)_n \qquad \to \Pi_{m+1}(S^n)\xrightarrow{k_n}\Pi_m(O(n))\xrightarrow{i_n}\Pi_m(O(n+1))\xrightarrow{j_n}\Pi_m(S^n)\to\cdots.$$

LEMMA (1.2). *This splits at $O(n+1)$ when $n+1$ is even, modulo 2-torsion; hence $\Pi^p_m(O(2k))\cong\Pi^p_m(O(2k-1))\oplus\Pi^p_m(S^{2k-1})$.*

**Proof.** Let $\iota_q$ generate $\Pi_q(S^q)$. According to ([10], p. 120), the composition

$$\Pi_{n+1}(S^{n+1})\xrightarrow{k_{n+1}}\Pi_n(O(n+1))\xrightarrow{j_n}\Pi_n(S^n)$$

is zero if $n+1$ is odd, and multiplication by $\pm 2$ if $n+1$ is even. Let $n+1$ be even, and let $\zeta_n=\pm k_{n+1}(\iota_{n+1})$ be so chosen that $j_n(\zeta_n)=2\iota_n$. Then

$$(\zeta_n)_*:\Pi_n(S^n)\to\Pi_n(O(n+1))$$

has the property that $j_n(\zeta_n)_*$ is an automorphism (modulo 2-primary torsion groups), for

$$j_n(\zeta_n)_*(\alpha) = (2\iota)\alpha,$$

and this is $2\alpha$ if $\alpha$ has odd order, and differs from $2\alpha$ by at most an element of order 2 in general. Hence

$$(1.2) \qquad \Pi^p_m(O(n+1)) \cong i_n\Pi^p_m(O(n))\oplus(\zeta_n)_*\Pi^p_m(S^n),$$

where $i_n$, $(\zeta_n)_*$ are monomorphisms.

By tensoring $(1.1)_n$ with the rationals $Q$ for all $n$, an exact couple is obtained; the associated spectral sequence converges to $\Pi_*(O)\otimes Q$, and it follows at once that, for odd $n$,

LEMMA (1.3). *$\Pi^p_m(O(n))$ is a finite $p$-group, except that $\Pi_{4q-1}(O(n))$ has a summand $Z$ for each $q$ and $n\geqq 2q+1$.*

In view of (1.2) it is only necessary to compute $\Pi_m(O(n))$ for odd $n$. The homotopy exact sequences of the fiberings $O(n) \to V_{n,2}$ form an exact couple $C=\langle D, E, \bar{i}, \bar{j}, \bar{k}\rangle$ where

$$D_{k,q} = \Pi_{2k+q}(O(2k+1)), \qquad E_{k,q} = \Pi_{2k+1}(V_{2k+1,2}).$$

LEMMA (1.4). *There is a map $\varphi_k : S^{4k-1} \to V_{2k+1,2}$ inducing an isomorphism of homotopy groups, modulo 2-torsion.*

For, $V_{2k+1,2}$ is an $S^{2k-1}$ bundle over $S^{2k}$ in which the boundary $\Delta(\iota_{2k})$ of $\iota_{2k}$ is $\pm 2\iota_{2k-1}$. If $P : \Pi_*(S^{4k-1}) \to \Pi_*(S^{2k})$ is obtained by composition with the Whitehead product $[\iota_{2k}, \iota_{2k}]$, we have

$$\Pi_m^p(S^{2k}) = E\Pi_{m-1}^p(S^{2k-1}) \oplus P\Pi_m^p(S^{4k-1}),$$

where $\Delta E$ is $\pm 2$, an automorphism for $m > 2k$. The lemma quickly follows.

In future, $\Pi_m^p(S^{4k-1})$ and $\Pi_m^p(V_{2k+1,2})$ will be identified by $(\varphi_k)_*$. There is, therefore, a spectral sequence $\{E^r, d^r\}$ converging to $\Pi_*^p(O)$, with

$$E_{k,q}^1 = \Pi_{2k+q}^p(S^{4k-1}), \qquad d^r : E_{k,q}^r \to E_{k-r,q-1+2k}^r$$

2. **Calculation of $\Pi_m^p(O(n))$.** In view of the identification $(\varphi_k)_*$, the following Theorem (2.1) determines $\bar{j} : \Pi_{4k-1}^p(SO(2k+1)) \to \Pi_{4k-1}^p(V_{2k+1,2})$.

Let $j_{2k} : \Pi_{4k-1}(SO(2k+1) \to \Pi_{4k-1}(S^{2k})$ be induced by the fibering $SO(2k+1) \to S^{2k}$ and consider the corresponding homomorphism

$$\partial : \Pi_{4k}(BSO(2k+1)) \to \Pi_{4k-1}(S^{2k})$$

on the classifying space. Let $\alpha$ generate the infinite cyclic group in $\Pi_{4k}(BSO(2k+1))$.

THEOREM (2.1). *$\partial(\alpha) = \lambda[\iota, \iota] + E\theta$ where $\lambda = (\eta(k)(2k-1)!)/8$ and $\eta(k) = 1$ or 2 according as $k$ is even or odd.*

**Proof.** Let $f : S^{4k} \to BSO(2k+1)$ be a representative of $\alpha$ and the bundle induced by $f$ be

$$\begin{array}{ccc} S^{2k} & \xrightarrow{\ \mathrm{Id}\ } & S^{2k} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\ g\ } & BSO(2k) \\ \downarrow{\scriptstyle p^1} & & \downarrow{\scriptstyle p} \\ S^{4k} & \xrightarrow{\ f\ } & BSO(2k+1). \end{array}$$

It can be determined that $Y = S^{2k} \cup_{\partial\alpha} e^{4k} \cup e^{6k}$ and that $p^{1*} : H^{4k}(S^{4k}) \to H^{4k}(Y)$ is an isomorphism; denote the generators of both the group by $h_{4k}$. Since the Hopf Invariant of $\partial\alpha$ is $2\lambda$ we have $(h_{2k})^2 = 2\lambda h_{4k}$ where $h_{2k}$ generates $H^{2k}(Y)$. It is known that $p^*(p_k) = \chi^2$ where $p_k$ is the Pontrjagan class in $H^{4k}(BSO(2k+1); Z)$, and $\chi$ is the Euler class. It is easy to see that $g^*(\chi) = 2h_{2k}$ and hence $f^*(p_k)=8\lambda h_{4k}$. But $f^*(p_k)$ can also be computed from the following homotopy-commutative diagram.

$$\begin{array}{ccc} S^{4k} & \xrightarrow{\quad f \quad} & BSO(2k+1) \\ {\scriptstyle \eta(k)\mu(2k)}\downarrow & \searrow^{\omega(k)} & \downarrow \\ BU & \longleftarrow & BSO \end{array}$$

where $\omega(k)$ and $\mu(2k)$ generate $\Pi_{4k}(BSO)$, and $\Pi_{4k}(BU)$ respectively. Theorem 1* in [3] states the commutativity of the top triangle and the commutativity of the lower triangle follows from Bott Periodicity Theorems. Now

$$\begin{aligned} 8\lambda h_{4k} &= f^*(p_k) = \omega(k)^*(p_k) = \eta(k)\mu(2k)^*(C_{2k}) \\ &= \eta(k)(2k-1)!\, h_{4k}, \end{aligned}$$

$C_{2k}$ being a Chern class. Q.E.D.

Hence it follows that $\Pi^p_{4k-2}(O(2k-1))=Z_{p^t}$ where $p^t$ is the highest power of $p$ dividing $(2k-1)!$ Notice here that $\Pi^p_{4k-2}(O(n))$, $n>2k-1$, does not contain $p$-torsion because it becomes stable. The groups $\Pi_{4k-2}(O(n))$ are isomorphic under $\bar{\imath}$ for $2k+p+2\leqq n<2k$ since $\Pi_{m+q}(S^q)$ has no $p$-torsion if $m<2p-3$. This proves Theorem H.

Corollary (2.2). *In the spectral sequence $(E^r, d^r)$, the subgroup of $\Pi_{4k-1}(S^{4k-1})$ generated by $\lambda\iota_{4k-1}$ converges to $\Pi_{4k-1}(O)$ and the corresponding quotient group of order $\lambda$ is annihilated.*

The principle underlying the determination of $p$-torsion in $\Pi_m(O(n))$ is as follows. Suppose $b$, in $\Pi^p_m(O(2k+1))$, is not in $\bar{\imath}(\Pi^p_m(O(2k-1)))$; then $\bar{\jmath}(b)\neq 0$. We know $b$ is unstable; suppose $(\bar{\imath})^r(b)\neq 0$ and $(\bar{\imath})^{r+1}(b)=0$ which would imply that $(\bar{\imath})^r(b)$ belongs to the image of $\bar{k}$. Since a differential $d^{r+1}=\bar{\jmath}(\bar{\imath})^{-r}\bar{k}$, the nonzero differentials correspond to elements of $\Pi^p_m(O(n))$. As we are interested in nonzero differentials we can replace $\Pi_{4k-1}(S^{4k-1})$ in the spectral sequence by a group of order $p^t$ generated by $\iota_{4k-1}$ where $p^t$ is the highest power of $p$ dividing $(2k-1)$; see Corollary (2.2). The calculations are based on knowledge of the homotopy of spheres as given in Proposition F.

We state an easy lemma without proof.

Lemma (2.3). *In the spectral sequence, if $d^s(\iota)=0$ for $s<r$ and $d^r(\iota)=\alpha$, then $d^r(E\beta)=d^r(\iota)\beta$.*

Theorem (2.4). *If $m<2p$, $\Pi_m(O(n))$ has no p-torsion.*

**Proof.** For a given $m<2p$, suppose $n$ is the smallest such that $\Pi_m(O(n))$ has $p$-torsion. Then $\bar{\jmath}:\Pi^p_m(O(n))\to\Pi^p_m(S^{2n-3})$ is a monomorphism on the $p$-torsion and $\Pi_m(S^{2n-3})$ has no $p$-torsion if $m-2n+3<2p-3$. The theorem follows.

THEOREM (2.5). *$\Pi_{2p}^{p}(O(n)) = Z_p$ for $3 \leqq n \leqq p$, and the groups are isomorphic under $\bar{\iota}$.*

**Proof.** This follows at once from the fact that $\alpha_1$ in $\Pi_{2p}^{p}(S^3)$ can be annihilated only by $\iota_{2p+1}$ and hence $d^s(\iota_{2p+1}) = \alpha_1$ where $2s+1 = p$.

THEOREM (2.6). *If $m < 2p(p-1)$ and $m \equiv 3$ or $0$ (mod 4), then $\Pi_m(O(n))$ has no $p$-torsion.*

**Proof.** For a specific $m$ in the given range let $n$ be the smallest such that $\Pi_m(O(n))$ has $p$-torsion. Then $\bar{j} : \Pi_m^p(O(n)) \to \Pi_m^p(S^{2n-3})$ is a monomorphism while the latter group has $p$-torsion only if $m-2n+3$ is of the form $2t(p-1)-1$ or $2t(p-1)-2$. Note that we consider only odd values of $n$.

Theorem $B$ is proved by 1.4 and 2.6.

We can determine how $\Pi_m^p(S^{4r-1})$ is annihilated in the spectral sequence for all $m \leqq 2p(p-1)+3$. If we determine how the unstable elements are killed, then it would become clear that there would only be one possible way in which the stable elements could be annihilated. Then the determination of $p$-torsion, stated in Theorem D, would be a straightforward computation.

Consider the unstable elements $\theta_r^{2k}$ in $\Pi_m(S^{4k-1})$, $m = 2r(p-1)-2+4k-1$, for all $k$ and $r$ such that $p > r \geqq 2k > 0$. Recall that $\theta_r^{2k}$ is the $2k$-fold Toda bracket $\{\alpha_1, \ldots, \alpha_1, \alpha_{r-2k+1}\}$ if $k > 1$ and $\theta_r^2 = \alpha_1\alpha_{r-1}$. Let $r-2k+1 = q$.

PROPOSITION (2.7). *The stable element $\alpha_q$ annihilates $\theta_r^{2k}$.*

This implies Theorem C, except for the action of $J$. The $J$ homomorphism will be considered in the next section.

**Proof of (2.7).** Proof is by induction on $r$. If $r=2$, then $k=1$ and $\theta_2^2 = \alpha_1\alpha_1$ generates $\Pi_{4p-3}^p(S^3)$. If $\alpha_1 \in \Pi_{4p-2}(S^{2p+1})$, $d^s(\alpha_1) = d^s(\iota \cdot \alpha_1) = d^s(\iota)\alpha_1 = \alpha_1\alpha_1$ from (2.5) where $2s+1 = p$. Since $\theta_2^2$ is the only unstable element for $r=2$, this starts the induction. Suppose the proposition is true for $2, 3, \ldots,$ and $r-1$.

*Case* 1. Suppose $q > 1$, that is, $r > 2k$. Then from induction hypothesis $\alpha_1$ annihilates $\theta_{2k}^{2k}$. It is an easy computation then to show that $\alpha_q = \{\alpha_1, p\iota, \alpha_{q-1}\}$ annihilates $\theta_r^{2k} = \{\theta_{2k}^{2k}, p\iota, \alpha_{q-1}\}$.

*Case* 2. Suppose $q=1$, then $r=2k$. In the spectral sequence the candidates for annihilating $\theta_r^{2k}$ are $\alpha_s$, $1 \leqq s < r$. We will show that $\bar{k}(\alpha_s) = 0$ for $1 < s < r$; then it follows that $d^n(\alpha_s) = 0$ for all $n$ and $s$ such that $1 < s < r$.

$$\begin{aligned} \bar{k}(\alpha_s) &= \bar{k}(\iota \circ \alpha_s) = \bar{k}(\iota)\alpha_s = \bar{k}(\iota)\{\alpha_1, p\iota, \alpha_{s-1}\} \\ &= \{\bar{k}(\iota)\alpha_1, p\iota, \alpha_{s-1}\}. \end{aligned}$$

Using induction hypothesis in the form of Theorem C we get $\bar{k}(\iota)\alpha_1 = 0$. That proves the proposition.

3. ***J* homomorphism.** The stable $J$ homomorphism is completely known; we consider $J : \Pi_m^p(O(n)) \to \Pi_{m+n}^p(S^n)$ on $p$-torsion only. In view of (1.2) and Proposition $G$ we need to consider only odd values of $n$.

THEOREM (3.1). *$J$ restricted to the torsion group in $\Pi_m^p(O(n))$ is zero provided $p>3$ and $m<2p(p-1)-2$.*

**Proof.** Suppose $m$ is not congruent to 2 (mod 4). Then $\Pi_{m+n}^p(S^n)$ is either zero or stable if $n$ is odd. Hence $J$ is zero.

Suppose $m\equiv 2$ (mod 4). Then $\Pi_{m+n}^p(S^n)=0$ unless $m$ is of the form $2t(p-1)-2$ and $n\leqq 2t-1$, $p>t\geqq 2$. We can assume that $m=2t(p-1)-2$ and $n\leqq 2t-1$. Let $\gamma$ generate the cyclic group $\Pi_{4p-6}^p(O(p))$; then $J(\gamma)=0$ if $p>3$. So the result is true if $t=2$; assume $t>2$. Now $\Pi_m^p(O(n))=0$ if $n<p$ and $\Pi_m^p(O(n))=Z_p$ if $p\leqq n\leqq 2p-1$ from Theorem D; since the latter groups are isomorphic under $\bar{\imath}$ and since $E^2J=J\bar{\imath}$, $E^2$ being double suspension, it is enough to show that $J$ is zero on $\Pi_m^p(O(p))$. It can be seen from the exact couple that $\bar{\jmath}:\Pi_m^p(O(p))\cong\Pi_m^p(S^{2p-3})$; the latter group is generated by $\alpha_{t-1}=\{\alpha_1, p\iota, \alpha_{t-2}\}$. It is an easy computation to show that $\Pi_m^p(O(p))$ is generated by $\{\gamma, p\iota, \alpha_{t-2}\}$ since $\bar{\jmath}(\gamma)=\alpha_1$. It follows that

$$J(\{\gamma, p\iota, \alpha_{t-2}\}) = 0$$

since $J(\gamma)=0$. That proves the theorem.

4. **The group** $N^p(m+1, n)$. Let $G_n$ be the space of maps $S^{n-1}\to S^{n-1}$ of degree $+1$ and $F_n$ be the subspace of $G_{n+1}$ of maps which preserve a base point. Then $G_n\subset F_n\subset G_{n+1}$. The composite of inclusions $\omega$: $SO(n)\to G_n\to F_n$ induces $J$ on homotopy groups. J. Levine [7] obtained, among other things, a formula for $N(m+1, n)$; $N(m+1, n)$ consists of the unstable elements of $\Pi_m(SO(n))$ which also belong to the kernel of $\omega(m, n):\Pi_m(SO(n))\to\Pi_m(G_n)$.

The bundle map of inclusions of the bundles $SO(n-1)\to SO(n)\to S^{n-1}$ and $F_{n-1}\to G_n\to S^{n-1}$ gives the commutative diagram:

$$(4.1)_n \qquad \begin{array}{ccccccccc} \to \Pi_{m+1}(S^{n-1}) & \to & \Pi_m(SO(n-1)) & \to & \Pi_m(SO(n)) & \to & \Pi_m(S^{n-1}) & \to\cdots \\ \downarrow \mathrm{Id} & & \downarrow J & & \downarrow \omega(m,n) & & \downarrow \mathrm{Id} & \\ \to \Pi_{m+1}(S^{n-1}) & \longrightarrow & \Pi_m(F_{n-1}) & \longrightarrow & \Pi_m(G_k) & \longrightarrow & \Pi_m(S^{n-1}) & \to\cdots \end{array}$$

The two rows are exact homotopy sequences of the bundles.

LEMMA (4.2). *Let $\zeta:\Pi_m^p(S^{2k-1})\to\Pi_m^p(O(2k))$ be the embedding. Then $\omega(m, 2k)$ restricted to $\varphi\Pi_m^p(S^{2k-1})$ is a monomorphism.*

This follows immediately from $(4.1)_{2k}$. It is obvious from (4.2) that $N(m+1, n)$ is a torsion group since the unstable infinite cyclic groups in $\Pi_m(O(n))$ occur as direct summands, $\varphi\Pi_{n-1}(S^{n-1})$, when $n$ is even and $m=n+1$. Hence Theorem B implies Theorem I.

LEMMA (4.3). *Let $\xi\in\Pi_m^p(O(n))$ and $\xi$ does not belong to the image of*

$$i:\Pi_m(O(n-1))\to\Pi_m(O(n)).$$

*Suppose* $J(\xi)=0$, $(i)^{r+1}(\xi)=0$, *and* $(i)^r(\xi)\neq 0$. *Then* $(i)^s(\xi)$ *belongs to the kernel of* $\omega(m, n+s)$ *for* $1\leqq s\leqq r$ *but* $\omega(m, n)(\xi)\neq 0$.

This follows easily from $(4.1)_n$ with appropriate values of $n$. The hypothesis of (4.3) is typical of the elements of $\Pi_m^p(O(n))$. Theorems H and K follow from Theorems C and D respectively by repeated application of (1.2), (4.2), and (4.3).

## References

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