A NONCONSTRUCTIBLE Δ_3^1 SET OF INTEGERS

BY
ROBERT M. SOLOVAY(1)

1. **Introduction.** In [6] and [10], Gaifman and Rowbottom independently proved the following result: If measurable cardinals exist, then there is a nonconstructible set of integers. In fact, Gaifman proved the following much stronger result: Let α be an ordinal definable in L, the universe of constructible sets. (For example, take $\alpha = \aleph_2^L$.) Then α is countable. Subsequently, Silver showed how to obtain these results under the weaker hypothesis that Ramsey cardinals exist [12]. (The definition of "Ramsey" will be recalled in §3. The least measurable cardinal is Ramsey [3].)

In this paper we show that the various countable sets mentioned above are Δ_3^1. (Cf. [11] for the definition of a Δ_3^1 set of natural numbers. There is an analogous notion of a Δ_3^1 subset of the power set of the natural numbers, which we shall use below.)

Definition. An ordinal γ is Δ_3^1 if it is finite or if it is order isomorphic to some Δ_3^1 ordering R of ω.

Our results are as follows. (We assume once for all that there is at least one Ramsey cardinal.)

Theorem 1. *The set of Gödel numbers of sentences true in L is a Δ_3^1 set of natural numbers.*

Theorem 2. *Let α be an ordinal definable in L. Then α is Δ_3^1.*

Theorem 2 has the following corollary.

Theorem 3. *There is a Δ_3^1 set of integers which is not constructible.*

On the other hand we have:

Theorem 4. *Every constructible set of integers is Δ_3^1.*

The following theorem answers a question of Azriel Levy.

Theorem 5. *There is a set of integers, A, with the following properties*:

(1) *A is Δ_3^1*;

(2) *A is not constructible*;

(3) *A is Δ_3^1 in $L[A]$*;

(4) *$L[A]$ has a well ordering definable in $L[A]$.*

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(Here $L[A]$ is the class of sets constructible from A.)

To state our next result, we need the following definition.

DEFINITION. Let A be a set of integers, and X a set of sets of integers. X is *constructible from A* if there exists a set-theoretical formula $\phi(x, y, z)$, and an ordinal λ such that

$$X = \{B \subseteq \omega \mid \phi(B, A, \lambda) \text{ holds in } L[A, B]\}.$$

(This concept is due to Dana Scott.)

THEOREM 6. *There is a Δ_3^1 set of sets of integers, X, which is not constructible from any set of integers A.*

REMARKS. (1) Every Δ_3^1 set of integers is, a fortiori, ordinal definable. Thus Theorem 3, for example, yields an ordinal-definable nonconstructible set of integers.

(2) In each of the results given above, Δ_3^1 is best possible. (I.e., it cannot be improved to Σ_2^1 or Π_2^1.) This follows easily from the following theorem of Shoenfield.

PROPOSITION 1.1 [11]. (1) *Let $A(\gamma)$ be a Σ_2^1 predicate. (Here γ ranges over sets of integers.) Let δ be a set of integers. Write $A^{L[\delta]}$ for the relativization of A to $L[\delta]$. Then if $\gamma \in L[\delta]$, we have*

$$A(\gamma) \equiv A^{L[\delta]}(\gamma).$$

Theorems 1 through 4 were first proved under the stronger hypothesis that there is a measurable cardinal. The proofs used ideas of Rowbottom and Gaifman and were fairly complicated. The present proof uses ideas of J. Silver [12] and is much simpler. It was discovered independently by J. Silver and myself in an attempt to weaken the hypothesis in the original proof. (I am grateful to W. Reinhardt for an illuminating discussion on Silver's work.)

This paper is organized as follows. In §2, Theorems 1 through 5 are deduced from a certain technical lemma (Lemma 2.8). The proof of Lemma 2.8 requires a detailed knowledge of Silver's work [12]. We review Silver's work in §3 and give a proof of Lemma 2.8 in §4. §5 gives the proof of Theorem 6. It is amusing to note that the proof uses Cohen's notion of a generic set of integers. This is probably the first application of Cohen's method to set theory yielding an absolute result rather than a relative consistency result.

1.1. At the referee's suggestion, we make a few remarks on the extent to which this paper is self-contained.

The portions of Silver's thesis used in this work are reviewed in detail in §3. Our principal omissions are the details of the implication (2) $\rightarrow$ (3) of Lemma 3.8 and the details of the discussion in subsection 3.9. If the reader can fill these details in, he will also be able to prove the following theorems of Silver on the basis of §3:

(1) The uncountable cardinals are a set of indiscernibles for L.

(2) If $\aleph_0 < \aleph < \aleph'$, then the inclusion map

$$\{L_{\aleph} \to L_{\aleph'}\}$$

is an elementary embedding.

(3) Lemma 2.7.

He will thus be able to completely understand this work without explicit reference to [12].

2. Proofs of Theorems 1–5.

2.0. The proofs of Theorem 1–5 are based on the following fact, which is Lemma 2.11 below. There is a countable ordinal λ_0 such that L_{λ_0} is an elementary submodel of L, and a Δ^1_3 relation R such that the relational systems

$$\langle \omega; R \rangle \quad \text{and} \quad \langle L_{\lambda_0}; \varepsilon \rangle$$

are isomorphic.

2.1. We assume that the reader is familiar with the theory of constructible sets [7]. If $\alpha \in On$ we let $L_\alpha = \{x \mid (\exists \beta < \alpha)(x = F(\beta))\}$. (Here F is the enumeration of the constructible sets given in [7].) If $x \in L$, let ord (x) be the least ordinal β such that $x = F(\beta)$. We define a well ordering $<$ of L by $x < y \equiv \text{ord}(x) < \text{ord}(y)$. Then $(L, <)$ and $(On, <)$ are order isomorphic. Let $G: L \to On$ give this isomorphism.

More generally, if (M, ε_M) is a model of $Z - F + V = L$, then the formal definitions of $<$ and G yield a canonical ordering, $<_M$, of M, and an order isomorphism $G: (M, <_M) \simeq (On_M, \varepsilon_M)$. When we speak of M as an ordered set, it will always be this ordering that we have in mind. The ordering on M restricts to the usual ordering on On_M. Thus M is well founded iff On_M is well ordered iff M is well ordered.

2.2. We now define the set of integers $O^\#$. Let $\mathscr{L}$ be a first order language with predicates $\in$ and $=$, and for each positive integer n, a constant c_n. We interpret $\mathscr{L}$ as follows: (1) the variables shall range over L; (2) $\in$ and $=$ have their usual meanings; (3) c_i shall denote the set $\aleph_i$. (Caution: $\aleph_i$ is the real cardinal and *not* $\aleph_i^L$.)

In general if $\mathscr{L}'$ is an interpreted language, we say that a sentence of $\mathscr{L}'$ is *true* if it is true under the intended interpretation. (The interpretation may be indicated by the context.)

DEFINITION. $O^\#$ is the set of Gödel numbers of true sentences of $\mathscr{L}$ (under the interpretation just given) relative to some Gödel numbering of $\mathscr{L}$ which we fix once for all.

The reader familiar with the "undefinability of truth" may wonder if this definition can be formalized in set theory. To handle this point, we use the following lemma.

LEMMA. *Let $\aleph$ be an uncountable cardinal. Then the inclusion map $\{L_{\aleph} \to L\}$ is an elementary embedding.* (*This is really a scheme of theorems.*)

This lemma is due to Gaifman in the measurable cardinal case [6], and to Silver in the present context [12]. (Recall that we are assuming throughout this paper that there is at least one Ramsey cardinal.)

The lemma shows that we get the same set of integers, $O^{\#}$, if we interpret the variables of $\mathscr{L}$ as ranging over $L_{\aleph_\omega}$ rather than L. The altered definition can clearly be formalized in set theory. Using this technique, Gaifman [6] shows that "satisfaction-in-L" can be formalized in set theory. In the future, we shall use this remark implicitly when we give definitions involving "true-in-L".

2.3. We wish to study the elementary submodel of L generated by

$$\{\aleph_1, \aleph_2, \ldots, \aleph_n, \ldots\}.$$

In order to do this we enlarge the language $\mathscr{L}$ by adding description terms (or μ-terms as we will call them later). The language $\mathscr{L}_\mu$ may be characterized as follows:

(1) the predicates of $\mathscr{L}_\mu$ are $\in$ and $=$;

(2) each constant of $\mathscr{L}$ is a constant of $\mathscr{L}_\mu$;

(3) let $\phi(y)$ be a formula of $\mathscr{L}_\mu$ containing free at most the variable y. Then $\mu y\phi(y)$ is a constant of $\mathscr{L}_\mu$.

(4) $\mathscr{L}_\mu$ has precisely those constants required by clauses (2) and (3).

To each term t of $\mathscr{L}_\mu$ we assign a nonnegative integer which is the height of t. If t is a term of $\mathscr{L}$, the height of t is zero; if t is $\mu y\phi(y)$, the height of t is

$$1+\max\{\text{height}(t') : t' \text{ appears in } \phi(y)\}.$$

We now extend the interpretation of $\mathscr{L}$ to an interpretation of $\mathscr{L}_\mu$ by giving a denotation to each constant term of $\mathscr{L}_\mu$. We do this by induction on the height of t. Namely, let $t=\mu y\phi(y)$. By our inductive assumption, we know the meaning of all terms appearing in ϕ. If $(\exists y)\phi(y)$ is true, let $\mu y\phi(y)$ be the least element $x\in L$ such that $\phi(x)$; otherwise, let $\mu y\phi(y)$ be 0. (Here 0 is the empty set.)

2.4. Let ϕ be a sentence of $\mathscr{L}_\mu$. We show how to construct a sentence ϕ' of $\mathscr{L}$ with the same truth value. The construction of ϕ' from ϕ will be recursive (given suitable Gödel numberings of $\mathscr{L}$ and $\mathscr{L}_\mu$).

Let n be the maximum height of any μ-term appearing in ϕ. We shall define a sequence of sentences, $\phi_0, \ldots, \phi_n$, equivalent to ϕ. Each term appearing in ϕ_j will have height $\leqq j$. If we can do this, then ϕ_0 will be the desired sentence ϕ' of $\mathscr{L}$.

Put $\phi_n=\phi$. Suppose now that $\phi_{j+1}(t_1, \ldots, t_m)$ has been constructed. Here $t_1, \ldots, t_m$ are the μ-terms appearing in ϕ_{j+1}. Say $t_r=(\mu y)\theta_r(y)$, $1\leqq r\leqq m$. We write

$$\theta_i'(y) \text{ for } \theta_i(y) \;\&\; (z)(z < y \rightarrow \neg\theta_i(z))$$

and $\psi_i(y)$ for $\theta_i'(y) .\vee. [(z)\neg\theta_i(z) \;\&\; y=0]$. Then $\mu y\theta_i(y)$ is the unique z such that $\psi_i(z)$. We take the following sentence for ϕ_j:

$$(z_1, \ldots, z_m)(\psi_1(z_1) \;\& \cdots \&\; \psi_r(z_r) \;\& \cdots\; \psi_m(z_m) .\rightarrow. \phi_{j+1}(z_1, \ldots, z_m)).$$

Clearly all terms in ϕ_j have height $\leqq j$ and ϕ_j is equivalent to ϕ_{j+1}.

2.5. DEFINITION. Let $\langle M; \varepsilon_M\rangle$ be a model of $Z-F+V=L$ and let A be a subset of M. Let N be an elementary submodel of M. We say that A *generates* N (or that N is the elementary submodel generated by A) if (1) $A\subseteq N$; (2) if N' is an elementary submodel of M such that $A\subseteq N'$, then $N\subseteq N'$.

N is uniquely determined by A. It is well known that every subset A of M generates an elementary submodel. (The generated submodel consists of the elements of M definable from A. Cf. the proof of the lemma below.)

Let D be the set of denotations of all terms of $\mathscr{L}_\mu$. Then we have the following lemma.

LEMMA. *D is the elementary submodel of L generated by* $\{\aleph_i: 0<i<\omega\}$.

Proof. Let D' be an elementary submodel of L containing $\aleph_i$ for $0<i<\omega$. One checks easily by induction that the denotation of each μ-term lies in D', i.e., $D\subseteq D'$.

It remains to prove that D is an elementary submodel of L. By examining the standard proof of the Skolem-Löwenheim theorem, we see that D will be an elementary submodel of L if the following criterion is satisfied. Let $\phi(x_0,\ldots,x_n)$ be a set-theoretical formula with free variables $x_0,\ldots,x_n$. Let $d_1,\ldots,d_n$ be elements of D. Suppose that $(\exists y)\phi(y, d_1,\ldots,d_n)$ is valid in L. Then for some d in D, $\phi(d, d_1,\ldots,d_n)$ is valid in L.

To see this, let t_i be a term of $\mathscr{L}_\mu$ denoting d_i. Take d to be the denotation of

$$\mu y\phi(y, t_1,\ldots,t_n).$$

This suffices.

2.6. Let M be a model of $Z-F$ with universe $|M|$; $M=\langle |M|; \varepsilon_M\rangle$. For each $n\in\omega$, let $g_M(n)$ be "the integer n in the model M." Thus $g_M: \omega\to|M|$.

LEMMA. *Let D be the elementary submodel of L described in Lemma* 2.5. *Then there is a model $M=\langle\omega; R\rangle$ isomorphic to $\langle D; \varepsilon\rangle$ such that R is recursive in $O^\#$. Moreover, the map*

$$g_M: \omega\to\omega$$

described above is recursive in $O^\#$.

Proof. Pick some fixed Gödel numbering for $\mathscr{L}_\mu$. In the following proof we identify a term or formula with its Gödel number.

We can effectively determine whether a number n is the Gödel number of a term of $\mathscr{L}_\mu$. If t_1 and t_2 are terms of $\mathscr{L}_\mu$ we can effectively determine from $O^\#$ the truth values of "$t_1=t_2$" and "$t_1\in t_2$." Consider, for example, "$t_1=t_2$." This is a statement of $\mathscr{L}_\mu$. The procedure outlined in 2.4 yields a sentence ϕ of $\mathscr{L}$ with the same truth value. One then looks up the truth value of ϕ in $O^\#$.

It is now easy to construct a function f (recursive in $O^\#$) such that (1) for each n,

$f(n)$ is a term of $\mathscr{L}_\mu$; (2) "$f(n)=f(m)$" is valid iff $n=m$; (3) if t is a term of $\mathscr{L}_\mu$, then "$t=f(n)$" is valid for some n. If we put

$$R = \{\langle r, s\rangle : \text{“}f(r) \in f(s)\text{” is valid}\},$$

then R is recursive in $O^\#$ and $\langle \omega; R\rangle$ is isomorphic to $\langle D; \varepsilon\rangle$.

Let $\phi_i(y)$ be a recursive sequence of formulas of $\mathscr{L}$ with one free variable such that ϕ_n defines the integer n in L. I.e., the sentence

$$(z)(\phi_n(z) \equiv z = n)$$

is true in L. Then $g_M(n)$ is the least integer r such that "$f(r)=\mu y\phi_n(y)$" is valid. This proves g_M is recursive in $O^\#$.

The proof that g_M is recursive in $O^\#$ has the following corollary:

COROLLARY. *There is a function h, recursive in $O^\#$, with the following property. Let r be the Gödel number of a set-theoretical formula, $\psi(y)$, with one free variable. Then if $(\exists y)\psi(y)$ is valid in L, then $\psi(h(r))$ is valid in $\langle \omega; R\rangle$.*

It is clear that the proofs of the lemma and its corollary are effective. We could, if we desired, explicitly describe Gödel numbers for R, g, and h in $O^\#$.

2.7. As an elementary submodel of L, $\langle D; \varepsilon\rangle$ is well founded. Thus it is isomorphic to a model L_{λ_0} for a certain countable ordinal λ_0. (Since λ_0 is order isomorphic to On_D, it is uniquely determined.)

The following result is due to Silver [12].

LEMMA. *The inclusion map $\{L_{\lambda_0} \to L\}$ is an elementary embedding.*

2.8. We now state the fundamental lemma. This lemma will be proved in §4.

LEMMA. *There is a Π_2^1 predicate $A(\gamma)$ such that*

$$(\gamma)(A(\gamma) \equiv \gamma = O^\#).$$

(Here γ ranges over sets of integers.)

2.9. LEMMA. *$O^\#$ is Δ_3^1.*

Proof. We have

(1) $n \in O^\# \equiv (\exists\gamma)(A(\gamma)$ and $n \in \gamma)$;

(2) $n \in O^\# \equiv (\forall\gamma)(A(\gamma) \to n \in \gamma)$.

(Here A is the Π_2^1 predicate provided by Lemma 2.8. Equations (1) and (2) show that $O^\#$ is respectively Σ_3^1 and Π_3^1.)

2.10. The following result is due to Kleene. (Cf. [8, §5.2].)

LEMMA. *Let γ, δ be sets of integers. Suppose that γ is recursive in δ and δ is Δ_3^1. Then γ is Δ_3^1.*

If we combine this lemma with Lemma 2.9, we get the following corollary.

COROLLARY. *Let γ be a set of integers recursive in $O^{\#}$. Then γ is Δ_3^1.*

2.11. LEMMA. *Let L_{λ_0} be the elementary submodel of L introduced in 2.7. The model $\langle L_{\lambda_0}; \varepsilon\rangle$ is isomorphic to $\langle\omega; R\rangle$ for some Δ_3^1 relation R. Moreover, if $g: \omega\to\omega$ is the canonical enumeration of the integers of the model $\langle\omega; R\rangle$, then g is Δ_3^1.*

Proof. By definition, $\langle L_{\lambda_0}; \varepsilon\rangle$ is isomorphic to $\langle D; \varepsilon\rangle$. Lemma 2.6 and Corollary 2.10 complete the proof.

2.12. **Proof of Theorem 1.** Each set-theoretical sentence is, a fortiori, a sentence of $\mathscr{L}$. Thus the set of Gödel numbers of L-true sentences is recursive in $O^{\#}$. (Cf. Definition 2.2.) Corollary 2.10 completes the proof.

2.13. **Proof of Theorem 2.** Let α be an infinite ordinal definable in L. Since L_{λ_0} is an elementary submodel of L, $\alpha\in L_{\lambda_0}$. Moreover, the order relation on α is

$$\{\langle\beta,\gamma\rangle: \beta\in\gamma \text{ and } \gamma\in\alpha\}.$$

By Lemma 2.11, there is an isomorphism

$$\phi: \langle L_{\lambda_0}; \varepsilon\rangle \simeq \langle\omega; R\rangle,$$

R is Δ_3^1. We put

$$A = \{m\in\omega: mR\phi(\alpha)\}$$

and $B=\{\langle m,n\rangle: mRn \text{ and } nR\phi(\alpha)\}$. Then $\langle\alpha; \varepsilon\rangle\simeq\langle A; B\rangle$. Let $f:\omega\to A$ be the enumeration of A in increasing order (without repetitions). Then if

$$S = \{\langle m,n\rangle: \langle f(m), f(n)\rangle\in B\},$$

then $\langle\omega; S\rangle\simeq\langle A; B\rangle\simeq\langle\alpha, \varepsilon\rangle$. Since A, B, f, and S are all recursive in R, they are Δ_3^1 by Lemma 2.10.

2.14. **Proof of Theorem 3.** By Theorem 2, there is a Δ_3^1 well ordering of ω, S, isomorphic to $\aleph_1^L$. Assume first that S is constructible. Then, a fortiori, every constructible subset of ω has an S-least element. Thus S well orders ω in L; therefore there is in L an isomorphism $\phi: \langle\omega; S\rangle\simeq\langle\xi; \varepsilon\rangle$ for some constructibly countable ordinal ξ. But this contradicts the fact that $\langle\omega; S\rangle$ is isomorphic to $\langle\aleph_1^L; \varepsilon\rangle$. Thus S is not constructible.

Put

$$A = \{n \mid n = 2^x3^y \text{ and } xSy\}.$$

Then A is a Δ_3^1-subset of ω. Since S is not constructible, neither is A.

2.15. **Proof of Theorem 4.** Let R be the Δ_3^1 relation on ω given by Lemma 2.11. Let

$$\phi: \langle L_{\lambda_0}; \varepsilon\rangle \simeq \langle\omega; R\rangle$$

be an isomorphism. By Lemma 2.11, the map $g=\phi|\omega$ is Δ_3^1.

Let A be a constructible set of integers. It is shown in [7] that $A=F(\xi)$ for some

$\xi < \aleph_1^L$. Since L_{λ_0} is an elementary submodel of L, $\aleph_1^L \in L_{\lambda_0}$. Therefore $\xi < \aleph_1^L < \lambda_0$, so $A \in L_{\lambda_0}$. Then

$$A = \{n \mid n \in A\} = \{n \mid g(n) R \phi(A)\}.$$

Thus A is recursive in g and R and therefore is Δ_3^1.

2.16. *On the effective versions of Theorems* 1–4. We consider, as a sample, Theorem 4. We sketch a proof, omitting many details, of the following "effective" version of Theorem 4. Let $\psi(x)$ be a set-theoretical formula with one free variable. Put

$$A = \{n \mid \psi(n) \text{ holds in } L\};$$

here n ranges over ω. We show how to effectively construct from ψ a Σ_3^1 predicate $C(n)$ and a Π_3^1 predicate $D(n)$ such that for all $n \in \omega$,

$$n \in A \equiv C(n) \equiv D(n).$$

The proof of Corollary 2.10 is effective, as can be seen by inspection. (Recall that Lemma 2.8 is proved in §4 and Lemma 2.10 in [8, §5.2]. It is necessary to inspect these proofs as well.) Thus, given a Gödel number, e, of A in $O^{\#}$, one can effectively construct the desired predicates C and D.

It remains to compute the Gödel number e from ψ. The proof of Theorem 4 in 2.15 shows that A is recursive in $O^{\#}$ but it is not effective. (There is no way to compute $\phi(A)$.)

We get around this difficulty using Corollary 2.6. Namely, let r be the Gödel number of the formula

$$(x)(x \in y \;.\!\equiv\!.\; x \in \omega \text{ and } \psi(x)).$$

Then

$$A = \{n \mid g(n) R h(r)\}.$$

From this description, it is easy to compute a Gödel number of A in $O^{\#}$. (Here h is the function defined in the last paragraph of 2.6.)

A similar discussion can be given for Theorem 2. Effective versions of Theorem 1 and Theorem 3 follow trivially.

2.17. **Proof of Theorem 5.** We take A to be $O^{\#}$. We already know that $O^{\#}$ is Δ_3^1. Our proof of Theorem 3 shows that some set recursive in $O^{\#}$ is not constructible. A fortiori, $O^{\#}$ is not constructible.

Next let $A(\gamma)$ be the Π_2^1 predicate guaranteed by Lemma 2.8:

(3) $(\gamma)(A(\gamma) \equiv \gamma = O^{\#})$.

Proposition 1.1 shows that (3) relativizes to $L[O^{\#}]$. The proof of Lemma 2.9 now shows that $O^{\#}$ is Δ_3^1 in $L[O^{\#}]$.

It remains to show that $L[O^{\#}]$ has a well ordering, definable in $L[O^{\#}]$. In general, if $a \subseteq \omega$, $L[a]$ has a well ordering definable within $L[a]$ from a. But $O^{\#}$ is Δ_3^1 in $L[O^{\#}]$ and, a fortiori, is definable in $L[O^{\#}]$.

3. An axiomatization of $O^{\#}$.

3.0. **Introduction.** In this section, we give a series of axioms for the set of Gödel numbers, $O^{\#}$, and show that they characterize $O^{\#}$. The axioms are theorems of Silver [12]. It is fairly easy to extract from Silver's work analytical properties of $O^{\#}$ which are equivalent to the axioms. In this way, one gets a proof of Lemma 2.8.

We now describe the axioms on $O^{\#}$. First let η be an ordinal. We let $\Gamma(O^{\#}, \eta)$ be the elementary submodel of L generated by the cardinals

$$\{\aleph_{1+\alpha}; \alpha < \eta\}.$$

(The proof given below will use a different definition of $\Gamma(O^{\#}, \eta)$ from that given in this sketch.) Let

$$j_\eta: \eta \to \Gamma(O^{\#}, \eta)$$

be defined by $j_\eta(\alpha) = \aleph_{\alpha+1}$. Then the pair Γ, j has the following properties:

(b) $\Gamma(O^{\#}, \eta)$ is well founded for each ordinal η;

(c) Let $\aleph$ be an uncountable cardinal. Then the ordinals of $\Gamma(O^{\#}, \aleph)$ are order isomorphic to $\aleph$;

(d) The map j_η imbeds η as a closed subset of the ordinals of $\Gamma(O^{\#}, \eta)$.

(With the definition of $\Gamma(O^{\#}, \eta)$ just given, (b) and (d) are clear, but (c) is not.)

As the notation suggests, the model $\Gamma(O^{\#}, \eta)$ can be reconstructed (up to canonical isomorphism) from a knowledge of η and $O^{\#}$. (This will be discussed in 3.3.) There is a simple arithmetical criterion, (a), on a set of integers t such that for t satisfying (a), the model

$$\Gamma(t, \eta)$$

is defined for all ordinals η. The properties (a)–(d) of $O^{\#}$ are the categorical set of axioms for $O^{\#}$ referred to above.

It turns out that axiom (b) can be expressed as a Π^1_2 condition on $O^{\#}$. Moreover, there are arithmetical properties (c′) and (d′) such that for t satisfying (a),

(b) and (c) $\equiv$ (b) and (c′);

(b) and (c) and (d) $\equiv$ (b) and (c′) and (d′).

Thus (a) and (b) and (c′) and (d′) will give a Π^1_2 axiomatization of $O^{\#}$.

Once one tries to extract a Π^1_2 characterization of $O^{\#}$ from Silver's work, the answer practically leaps to the eye. This is a tribute to the power of Silver's ideas; the original Gaifman-Rowbottom style proof was much less transparent. All the results in this section are due to Silver. My goal has been to present [12] in enough detail to make Lemma 2.8 clear.

3.1. We first recall the definition of a Ramsey cardinal. We assume the reader is familiar with the notion of a relational system. Let $\mathscr{A} = \langle X; R_1, \ldots, R_n \rangle$ be a relational system. X is the universe of $\mathscr{A}$ and $R_1, \ldots, R_n$ are finitary relations on X. Let $\mathscr{L}_{\mathscr{A}}$ be the first order language associated to the relational type of $\mathscr{A}$; $\mathscr{L}_{\mathscr{A}}$ has an equality predicate $=$, together with a predicate P_i corresponding to each of the relations R_i of $\mathscr{A}$. If $\phi(x_1, \ldots, x_n)$ is a formula of $\mathscr{L}_{\mathscr{A}}$ containing at most

$x_1, \ldots, x_n$ free, and $y_1, \ldots, y_n$ is a sequence of elements of X, then $\phi(y_1, \ldots, y_n)$ has a definite truth value in $\mathscr{A}$ (2).

$\mathscr{A}$ is an *ordered relational system* if R_1 linearly orders X. In that case, we write $<$ in place of R_1 or P_1.

DEFINITION 1. Let X be a linearly ordered set. An n-tuple of elements of X,

$$\langle x_1, \ldots, x_n \rangle$$

is *ordered* if $x_i < x_j$ when $1 \leqq i < j \leqq n$.

DEFINITION 2. Let $\mathscr{A}$ be an ordered relational system with universe X. A subset Y of X is a set of indiscernibles for $\mathscr{A}$ if for every formula $\phi(x_1, \ldots, x_n)$ of $\mathscr{L}_{\mathscr{A}}$ and every pair of ordered n-tuples

$$\langle y_1, \ldots, y_n \rangle, \langle y_1', \ldots, y_n' \rangle$$

of elements of Y, we have

$$\phi(y_1, \ldots, y_n) \equiv \phi(y_1', \ldots, y_n').$$

REMARKS. 1. It is a theorem of [12] that the (true) uncountable cardinals form a set of indiscernibles for the constructible universe.

2. The following special case of Definition 3.1 is worth noting. Let y, y' be members of the set of indiscernibles Y, and let $\phi(x)$ be a formula of $\mathscr{L}_{\mathscr{A}}$. Then

$$\phi(y) \equiv \phi(y');$$

i.e., y and y' are indiscernible with regard to properties expressible in $\mathscr{L}_{\mathscr{A}}$.

Let κ and λ be infinite cardinals with $\kappa \geqq \lambda$. We say that

$$\kappa \to (\lambda)^{<\aleph_0}$$

if each ordered relational system whose universe has cardinal κ possesses a set of indiscernibles, Y, of cardinality λ. (This definition is equivalent to the one given in [4].)

DEFINITION 3. An infinite cardinal κ is Ramsey if

$$\kappa \to (\kappa)^{<\aleph_0}.$$

REMARKS. (1) An uncountable cardinal κ is measurable if there is a two valued measure

$$\mu: S(\kappa) \to \{0, 1\}$$

(here $S(\kappa)$ is the algebra of subsets of κ) such that: (1) the measure of any one-point set is zero; (2) the measure of κ is 1; (3) if $\mathscr{F}$ is a family of sets of measure zero, and $\mathscr{F}$ has cardinality less than κ, then

$$\mu\left(\bigcup \mathscr{F}\right) = 0,$$

(i.e., μ is κ-additive). Every measurable cardinal is Ramsey [3].

(2) The conscientious reader will detect an "abuse of language."

(2) Every Ramsey cardinal is strongly inaccessible [3]. (In fact, a Ramsey cardinal is weakly compact.)

(3) Silver has shown that the arguments of the present section can be modified so that the proof of Lemma 2.8 can be deduced from the existence of a cardinal κ such that

$$\kappa \to (\aleph_1)^{<\aleph_0}.$$

From now on, κ denotes a fixed Ramsey cardinal.

3.2. We first describe the property (a) mentioned in subsection 3.0. Let t be a set of integers. The property (a) is the conjunction of the following five conditions on t:

(1) If $n \in t$, n is the Gödel number of some sentence of $\mathscr{L}$.

(We shall not usually distinguish between a sentence and its Gödel number.)

(2) t is a complete consistent theory.

(3) t extends the theory $Z-F+V=L$.

(4) Let $\langle i_1, \ldots, i_n \rangle$ and $\langle j_1, \ldots, j_n \rangle$ be ordered (cf. Definition 3.1.1) n-tuples of positive integers. Let $\phi(x_1, \ldots, x_n)$ be a formula of $\mathscr{L}$ containing at most $x_1, \ldots, x_n$ free and not containing any of the c_i's. Then the sentence

$$\phi(c_{i_1}, \ldots, c_{i_n}) \equiv \phi(c_{j_1}, \ldots, c_{j_n})$$

lies in t. (In effect, (4) says that the c_i's are a set of indiscernibles. Cf. Definition 3.1.2.)

(5) The sentence

$$c_1 < c_2$$

lies in t. (Here $<$ is the ordering of L discussed in 2.1.)

The property (a) is clearly an arithmetical property of t.

Lemma 1. *The set* $O^\#$ *satisfies* (a).

Proof. For all the clauses except (4) this is clear from the definition of $O^\#$ (Definition 2.2). For (4), we quote the theorem of Silver that the uncountable cardinals are indiscernible in L.

We pick a model M for the theory t. Since M is a model of $Z-F+V=L$, the interpretation of $\mathscr{L}$ in M extends to an interpretation of $\mathscr{L}_\mu$, exactly as in 2.3. (We can do this even though the model M may not be well ordered by $<$. The point is that it is a theorem of $Z-F+V=L$ that "if there is a y such that $\phi(y)$, then there is a least such y (with respect to $<$).") It follows from clause (4) of (a) that the elements of M denoted by the c_i's form a set of indiscernibles for M. Thus the following lemma is clear.

Lemma 2. *Let* $\langle i_1, \ldots, i_n \rangle$ *and* $\langle j_1, \ldots, j_n \rangle$ *be ordered n-tuples of positive integers. Let* $\phi(c_{i_1}, \ldots, c_{i_n})$ *be a sentence of* $\mathscr{L}_\mu$ *containing at most* $c_{i_1}, \ldots, c_{i_n}$ *among the*

c_i's. Let $\phi(c_{j_1}, \ldots, c_{j_n})$ be the sentence resulting from $\phi(c_{i_1}, \ldots, c_{i_n})$ by making the indicated substitutions. Then the sentence

$$\phi(c_{i_1}, \ldots, c_{i_n}) \equiv \phi(c_{j_1}, \ldots, c_{j_n})$$

holds in M.

We introduce the following conventions. If we say "let $\phi(c_{i_1}, \ldots, c_{i_n})$ be a sentence of $\mathscr{L}_\mu$" then it is understood, first, that $\langle i_1, \ldots, i_n \rangle$ is an ordered n-tuple, and second, no other c_i than $c_{i_1}, \ldots, c_{i_n}$ appears in ϕ. The sentence $\phi(c_{j_1}, \ldots, c_{j_n})$ is the sentence resulting from $\phi(c_{i_1}, \ldots, c_{i_n})$ after simultaneously substituting c_{j_r} for c_{i_r} for $1 \leqq r \leqq n$. A similar remark applies to terms of $\mathscr{L}_\mu$ and to the language $\mathscr{L}_{A,\mu}$ to be constructed in a moment.

3.3. Let t be a set of integers satisfying (a). Let A be an ordered set. We are going to construct the following:

(1) A model $\Gamma(t, A)$ of $Z-F+V=L$.

(2) An order-preserving map

$$j\colon A \to |\Gamma(t, A)|.$$

($|\Gamma(t, A)|$ is the underlying set of $\Gamma(t, A)$.)

(Recall from 2.1 that each model of $Z-F+V=L$ is canonically ordered.)

The construction will have the following properties:

(a) $j[A]$ is a set of indiscernibles for $\Gamma(t, A)$.

(b) Let $\phi(x_1, \ldots, x_n)$ be a formula of $\mathscr{L}$, and $\langle a_1, \ldots, a_n \rangle$ an ordered n-tuple of elements of A. Then

$$\models_{\Gamma(t,A)} \phi(j(a_1), \ldots, j(a_n)) \equiv \phi(c_1, \ldots, c_n) \in t.$$

(Here "$\models_{\mathscr{A}} \phi$" means that ϕ holds in the relational system $\mathscr{A}$.)

(c) $\Gamma(t, A)$ is generated by $j[A]$. (Cf. Definition 2.5.)

It will be clear from our construction that the pair $(\Gamma(t, A), j)$ is determined up to canonical isomorphism by (a)–(c).

We first construct a language $\mathscr{L}_A$. $\mathscr{L}_A$ will be first order language with two two-place predicates, $\in$ and $=$, and for each $a \in A$ a constant c_a. We enlarge $\mathscr{L}_A$ to a language $\mathscr{L}_{A,\mu}$ with μ-terms, analogously to 2.3.

Now fix a model, M, of the theory t. We define an equivalence relation on the set of terms of $\mathscr{L}_{A,\mu}$ as follows. Let $\langle a_1, \ldots, a_n \rangle$ be an ordered n-tuple of elements of A, and $\langle i_1, \ldots, i_n \rangle$ an ordered n-tuple of positive integers. Then we put

$$f_1(c_{a_1}, \ldots, c_{a_n}) \equiv f_2(c_{a_1}, \ldots, c_{a_n})$$

iff the statement

$$f_1(c_{i_1}, \ldots, c_{i_n}) = f_2(c_{i_1}, \ldots, c_{i_n})$$

is valid in M. (We use the letter f to denote terms of languages such as $\mathscr{L}_\mu$. We think of f as a "Skolem function.")

By a previous observation, clause (4) of (a) makes the particular choice of an ordered n-tuple, $\langle i_1, \ldots, i_n \rangle$, irrelevant. Using this, it is easy to check that $\equiv$ is an equivalence relation.

The universe of $\Gamma(t, A)$, $|\Gamma(t, A)|$, will be the set of equivalence classes of terms of $\mathscr{L}_{A,\mu}$ under the equivalence relation just defined. We shall denote the equivalence class of the term f by $[f]$. The map

$$j: A \to |\Gamma(t, A)|$$

is given by the formula

$$j(a) = [c_a].$$

The ε-relation on $\Gamma(t, A)$ is determined in a similar way: We put

$$[f_1(c_{a_1}, \ldots, c_{a_n})] \in [f_2(c_{a_1}, \ldots, c_{a_n})]$$

iff

$$\models_M f_1(c_{i_1}, \ldots, c_{i_n}) \in f_2(c_{i_1}, \ldots, c_{i_n}).$$

It is not difficult to check that this definition is valid (i.e., that the various choices made are irrelevant).

LEMMA 1. *Let $\langle a_1, \ldots, a_n \rangle$ and $\langle i_1, \ldots, i_n \rangle$ be respectively ordered n-tuples from A and the positive integers. Then if $\phi(c_1, \ldots, c_n)$ is a formula of $\mathscr{L}_\mu$, we have*

$$\models_M \phi(c_{i_1}, \ldots, c_{i_n}) \equiv \models_{\Gamma(t,A)} \phi(c_{a_1}, \ldots, c_{a_n}).$$

Proof. Left to the reader. (The proof proceeds by induction on the number of logical operators in ϕ. In handling the quantifiers, μ-terms play a vital role.)

We remark next that the model $\Gamma(t, A)$ is independent of the choice of the model M. For if $\phi(c_{a_1}, \ldots, c_{a_n})$ is a sentence of $\mathscr{L}_{A,\mu}$, let $\phi'(c_{a_1}, \ldots, c_{a_n})$ be the formula of $\mathscr{L}_A$ resulting from eliminating μ-terms. (Cf. 2.4.) By Lemma 1, $\Gamma(t, A)$ is a model of $Z-F+V=L$. Thus the following sequences of statements are equivalent to one another.

(1) $\models_{\Gamma(t,A)} \phi(c_{a_1}, \ldots, c_{a_n})$;
(2) $\models_{\Gamma(t,A)} \phi'(c_{a_1}, \ldots, c_{a_n})$;
(3) $\models_M \phi'(c_1, \ldots, c_n)$;
(4) $\phi'(c_1, \ldots, c_n) \in t$.

Applying this observation to atomic ϕ, we see that $\Gamma(t, A)$ depends only on t and A.

We leave the verification of the properties (a)–(c) of $\Gamma(t, A)$ discussed above to the reader. They follow easily from Lemma 1. (The proof of (c) is similar to that of Lemma 2.5.)

3.4. We next discuss the functorial properties of the construction $\Gamma(t, A)$ in A. Let A, A' be ordered sets and $h: A \to A'$ an order-preserving map. We define a map

$$h_*: |\Gamma(t, A)| \to |\Gamma(t, A')|$$

by the formula:

$$h_*[f(c_{a_1}, \ldots, c_{a_n})] = [f(c_{h(a_1)}, \ldots, c_{h(a_n)})].$$

Using Lemma 3.3.1, it is not difficult to prove that h_* is well defined.

LEMMA. *The map h_* is an elementary embedding. The following diagram is commutative*

$$\begin{array}{ccc} A & \xrightarrow{h} & A' \\ \downarrow{\scriptstyle j_A} & & \downarrow{\scriptstyle j_{A'}} \\ |\Gamma(t, A)| & \xrightarrow{h_*} & |\Gamma(t, A')| \end{array}$$

Proof. This follows immediately from Lemma 3.3.1.

Using this lemma, it is not difficult to show that

$$\{A \to \Gamma(t, A)\}$$

is a functor from the category of ordered sets (and order-preserving maps) to the category of models of set theory (and elementary embeddings).

3.5. Now let N be a model of $Z-F+V=L$, and let $X \subseteq |N|$ be an infinite set of indiscernibles. The set $|N|$ is canonically ordered (cf. 2.1) and we give X the induced ordering. We define a set of integers, t_X, as follows: t_X is the set of Gödel numbers of sentences $\phi(c_1, \ldots, c_n)$ of $\mathscr{L}$ such that if $\langle x_1, \ldots, x_n \rangle$ is an ordered n-tuple of elements of X, we have

$$\models_N \phi(x_1, \ldots, x_n).$$

Since X is a set of indiscernibles, the following lemma is clear.

LEMMA 1. *t_X has property* (a).

We now interpret $\mathscr{L}_{X,\mu}$ in N in the obvious way. (c_x denotes x, for $x \in X$.) In this way we get a map

$$\psi: |\Gamma(t_X, X)| \to |N|$$

by sending $[f(c_{x_1}, \ldots, c_{x_n})]$ into the element of N denoted by $f(c_{x_1}, \ldots, c_{x_n})$.

LEMMA 2. *The map ψ is an elementary embedding. Its image is the elementary submodel of N generated by X.*

(The proof is similar to the proof of Lemma 2.5.)

Using Lemma 2 it is not difficult to check that $\Gamma(O^{\#}, \eta)$ is canonically isomorphic to the elementary submodel of L generated by

$$\{\aleph_\alpha \mid 1 \leqq \alpha < 1+\eta\}.$$

(It is necessary to use Silver's result that the uncountable cardinals form a class of indiscernibles for L. If $\eta<\omega$, Lemma 2 does not quite apply, but the result is still true and easy to check.)

3.6. The following lemma says, in effect, that $\Gamma(t, A)$ is uniformly recursive in t, A. We consider the following situation: (1) t is a set of integers satisfying (a); (2) R is a linear ordering of ω. (We use ω_R as a notation for the ordered set $\langle\omega; R\rangle$.)

LEMMA. *There is a relation S on ω and a function $h\colon \omega\to\omega$ such that*

(1) *$\langle\omega; S\rangle$ is a model of $Z-F+V=L$,*

(2) *h induces, by passage to quotients, an isomorphism*

$$h_*\colon \Gamma(t, \omega_R) \simeq \langle\omega; S\rangle.$$

(We think of terms of $\mathscr{L}_{\omega,\mu}$ as being identified with their Gödel numbers. Then h_* is given by the formula

$$h_*([f]) = h(f)$$

for f a term of $\mathscr{L}_{\omega,\mu}$.)

(3) *S and h are uniformly recursive in t and R. (Note that the map*

$$h_*\circ j_{\omega_R}\colon \omega\to\omega$$

is uniformly recursive in t and A. In fact,

$$h_*\circ j_{\omega_R}(n) = h(c_n).)$$

Proof. (Similar to the proof of Lemma 2.6.)

3.7. The following lemma is the key step in proving that (b) expresses an analytical property of t.

LEMMA. *Let t be a set of integers satisfying* (a). *Then the following conditions are equivalent.*

(b) *For every ordinal λ, $\Gamma(t, \lambda)$ is well founded.*

(b′) *For every countable ordinal η, $\Gamma(t, \eta)$ is well founded.*

Proof. We sketch the proof and refer the reader to [12] for details. Suppose for some ordinal λ, we can find a decreasing sequence of ordinals $\{\alpha_n\}$ of the model $\Gamma(t, \lambda)$. We find a countable subset N of λ, with inclusion map $i\colon N\to\lambda$ such that for all n, α_n is in the image of the map

$$i_*\colon \Gamma(t, N)\to\Gamma(t, \lambda).$$

It follows that $\Gamma(t, N)$ is not well founded. Let η be the ordinal order isomorphic to N. Then η is countable and $\Gamma(t, \eta)$ is not well founded.

Let κ be a Ramsey cardinal and X a set of indiscernibles for $\langle L_\kappa; \in, =\rangle$ of power $\geqq\aleph_1$. Define t_X as in 3.5. Using Lemma 3.5.2, one checks easily that t_X has property (b′). It follows from the Lemma that $\Gamma(t_X, \eta)$ is well founded for all ordinals η.

3.8. Let A be an ordered set and t a set of integers satisfying (a) and (b). We put

$$On(t, A) = \{x \in |\Gamma(t, A)| : \models_{\Gamma(t,A)} x \text{ is an ordinal}\}.$$

If η is an ordinal, $On(t, \eta)$ is well ordered by (b). Let $|On(t, \eta)|$ be the ordinal order isomorphic to $On(t, \lambda)$.

Lemma. *Let t be a set of integers satisfying* (a) *and* (b). *Then the following properties of t are equivalent.*

(1) *For each uncountable cardinal $\aleph$, $|On(t, \aleph)| = \aleph$. (This is property* (c).)

(2) *We have $t = t_X$ for some set of indiscernibles, X, for*

$$\langle L_\kappa; \in, = \rangle$$

of power κ.

(3) *The following recursive set of sentences lies in t: (This is property* (c′).)

(i) *every sentence of the form*

$$f(c_1, \ldots, c_n) < c_{n+1}.$$

(*Here $f(c_1, \ldots, c_n)$ is a term of $\mathscr{L}_\mu$. Strictly speaking, the sentence associated to this by eliminating μ-terms should lie in t.*)

(ii) *Every sentence of the form*

$$f(c_1, \ldots, c_n, c_{n+i_1}, \ldots, c_{n+i_k}) < c_n \to f(c_1, \ldots, c_n, c_{n+i_1}, \ldots, c_{n+i_k})$$
$$= f(c_1, \ldots, c_n, c_{n+j_1}, \ldots, c_{n+j_k}).$$

Here $\langle i_1, \ldots, i_k \rangle$ and $\langle j_1, \ldots, j_k \rangle$ are ordered k-tuples of positive integers.

Proof. Again we sketch the proof and refer the reader to [12] for details.

(1) → (2). We have $\Gamma(t, \kappa) \simeq L_\kappa$ by (1). The isomorphism takes the image of j onto a set of indiscernibles for L_κ, say X. It is easy to see that $t = t_X$.

(2) → (3). We interpret $\mathscr{L}_{\kappa,\mu}$ in L_κ so that X is the set of elements denoted by the c_λ's. Consider first an element $f(c_1, \ldots, c_n)$. Surely for some $\eta < \kappa$, $f(c_1, \ldots, c_n) < c_\eta$ and $\eta > n$ (since X is unbounded in κ). Since the c_λ's are indiscernibles, we must have

$$f(c_1, \ldots, c_n) < c_{n+1}.$$

The proof that every sentence of type (ii) lies in t is more difficult. To illustrate the idea suppose

$$f(c_1, c_2) < c_1.$$

We shall show that

$$f(c_1, c_2) = f(c_1, c_3).$$

If $f(c_1, c_2) \neq f(c_1, c_3)$, then

$$\{f(c_1, c_\eta) : 1 < \eta < \kappa\}$$

would have power κ. This is absurd since

$$f(c_1, c_\eta) < c_1$$

and the set of elements of L_κ less than c_1 have power less than κ. Thus

$$f(c_1, c_2) = f(c_1, c_3).$$

(3) $\to$ (1) Let $\aleph$ be an uncountable cardinal. Then by (i), the set $\{c_\eta \mid \eta < \aleph\}$ forms a cofinal subset of length $\aleph$ of $\Gamma(t, \aleph)$. If $\eta < \aleph$, then if $x \in \Gamma(t, \aleph)$ and $x < c_\eta$, x can be written in the form $g(c_{\alpha_1}, \ldots, c_{\alpha_n}, c_{\eta+1}, \ldots, c_{\eta+k})$ where $\alpha_1 < \cdots < \alpha_n \leqq \eta$. It follows that there are fewer than $\aleph$ predecessors to c_η. So $|\Gamma(t, \aleph)|$ is an ordinal of power $\aleph$ such that every proper initial segment has power less than $\aleph$. Thus $|\Gamma(t, \aleph)| = \aleph$.

REMARK. Let $\aleph$ and $\aleph'$ be uncountable cardinals with $\aleph < \aleph'$. Let

$$i_*: \Gamma(t, \aleph) \to \Gamma(t, \aleph')$$

be the map induced by the inclusion of $\aleph$ in $\aleph'$. The arguments presented above show that i_* maps $\Gamma(t, \aleph)$ onto a proper segment of $\Gamma(t, \aleph')$ if t satisfies (a) and (b). It follows that the inclusion map

$$L_\aleph \to L_{\aleph'}$$

(which may be identified with i_*) is an elementary embedding. This is a result of Silver which we mentioned earlier.

In a similar way, one can show that the elementary embedding of $\Gamma(O^\#, \omega)$ into $\Gamma(O^\#, \aleph_1)$ induced by the inclusion of ω in $\aleph_1$ may be identified with the inclusion map

$$L_{\lambda_0} \to L_{\aleph_1}.$$

This is how Lemma 2.7 is proved.

3.9. Let κ be a Ramsey cardinal. Then there is a set of indiscernibles for L_κ consisting entirely of ordinals. Indeed let $X \subseteq L_\kappa$ be an arbitrary set of indiscernibles of power κ. Let $G: L_\kappa \simeq \kappa$ be as in 2.1. Then $\{G(x) : x \in X\}$ is a set of indiscernibles for L_κ of power κ which is a set of ordinals.

If $X \subseteq \kappa$ has power κ and $\lambda < \kappa$, it makes sense to speak of the λth member of X. Following Silver, we let X_0 be a set of ordinal indiscernibles for which the ωth element is as small as possible. Silver showed that t_{X_0} (which by earlier results satisfies (a)–(c)) has the following additional property:

(d′) (1) The sentence

$$\text{“}c_1 \text{ is an ordinal”}$$

is in t.

(2) Suppose that the sentences

$$\text{“}f(c_1, \ldots, c_{n+k}) \text{ is an ordinal”}$$

and

$$\text{“}f(c_1, \ldots, c_{n+k}) < c_{n+1}\text{”}$$

are in t. Then the sentence

$$f(c_1, \ldots, c_n, c_{n+i_1}, \ldots, c_{n+i_k}) = f(c_1, \ldots, c_n, c_{n+j_1}, \ldots, c_{n+j_k})$$

lies in t.

(Note that (d′) (2) is a strengthening of (c′) (ii) of 3.8 in a special case.)

The idea behind the proof of (d′) (2) is exhibited in the following special case. Let $\{x_\lambda\colon \lambda<\kappa\}$ be the elements of X_0 arranged in their natural order. Suppose that $f(c_1)$ is a term such that the statements "$f(c_1)$ is an ordinal" and "$f(c_1)<c_1$" lies in t_{X_0}. We show that the statement

$$f(c_1) = f(c_2)$$

also lies in t_{X_0}. This is equivalent to a special case of (d′).

If "$f(c_1)>f(c_2)$" lies in t_{X_0}, then $\{f(x_i)|i<\omega\}$ is a strictly decreasing sequence of ordinals, which is absurd. Thus

$$\text{“}f(c_1) \leqq f(c_2)\text{”}$$

lies in t_{X_0}. If "$f(c_1)<f(c_2)$" lies in t_{X_0}, then

$$\{f(x_\lambda)\colon \lambda < \kappa\}$$

would be a set of κ indiscernible ordinals whose ωth member is strictly smaller than the ωth element of X_0. This contradicts the definition of X_0. Thus the sentence

$$\text{“}f(c_1) = f(c_2)\text{”}$$

lies in t_{X_0}. (If we apply this to the term $f(c_1)=$"the cardinal of c_1," we conclude easily that the statement "c_1 is a cardinal" lies in t_{X_0}.)

Lemma. *Let t be a set of integers satisfying* (a)–(c). *Then the following are equivalent*:

(1) *t satisfies condition* (d′).

(2) *For every ordinal η, the image of $j\colon \eta \to |\Gamma(t, \eta)|$ is a closed subset of $On(t, \eta)$.*

(3) *$t=O\#$.* (*In particular, since t_{X_0} satisfies* (a)–(c) *and* (d′), *it follows that $t_{X_0}=O\#$ so $O\#$ satisfies property* (c).)

Proof. Once again, the result is essentially contained in Silver's work so I omit some details.

(1) $\rightarrow$ (2). We interpret $\mathscr{L}_{\eta,\mu}$ in $\Gamma(t, \eta)$ in the obvious way. Let λ be a limit ordinal less than η. We must show that

$$j(\lambda) = \text{l.u.b. } \{j(\theta)\colon \theta < \lambda\}$$

where the l.u.b. is computed in $On(t, \eta)$. Suppose that this is not so. Then for some term $f(c_{\alpha_1}, \ldots, c_{\alpha_k})$, the following sentences hold in $\Gamma(t, \eta)$:

(1) For each $\theta < \lambda$, $c_\theta < f(c_{\alpha_1}, \ldots, c_{\alpha_k})$;

(2) $f(c_{\alpha_1}, \ldots, c_{\alpha_k}) < c_\lambda$;

(3) $f(c_{\alpha_1}, \ldots, c_{\alpha_k})$ is an ordinal.

Using part (2) of condition (d′), and the fact that λ is a limit ordinal, one can show the following:

There are ordinals $\beta_1 < \beta_2 < \beta_3 < \cdots < \beta_k < \lambda$ such that

(4) $f(c_{\alpha_1}, \ldots, c_{\alpha_k}) = f(c_{\beta_1}, \ldots, c_{\beta_k})$.

But (2) and (4) yield

$$c_{\beta_k+1} < f(c_{\beta_1}, \ldots, c_{\beta_k}).$$

This contradicts the fact that t has property (c). (Cf. clause (i) of the definition of property (c′) in Lemma 3.8.)

(2) $\rightarrow$ (3). By property (c), there is an isomorphism

$$\Psi\colon \Gamma(t, \kappa) \simeq \langle L_\kappa; \varepsilon\rangle.$$

Let θ be the image of κ under $\Psi \circ j$. By (2), θ is a closed subset of On of power κ. Moreover the proof that (3) $\rightarrow$ (1) in Lemma 3.8 establishes the following: Let η be an infinite ordinal less than κ. Let λ be the cardinal of η. Then there are at most λ ordinals less than $\Psi \circ j(\eta)$. It follows that if $\aleph$ is an uncountable cardinal less than κ, and $\eta < \aleph$, then

$$\eta \leqq \Psi \circ j(\eta) < \aleph.$$

Since θ is closed, we have $\aleph$ in θ. Thus θ contains all uncountable cardinals. (This is the argument Silver uses to show that the uncountable cardinals form a set of indiscernibles.)

Now let $\phi(c_1, \ldots, c_n)$ be a sentence of $\mathscr{L}$. Since θ is the image of $j[\kappa]$ under the isomorphism Ψ, we have

$$(c_1, \ldots, c_n) \in t \equiv (\aleph_1, \ldots, \aleph_n) \text{ holds in } \langle L_\kappa, \varepsilon\rangle.$$

Since the inclusion map $i\colon L_\kappa \rightarrow L$ is an elementary embedding, we have

$$\phi(c_1, \ldots, c_n) \in t \equiv \models_L \phi(\aleph_1, \ldots, \aleph_n).$$

In other words, $t = O^\#$.

(3) $\rightarrow$ (1). We have to show that $O^\#$ satisfies (a), (b), (c), and (d′). Let X_0 be the set of indiscernibles introduced above. By Lemma 3.8, t_{X_0} satisfies (a)–(c). By the result of Silver previously cited, t_{X_0} satisfies (d′).

It follows now from the part of the Lemma already proved that $t_{X_0}=O\#$. Thus if $t=O\#$, $t=t_{X_0}$ so t has property (d′).

4. **Proof of Lemma 2.8.**

4.1. The following statement is an immediate consequence of Lemmas 3.7, 3.8, and 3.9.

LEMMA. *A set of integers t is equal to $O\#$ iff it satisfies conditions* (a), (b′), (c′), *and* (d′).

Conditions (a), (c′), and (d′) are arithmetical properties of t. To complete the proof of Lemma 2.8 it suffices to show that condition (b′) is Π_2^1.

4.2. If R is a set of integers, let $\hat{R}$ be the following binary relation on ω:

$$\hat{R} = \{\langle m, n\rangle : 2^m 3^n \in \omega\}.$$

(We say that $\hat{R}$ is the binary relation determined by R.)

It is well known (cf. [13]) that there is a Π_1^1 predicate $P_1(R)$ such that

$$P_1(R) \equiv \hat{R} \text{ is a well ordering of } \omega.$$

4.3. Lemma 3.6 can be used to construct an arithmetic predicate $B_1(A, t, m, n)$ with the following property: If t has property (a) and A determines a linear ordering $\hat{A}$ of ω, then

$$\{\langle m, n\rangle : B_1(A, t, m, n)\}$$

is a linear ordering of ω order isomorphic to $\Gamma(t, \hat{A})$.

It follows that there is a Π_1^1 predicate $P_2(t, A)$ such that if t has property (a) and $\hat{A}$ linearly orders ω,

$$P_2(t, A) \equiv \Gamma(t, A) \text{ is well founded.}$$

4.4. By Lemma 3.4, $\Gamma(t, n)$ is isomorphic to an elementary submodel of $\Gamma(t, \omega)$. Thus (b′) is equivalent to the following proposition (for t satisfying (a)):

$$(A)[P_1(A) \rightarrow P_2(t, A)].$$

4.5. We write $\Pi_1^1(A)$, for example, to indicate a Π_1^1 predicate containing the variable A free. The reader may verify that (b′) is equivalent to a predicate of each of the types listed below:

$$(A)[\Pi_1^1(A) \rightarrow \Pi_1^1(t, A)]$$

$$(A)[\Sigma_1^1(A) \vee \Pi_1^1(t, A)]$$

$$(A)\Pi_2^1(t, A)$$

$$\Pi_2^1(t).$$

Thus (b′) is Π_2^1. This completes the proof of Lemma 2.8.

5. **Proof of Theorem 6.**

5.1. We shall need the relative version of the results of §§1–4. Let $a \subseteq \omega$ be a set of integers. Let $L[a]$ be the class of sets constructible from a. One can define a function $F_a(\lambda)$ in close analogy with the function $F(\lambda)$ used to enumerate the constructible sets. (For example, one can modify Gödel's definition of F by introducing, as a new fundamental operation, the intersection of a set with a.)

We have

$$L[a] = \{F_a(\gamma) \mid \gamma \in On\}.$$

If λ is an ordinal, we put $L_\lambda[a] = \{F_a(\gamma) \mid \gamma < \lambda\}$.

We introduce a first-order language $\mathscr{L}'$ as follows. The predicates of $\mathscr{L}'$ are $\in$, $=$, and a unary predicate A. For each positive integer i, there is a constant, c_i. We interpret $\mathscr{L}'$ as follows. The variables of $\mathscr{L}'$ shall range over $L[a]$. The predicates $\in$ and $=$ shall have their usual meaning. We interpret the predicate A so that

$$Ax \equiv x \in a \qquad (\text{for } x \in L[a]).$$

Finally, we let c_i denote the (true) cardinal $\aleph_i$.

Definition 1. $a^\#$ is the set of Gödel numbers of true sentences of $\mathscr{L}'$ (under the interpretation just given).

As we indicated above, all of the results proved in §§1–4 have relativizations to results about $L[a]$. In 5.2, we list the relativizations that we shall use below. For the most part, relativizing the proofs is routine; we discuss one slightly tricky point in 5.3.

(We remark that if 0 is the empty set, then $0^\#$ and $O^\#$ are recursive in each other. Thus, for all practical purposes, they can be identified.)

5.2. Lemma 1. *The set a is uniformly recursive in $a^\#$.*

Proof. Let $\Psi_n(x)$ be a recursive sequence of formulas with one free variable such that

$$\models_{L[a]} (x)(\Psi_n(x) \equiv x = n).$$

Then $n \in a$ iff the sentence

$$(\exists x)(\Psi_n(x) \ \& \ Ax)$$

lies in $a^\#$.

Lemma 2. *Let α be an ordinal definable in $L[a]$. Then α is countable in $L[a^\#]$.*

Proof. By relativizing the proof of Theorem 2 one sees in fact that α is recursive in $a^\#$.

Lemma 3. *There is a Π^1_2 predicate $R_1(x, y)$ such that*

$$R_1(a, b) \equiv b = a^\#.$$

Proof. This is just the relativized version of Lemma 2.8.

5.3. Let $\mathscr{M}$ be a transitive model of $Z-F+V=L$. Then one can define the function F within $\mathscr{M}$. Moreover, if λ is an ordinal of $\mathscr{M}$, then

$$F(\lambda) = F^{\mathscr{M}}(\lambda).$$

Moreover, if $\mathscr{M}_1$ and $\mathscr{M}_2$ are transitive models of $Z-F+V=L$ and $On_{\mathscr{M}_1}=On_{\mathscr{M}_2}$, then $\mathscr{M}_1=\mathscr{M}_2$. These facts played an important role in the proofs of results about L given in §§1–4.

The following theory, T_a will substitute for $Z-F+V=L$ in the proofs of the results about $L[a]$ quoted in 5.2. The theory T_a has as predicates $\in$, $=$, and a one place predicate A. In the following description, we use symbols for ω and the non-negative integers. These must be eliminated in some standard way to get the actual axioms in T_a.

Axioms for T_a. *Group* I.

(1) If ϕ is an axiom of $Z-F$, ϕ is an axiom of T_a.

(2) Let $n \in \omega$. Then if $n \in a$, An is an axiom of T_a; if $n \notin a$, then $\neg An$ is an axiom of T_a.

(3) The following is an axiom of T_a:

$$(\exists y)(y \subseteq \omega \wedge (x)(x \in y \leftrightarrow Ax)).$$

The axioms of Group I allow us to define the function F_a in the theory T_a. The following axiom completes the description of the theory T_a:

(4) $(\forall x)(\exists \lambda \in On)(x=F_a(\lambda))$.

If $\mathscr{M}$ is a transitive model of T_a, then for all ordinals $\lambda \in \mathscr{M}$, we have

$$F_a(\lambda) = F_a^{\mathscr{M}}(\lambda).$$

If $\mathscr{M}_1$ and $\mathscr{M}_2$ are transitive models of T_a and $On_{\mathscr{M}_1}=On_{\mathscr{M}_2}$ then $\mathscr{M}_1=\mathscr{M}_2$. (The proofs are quite similar to the proofs of the corresponding facts about $Z-F+V=L$.) Thus T_a can substitute for $Z-F+V=L$ in the proofs of results cited in 5.2.

5.4. We now describe the subset of $P(\omega)$ (the power set of ω) used to prove Theorem 6.

If $a \subseteq \omega$, let

$$a_0 = \{x \in \omega \mid 2x \in a\} \quad \text{and} \quad a_1 = \{x \in \omega \mid 2x+1 \in a\}.$$

Then the map

$$a \to \langle a_0, a_1 \rangle$$

sets up a 1-1 correspondence between $P(\omega)$ and $P(\omega) \times P(\omega)$. Let

$$W = \{a | a_1 \text{ is constructible from } (a_0)^{\#}\}.$$

We shall show that W is Δ_3^1 but W is not constructible from any set of integers b. This will establish Theorem 6.

(The definition of a subset of $P(\omega)$ being constructible from a given set of integers is given directly before the statement of Theorem 6 (in §1).)

5.5. We show first that W is Δ_3^1. Recall first that there is a Σ_2^1 predicate $R_2(x, y)$ such that

$$R_2(x, y) \equiv y \text{ is constructible from } x. \qquad \text{(Cf. [1].)}$$

We have

$$\begin{aligned} a \in W &\equiv (\exists b)(b = a_0^{\#} \;\&\; a_1 \text{ is constructible from } b) \\ &\equiv (\exists b)(R_1(a_0, b) \;\&\; R_2(b, a_1)) \\ &\equiv (\exists b)(\Pi_2^1(a, b) \;\&\; \Sigma_2^1(a, b)) \\ &\equiv \Sigma_3^1(a). \end{aligned}$$

Similarly,

$$\begin{aligned} a \in W &\equiv (\forall b)(R_1(a_0, b) \rightarrow R_2(b, a_1)) \\ &\equiv (\forall b)(\Pi_2^1(a, b) \rightarrow \Sigma_2^1(a, b)) \\ &\equiv (\forall b)(\Sigma_2^1(a, b)) \\ &\equiv \Pi_3^1(a). \end{aligned}$$

Thus we have proved the following lemma.

LEMMA. *W is a Δ_3^1 subset of $P(\omega)$.*

5.6. We now sketch the proof that W is not constructible from any set of integers a. The proofs will be based on Cohen's notion of a generic set of integers. (Cf. [2], [5].) It will turn out that there are sets of integers, b, generic over $L[a]$ such that b is constructible from $a^{\#}$; there are also b generic over $L[a]$ such that b is not constructible from $a^{\#}$.

Now suppose that W is constructible from the set of integers a. By definition this means there is an ordinal λ and a set-theoretical formula $\phi_1(x, y, z)$ such that

$$b \in W \equiv \models_{L[a,b]} \phi_1(a, b, \lambda).$$

Recalling the definition of W, one deduces the existence of a formula $\phi_2(x, y, z)$ such that

(1) $$b \text{ is constructible from } a^{\#} \equiv \models_{L[a,b]} \phi_2(a, b, \lambda).$$

(Take $\phi_2(x, y, z)$ to be the formula:

$$y \subseteq \omega \;\&\; x \subseteq \omega \;\&\; (\exists w)(w_0 = x \;\&\; w_1 = y \;\&\; \phi_1(x, w, z)).)$$

Now let b be generic over $L[a]$ and constructible from $a^{\#}$. By (1), we have

$$\models_{L[a][b]} \phi_2(a, b, \lambda).$$

Since b is generic there exists a condition P true of b which forces $\phi_2(a, b, \lambda)$:

(2) $$P \Vdash \phi_2(a, b, \lambda).$$

We now select b' generic over $L[a]$ such that (a) P is true of b' and (b) b' is not constructible from $a^{\#}$. In view of (2) and (a), we have

(3) $$\models_{L[a][b']} \phi_2(a, b', \lambda).$$

In view of (1), we conclude that b' is constructible from $a^{\#}$. But this contradicts property (b) of b'.

To complete the proof we shall do the following:

(1) Show how Cohen's techniques can be adapted to study extensions of $L[a]$. (To do this, it is necessary to assume the existence of Ramsey cardinals.)

(2) Prove the existence of b generic over $L[a]$ such that b is constructible from $a^{\#}$.

(3) Prove that if P is a condition, there is a set of integers b', generic over $L[a]$, such that (a) P is true of b' and (b) b' is not constructible from $a^{\#}$.

5.7. Let $\mathcal{M}=L[a]$. We seek to enlarge $\mathcal{M}$ by adding a set of integers b generic over $\mathcal{M}$. The present situation differs from that considered by Cohen in two respects: first the model $\mathcal{M}$ does not satisfy $V=L$; second, the model $\mathcal{M}$ is not countable, and in fact is a proper class.

In [9] it is shown how to adapt Cohen's method to an arbitrary countable standard model $\mathcal{M}$ of $Z-F$. Thus the fact that $V=L$ fails in $\mathcal{M}$ causes no problems.

We next show how to handle the uncountability of $\mathcal{M}$. The key tool will be Lemma 5.2.2.

The definition of forcing takes place within $\mathcal{M}$ and all the usual formal properties of forcing are true. (A condition on b is a finite consistent set of sentences of the form $n \in \mathbf{b}$, or $n \notin \mathbf{b}$. Here $\mathbf{b}$ is a symbol used to denote the set b to be added to $\mathcal{M}$.) Let $\mathcal{P}$ be the set of conditions. Following [15], we make the following definitions:

Definition 1. A subset X of $\mathcal{P}$ is *dense* if

(1) For all $P \in \mathcal{P}$, there exists $Q \supseteq P$ such that $Q \in X$;

(2) if $P \in X$, $Q \in \mathcal{P}$, and $P \subseteq Q$, then $Q \in X$.

Definition 2. An increasing sequence of conditions,

$$P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots$$

is *complete* if for each dense subset X of $\mathcal{P}$, *lying in* $\mathcal{M}$, we have

$$P_n \in X$$

for all sufficiently large n.

(It is not clear, a priori, that any complete sequences exist.)

Let $\{P_n\}$ be a complete sequence of conditions. We say that $\{P_n\}$ converges to the set of integers b if $m \in b$ iff the statement "$m \in \mathbf{b}$" appears in P_n for n sufficiently large. It is easy to see that every complete sequence converges to exactly one subset of ω.

DEFINITION 3. A set of integers b is generic over $\mathcal{M}$ if there is a complete sequence $\{P_n\}$ converging to b.

The connection between forcing and truth may be proved in the usual way. In particular we have the following lemma. (Cf. [15, §2].)

LEMMA. *Let $\{P_n\}$ be a complete sequence converging to a generic set of integers b. Let $\phi(x, y, z)$ be a set-theoretical formula (containing free at most x, y, and z). Let λ be an ordinal. If $\models_{\mathcal{M}[b]} \phi(a, b, \lambda)$, then for n sufficiently large we have $P_n \Vdash \phi(a, b, \lambda)$; conversely, if $P_n \Vdash \phi(a, b, \lambda)$ for some n, then $\models_{\mathcal{M}[b]} \phi(a, b, \lambda)$.*

5.8. LEMMA 1. *There are only countably many dense subsets of $\mathcal{P}$ lying in $\mathcal{M}$.*

Proof. Let α be the cardinality of the power set of $\mathcal{P}$, as computed in $\mathcal{M}$. By Lemma 5.2.2, α is countable. (In fact, $\alpha=(2^{\aleph_0})^{\mathcal{M}}$.)

LEMMA 2. *Let P be a condition, and let $\{b_i\}$ be a countable sequence of subsets of ω. Then there is a complete sequence $\{P_n\}$ converging to a generic set of integers b; moreover, we have $P_0=P$, and $b \neq b_i$ for any $i \in \omega$.*

Proof. By Lemma 1, we can enumerate the dense subsets of $\mathcal{P}$ in a sequence, $\{X_n\}$. We define $\{P_i\}$ inductively so that (1) $P_0=P$; (2) $P_{2n+1} \in X_n$ (possible by (1) of Definition 5.7.1); (3) For some integer r, P_{2n+2} decides whether or not $r \in b$, and

$$r \in b_n \equiv \text{“} r \notin \mathbf{b} \text{” is in } P_{2n+2}.$$

This sequence has all the desired properties.

LEMMA 3. *Let P be a condition. Then there exists b' generic over $L[a]$ such that (1) P is true of b'; (2) b' is not constructible from $a^{\#}$.*

Proof. By Lemma 5.2.2, there are only countably many subsets of ω in $L[a^{\#}]$. Thus Lemma 3 follows from Lemma 2.

LEMMA 4. *There is a set of integers b constructible from $a^{\#}$ but generic over $L[a]$.*

Proof. By Lemma 5.2.2, every ordinal definable in $L[a]$ is countable in $L[a^{\#}]$. Thus the construction of a complete sequence given by the proof of Lemma 2 can be carried out in $L[a^{\#}]$. This proves Lemma 4.

Lemma 5.7 and Lemmas 3 and 4 complete the proof of Theorem 6 sketched in 5.6.

REMARK. It is clear that the proof of Lemma 2 provides a general technique for constructing Cohen extensions of uncountable models $\mathcal{M}$. The technique applies when the family

$$\{X \mid X \in \mathcal{M} \text{ and } X \subseteq \mathcal{P}\}$$

is countable. (Here $\mathcal{P}$ is the set of conditions relevant to the class of Cohen extensions at hand.)

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University of California,
Berkeley, California