A SUFFICIENT CONDITION FOR TOTAL MONOTONICITY: CORRECTIONS

BY

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In the expression for μ_n in the proof of Theorem 5 [2, p. 315] the term $\Gamma(n+b)$ was incorrectly written as $\Gamma(n+\alpha)$. This paper corrects that mistake. I am indebted to Dr. S. K. Basu for bringing the error to my attention. The reader is referred to [2] for pertinent notation and definitions. Theorem 1 of this paper replaces Theorem 5 of [2].

THEOREM 1. (i) *For* $-1<\alpha<0$, $2b\geqq 3-\alpha$, C_b^α t.s. H^α.

(ii) *For* $-1<\alpha<0$, $1<b\leqq 1-\alpha$, H^α t.s. C_b^α.

(iii) *For* $0<\alpha<1$, $2b\geqq 3-\alpha$, H^α t.s. C_b^α.

(iv) *For* $\alpha>1$, $0<a\leqq(3-\alpha)/2$, C_a^α t.s. H^α.

Let

$$\mu(t) = \frac{\Gamma(b+\alpha)\Gamma(t+b)(t+1)^\alpha}{\Gamma(b)\Gamma(t+b+\alpha)} = e^{\sigma(t)}.$$

Using the series expansion for the logarithmic derivative of the gamma function,

$$\begin{aligned}(-1)\sigma'(t) &= \frac{\Gamma'(t+b+\alpha)}{\Gamma(t+b+\alpha)}-\frac{\Gamma'(t+b)}{\Gamma(t+b)}-\frac{\alpha}{t+1}\\ &= \sum_{m=0}^{\infty}\left(\frac{1}{m+t+1}-\frac{1}{m+t+b+\alpha}\right)\\ &\qquad -\sum_{m=0}^{\infty}\left(\frac{1}{m+t+1}-\frac{1}{m+t+b}\right)-\frac{\alpha}{t+1}\\ &= \sum_{m=0}^{\infty}\left(\frac{1}{m+t+b}-\frac{1}{m+t+b+\alpha}\right)-\alpha\sum_{m=0}^{\infty}\left(\frac{1}{m+t+1}-\frac{1}{m+t+2}\right)\end{aligned}\tag{1}$$

$$(-1)\sigma'(t) = \alpha\sum_{m=0}^{\infty} f_m(t)g_m(t),$$

where

$$1/g_m(t) = (m+t+b)(m+t+b+\alpha)(m+t+1)(m+t+2),\tag{2}$$

and

$$\begin{aligned}f_m(t) &= (m+t+1)(m+t+2)-(m+t+b)(m+t+b+\alpha)\\ &= (3-2b-\alpha)(m+t)+2-b(b+\alpha).\end{aligned}\tag{3}$$

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Replacing b in (3) by $(3-\alpha)/2$, then, from (1),

$$(4)\qquad (-1)\sigma'(t) = \frac{\alpha(\alpha^2-1)}{4}\sum_{m=0}^{\infty} g_m(t).$$

From (2) it is clear that $\operatorname{sgn} g_m^{(n)}(t)=(-1)^n$ for all $t\geqq 0$. From (4) we have, for all $t\geqq 0$,

$$(-1)^n\sigma^{(n)}(t) \geqq 0 \quad \text{for} \quad -1<\alpha<0 \text{ or } \alpha>1,$$
$$(-1)^{n+1}\sigma^{(n+1)}(t) \geqq 0 \quad \text{for} \quad 0<\alpha<1,$$

and (i), (iii), and (iv) have been proved for equality.

To finish the proof of (i), for $-1<\alpha<0$, $c\geqq b$, C_c^α t.s. C_b^α [2, p. 313, Theorem 2(ii)]. Since t.s. is transitive, (i) is now proved. The proofs of (iii) and (iv) are completed using [2, p. 313, Theorem 2(iii)].

To prove (ii), let $b=1-\alpha$ in (3). Then (4) becomes

$$(5)\qquad (-1)\sigma'(t) = \alpha(1+\alpha)\sum_{m=0}^{\infty} h_m(t),$$

where $h_m(t)=(m+t+1)g_m(t)$. Clearly $\operatorname{sgn} h_m^{(n)}(t)=(-1)^n$ for $t\geqq 0$. Therefore, from (5),

$$(-1)^{n+1}\sigma^{(n)}(t) \geqq 0 \quad \text{for all} \quad t\geqq 0,\ -1<\alpha<0;$$

i.e., H^α t.s. C_b^α for $b=1-\alpha$. The proof is completed by again using [2, p. 313, Theorem 2(ii)].

Theorem 1 leaves unanswered the question of other comparisons for the remaining parameter values of a and b. This question is settled by the negative results of the following theorem.

THEOREM 2. (i) *For* $-1<\alpha<0$, $1<b<(3-\alpha)/2$, C_b^α n.t.s. H^α.
(ii) *For* $-1<\alpha\leqq\gamma<0$, $b>1-\alpha$, H^γ n.t.s. C_b^α.
(iii) *For* $0<\alpha<1$, $1<b<(3-\alpha)/2$, H^α n.t.s. C_b^α.
(iv) *For* $0<\alpha\leqq\beta<1$, $b>1$, C_b^β n.t.s. H^α.
(v) *For* $\alpha>1$, $(3-\alpha)/2<a<1$, C_a^α n.t.s. H^α.
(vi) *For* $1<\alpha\leqq\beta$, $0<a<1$, H^β n.t.s. C_a^α.

We shall first prove (i), (iii), and (v). From (2), $g_m(t)>0$ for all $t\geqq 0$, $m=0, 1, 2,\ldots$. If we let $c=3-2b-\alpha$, then $c>0$ for the values of b stated in (i) and (iii) and $c<0$ for the values stated in (v), with b replaced by a. From (3), for all t sufficiently large, $f_n(t)>0$ for $|\alpha|<1$ and $f_n(t)<0$ for $\alpha>1$; $n=0, 1, 2,\ldots$. Therefore $(-1)\sigma'(t)$ in (1) is negative for $-1<\alpha<0$ or $\alpha>1$ and positive for $0<\alpha<1$.

To prove the remaining parts we shall show that there exists a positive integer k for which $(-1)^k\mu^{(k)}(t)<0$ for the moment function under consideration. The procedure will be to examine the coefficient of t^k in an infinite series expansion for $\mu(t)$ that is valid for $0<t<1$, and to use the fact that the sign of the kth derivative of $\mu(t)$ is determined by the sign of the coefficient of t^k for values of t sufficiently small.

Proof of (ii). Let $\beta=-\alpha$, $\delta=-\gamma$, $0<t<1$, and define

$$\mu(t) = \Gamma(b)\mu_1(t)\mu_2(t)/\Gamma(\beta)\Gamma(b-\beta),$$

where $\mu_2(t)=(1+t)^\delta$, and

$$\begin{aligned}\mu_1(t) &= \frac{\Gamma(t+b-\beta)\Gamma(\beta)}{\Gamma(t+\beta)} = \int_0^1 u^{t+b-\beta-1}(1-u)^{\beta-1}\,du \\ &= \int_0^1 \Big(u^{t+b-\beta-1}-(\beta-1)u^{t+b-\beta}+\cdots \\ &\qquad +\frac{(\beta-1)(\beta-2)\cdots(\beta-n)u^{t+b+n-\beta-1}}{n!}+\cdots\Big)\,du \\ &= \frac{1}{t+b-\beta}+\sum_{n=1}^{\infty}\frac{(1-\beta)(2-\beta)\cdots(n-\beta)}{n!(t+b-\beta+n)},\end{aligned}$$

the term by term integration being justified as in [1].

Expanding $(t+b-\beta+n)^{-1}$ in powers of t we have

$$\begin{aligned}\mu_1(t) &= \frac{1}{t+b-\beta+n}+\sum_{n=1}^{\infty}\frac{(1-\beta)(2-\beta)\cdots(n-\beta)}{n!}\sum_{k=0}^{\infty}\frac{(-1)^k t^k}{(b-\beta+n)^{k+1}} \\ &= \frac{1}{t+b-\beta+n}+\sum_{k=0}^{\infty}(-1)^k t^k\sum_{n=1}^{\infty}\frac{(1-\beta)(2-\beta)\cdots(n-\beta)}{n!(b-\beta+n)^{k+1}}.\end{aligned}$$

The inversion of the order of summation can be justified as in [1, p. 455].

Now let

$$C_{k+1}(\beta) = \sum_{n=1}^{\infty}\frac{(1-\beta)(2-\beta)\cdots(n-\beta)}{n!(b-\beta+n)^{k+1}} \qquad (k = 0, 1, 2, \ldots),$$

and expand $(t+b-\beta)^{-1}$ in powers of t to obtain

$$\mu_1(t) = \sum_{k=0}^{\infty}(-1)^k[(b-\beta)^{-k-1}+C_{k+1}(\beta)]t^k.$$

If we expand $\mu_2(t)$ in powers of t, then the coefficients of t^k in the power series expansion for the product is

$$\begin{aligned}&(-1)^k[(b-\beta)^{-k-1}+C_{k+1}(\beta)]+(-1)^{k-1}[(b-\beta)^{-k}+C_k(\beta)]\delta \\ &\qquad +\sum_{r=2}^{k}(-1)^{k-r}[(b-\beta)^{-k+r-1}+C_{k-r+1}(\beta)]\frac{(-1)^{r-1}\delta(1-\delta)\cdots(r-1-\delta)}{r!} \\ &= (-1)^k(b-\beta)^{-k-1}\Big[1-(b-\beta)-\delta\sum_{r=2}^{k}\frac{(1-\delta)(2-\delta)\cdots(r-1-\delta)(b-\beta)^r}{r!} \\ &\qquad +(b-\beta)^{k+1}\Big\{C_{k+1}(\beta)-\delta C_k(\beta)-\delta\sum_{r=2}^{k}\frac{(1-\delta)\cdots(r-1-\delta)C_{k-r+1}(\beta)}{r!}\Big\}\Big].\end{aligned}$$

Since $b-\beta=b+\alpha>1$ by hypothesis, the first series diverges. Since $0<\beta<1$, $(b-\beta)^{k+1}C_{k+1}(\beta)$ is uniformly bounded in k. If the second series converges, then the quantity in brackets is negative for all k sufficiently large. If the second series diverges then, a fortiori, the quantity in brackets is negative for all k sufficiently large.

Proof of (iv). For $0<t<1$ now define

$$\mu(t) = \frac{\Gamma(b+\beta)\Gamma(t+b)(t+1)^\alpha}{\Gamma(b)\Gamma(t+b+\beta)} = \frac{\Gamma(b+\beta)}{\Gamma(b)}\mu_1(t)\mu_2(t),$$

where

$$\mu_2(t) = (1+t)^\alpha$$

and

$$\mu_1(t) = \frac{\Gamma(b)\Gamma(t+b)}{\Gamma(t+b+\beta)} = \int_0^1 u^{t+b-1}(1-u)^{\beta-1}\,du.$$

Using the same procedure as in the proof of (ii), we may write

$$\mu_1(t) = \sum_{k=0}^{\infty} (-1)^k[b^{-k-1}+d_{k+1}(\beta)]t^k,$$

where

$$d_{k+1}(\beta) = \sum_{n=1}^{\infty} \frac{(1-\beta)(2-\beta)\cdots(n-\beta)}{n!(b+n)^{k+1}}.$$

Expanding $\mu_2(t)$ in powers of t, the coefficient of t^k in the power series expansion for the product can be written in the form

$$(-1)^k b^{-k-1}\left[1-b\alpha-\alpha\sum_{r=2}^{k}\frac{(1-\alpha)(2-\alpha)\cdots(r-1-\alpha)b^r}{r!}\right.$$
$$\left.+b^{k+1}\left\{d_{k+1}(\beta)-\alpha d_k(\beta)-\alpha\sum_{r=2}^{k}\frac{(1-\alpha)(2-\alpha)\cdots(r-1-\alpha)}{r!}d_{k-r+1}(\beta)\right\}\right].$$

Since $b>1$, the first series diverges. Since $1-\beta>0$, $b^{k+1}d_{k+1}(\beta)$ is uniformly bounded in k. Whether or not the second series converges or diverges the quantity in brackets will be negative for all values of k sufficiently large.

Proof of (vi). We first prove that H^β n.t.s. C_a^1 for $\beta>1$, $0<a<1$. Note that $C_a^1=\Gamma_a^1$. Let

$$\mu(t) = (t+a)/a(t+1)^\beta.$$

Then, for $0<t<1$,

$$\mu(t) = (1+t/a)(1+t)^{-\beta}$$
$$= (1+t/a)\left[1+\sum_{k=1}^{\infty}\frac{(-1)^k\beta(\beta+1)\cdots(\beta+k-1)t^k}{k!}\right].$$

For $k>1$, the coefficient of t^k is

$$\frac{(-1)^k\beta(\beta+1)\cdots(\beta+k-2)}{k!}\,[\beta+k-1-k/a].$$

Since $\beta>1$, $a<1$, the quantity in brackets will be negative for all k sufficiently large, and $\mu(t)$ is not totally monotone.

Now suppose H^β t.s. C^α_a for $1<a\leqq\beta$. From [2, p. 313, Theorem 2(i)], C^α_a t.s. C^1_a for $\alpha>1$. Since t.s. is transitive, H^β t.s. C^1_a, a contradiction.

We conclude by listing a new total comparison table to replace the one on p. 316 of [2]. The arrow points toward the weaker method. Let $-1<\alpha<0$, $0<a<1$ and such that $a+\alpha>0$, $0<a''\leqq(\alpha+1)/2<1+\alpha\leqq a'<1$, $1<b\leqq1-\alpha<(3-\alpha)/2\leqq b'$. Then

$$\begin{tikzcd}
& & & \Gamma^\alpha_{a'} \arrow[dr] & & \\
\Gamma^\alpha_{b'} \arrow[r] & C^\alpha_{b'} \arrow[r] & H^\alpha \arrow[ur] \arrow[r] \arrow[dr] & C^\alpha_b \arrow[r] & C^\alpha_1 \arrow[r] \arrow[dr] & \Gamma^\alpha_{a''} \\
& & & \Gamma^\alpha_a \arrow[rr] & & C^\alpha_a
\end{tikzcd}$$

If $a''\geqq a$, then, of course $\Gamma^\alpha_{a''}$ t.s. C^α_a. If $a''<a$, then $\Gamma^\alpha_{a''}$ and C^α_a are not totally comparable.

Let $0<\alpha<1$, $0<a\leqq(\alpha+1)/2<1<(3-\alpha)/2\leqq b$. Then

$$\begin{tikzcd}
C^\alpha_a \arrow[r] & \Gamma^\alpha_a \arrow[r] & C^\alpha_1 \arrow[r] \arrow[dr] & H^\alpha \arrow[d] \arrow[dr] & \\
& & & C^\alpha_b \arrow[r] & \Gamma^\alpha_b
\end{tikzcd}$$

Let $\alpha>1$, $0<a\leqq(3-\alpha)/2<1<\alpha+1\leqq2b$. Then

$$\begin{tikzcd}
\Gamma^\alpha_a \arrow[r] \arrow[dr] & H^\alpha \arrow[r] & C^\alpha_1 \arrow[r] & \Gamma^\alpha_b \arrow[r] & C^\alpha_b \\
& C^\alpha_a \arrow[u] \arrow[ur] & & &
\end{tikzcd}$$

References

1. S. K. Basu, *On the total relative strength of the Hölder and Cesàro methods*, Proc. London Math. Soc. **50** (1948–1949), 447–462.

2. B. E. Rhoades, *A sufficient condition for total monotonicity*, Trans. Amer. Math. Soc. **107** (1963), 309–319.

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