NONAUTONOMOUS DIFFERENTIAL EQUATIONS AND TOPOLOGICAL DYNAMICS. I. THE BASIC THEORY

BY
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1. **Introduction.** Probably one of the greatest contributions Professor George Birkhoff made to mathematics was his work on the theory of dynamical systems [5]. His work in the early part of this century formed the foundation of the modern (axiomatic) theory of topological dynamics [8], [15], [18]. It has long been realized that this theory has direct and important applications to autonomous differential equations, in fact, it is precisely this observation that underlies the important work of Professor Birkhoff.

It has been observed (see [18], for example) that the solutions of a nonautonomous ordinary differential equation can be viewed as a dynamical system by imbedding the given differential equation in a higher dimensional phase space and treating the independent variable as a new coordinate. While this construction is valid, it has the effect of destroying some of the latent structure of the original equation. For example, the new equation will not have any bounded motions, nor any periodic motions, nor any almost periodic motions. Because of this fact, the theory of topological dynamics has not developed into a powerful technique in applications to nonautonomous equations.

Recently however, some authors (L. G. Deysach and G. R. Sell [7], R. K. Miller [17], G. R. Sell [23]) have shown that applications of topological dynamics are possible when treating nonautonomous differential equations that are either periodic, or almost periodic, in t. In addition L. Markus [14] and Z. Opial [20] have used some techniques of topological dynamics to discuss the asymptotic behavior of solutions of equations that are "asymptotically autonomous" or "nearly autonomous." J. P. LaSalle [13] and R. K. Miller [16] have used the concept of the limit set for solutions of periodic and almost periodic equations. In this paper we shall show that there is a way of viewing the solutions of a nonautonomous differential equation as a dynamical system. The above results are included and generalized in this context. We shall see that this viewpoint is very general and includes all differential equations satisfying only the weakest hypotheses.

In the present paper we shall develop the basic theory for viewing the solutions of nonautonomous equations as dynamical systems. In the following paper [24], we shall introduce the concept of the "set of limiting equations" for a given

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differential equation and shall investigate some of the implications of the theory of topological dynamics in this setting.

Let us briefly describe the results of this paper. We shall consider differential equations of the form

$$(1) \qquad x' = f(x, t) \qquad (x' = dx/dt),$$

where f: $W \times R \to R^n$ is continuous, W is open in R^n, R is the set of real members and (1) satisfies some uniqueness condition. In order to define a dynamical system, one still wants to enlarge the phase space. However, it appears to be more natural to use a technique suggested by R. K. Miller [17]. Instead of treating $W \times R$ as the new phase space, one can consider the product $W \times \mathfrak{F}$, where $\mathfrak{F}$ is the function space consisting of all translates of f. (That is, a typical element of $\mathfrak{F}$ would be f_τ where $f_\tau(x, t) = f(x, \tau + t)$.) By introducing an appropriate topology on $\mathfrak{F}$, one can define a "local dynamical system" on $W \times \mathfrak{F}$. If the solutions of (1) are defined for all time t, this becomes a dynamical system on $W \times \mathfrak{F}$. Moreover, this dynamical system does not have the defects mentioned above. That is, under appropriate restrictions on f, the dynamical system will now have "bounded motions," as well as periodic or almost periodic motions.

In order to use the theory of α- and ω-limit sets, we introduce the concept of the "hull" of f, which we denote by $\mathfrak{F}^*$. (This is the closure of $\mathfrak{F}$ in an appropriate topological space.) We show that the original local dynamical system on $W \times \mathfrak{F}$ can be extended to $W \times \mathfrak{F}^*$ if and only if f satisfies a certain regularity condition.

The projection of the dynamical system on $W \times \mathfrak{F}$ onto $\mathfrak{F}$ defines a dynamical system on $\mathfrak{F}$. We conclude this paper by investigating various topologies on $\mathfrak{F}$ and the relationships with topological dynamics. This study is a generalization of some work of L. Auslander and F. Hahn [2], and V. V. Nemyckiĭ [19].

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2. **Basic definitions and notation.** In this section we recall some basic concepts from the theory of topological dynamics. Let X be a topological space and let R denote the real numbers, with the usual topology. A *dynamical system* on X is defined to be a mapping

$$\pi\colon X \times R \to X$$

that satisfies the following properties:

(i) $\pi(p, 0) = p, \quad (p \in X)$;

(ii) $\pi(\pi(p, \tau), \sigma) = \pi(p, \tau + \sigma), \quad (p \in X, \tau \in R, \sigma \in R)$;

(iii) π is continuous.

If we let π_t: $X \to X$ denote the mapping defined by

$$\pi_t(p) = \pi(p, t),$$

then $\{\pi_t : t \in R\}$ is a group of homeomorphisms of X with $\pi_0 = 1$ and $(\pi_t)^{-1} = \pi_{-t}$.

Although the theory of dynamical systems has been greatly developed in this general context (see [8] and [15]), we will only be concerned with metrizable topological spaces in this paper. Therefore, we shall now assume that X is a metric space and let d denote the metric on X.

For p fixed, the mapping $\pi(p, t)$: $R \to X$ is defined to be the *motion through* p and $\gamma(p) = \{\pi(p, t) : t \in R\}$ is the *trajectory through* p. Similarly,

$$\gamma^+(p) = \{\pi(p, t) : t \geqq 0\} \quad \text{and} \quad \gamma^-(p) = \{\pi(p, t) : t \leqq 0\}$$

are defined to be, respectively, the *positive* and *negative semitrajectory through* p. The *limit sets* of a motion $\pi(p, t)$ are defined by

$$\Omega_p = \bigcap_\tau \mathrm{Cl}\, \gamma^+(\pi(p, \tau)), \qquad A_p = \bigcap_\tau \mathrm{Cl}\, \gamma^-(\pi(p, \tau)),$$

where Cl denotes the closure operation on X. A point q is in Ω_p {or A_p} if and only if there is a sequence $\{t_n\}$ in R with $t_n \to +\infty$ {or, respectively, $t_n \to -\infty$} and $\pi(p, t_n) \to q$.

A motion $\pi(p, t)$ is said to be *positively compact* {or, *negatively compact*} if $\gamma^+(p)$ {or, respectively, $\gamma^-(p)$} lies in a compact subset in X. The motion is *compact* if $\gamma(p)$ lies in a compact subset in X. To say that a motion $\pi(p, t)$ is compact does not mean that the trajectory $\gamma(p)$ is compact, but rather that $\mathrm{Cl}\, \gamma(p)$ is a compact set. (Compactness, in this sense, is the same as "Lagrange-stability" as used in [7], [18], [24].) If $\pi(p, t)$ is positively compact, then it is known (see [18, pp. 340–342]) that Ω_p is nonempty, compact and connected.

A motion $\pi(p, t)$ is said to be *positively Poisson-stable* {or *negatively Poisson-stable*} if $p \in \Omega_p$ {or, respectively, $p \in A_p$}. The motion $\pi(p, t)$ is said to be *Poisson-stable* if it is both positively and negatively Poisson-stable.

A motion $\pi(p, t)$ is said to be *recurrent* if for every $\varepsilon > 0$ there is an $L > 0$ such that for every t in R and every interval I of R of length greater than L, there is an s in I such that $d(\pi(p, t), \pi(p, s)) < \varepsilon$. It is known (see [18, p. 378]) that a positively compact motion $\pi(p, t)$ is recurrent if and only if for every $\varepsilon > 0$, the set

$$\{\tau \in R : d(p, \pi(p, \tau)) < \varepsilon\}$$

is relatively dense in R. A motion $\pi(p, t)$ is said to be *almost periodic* if for every $\varepsilon > 0$ the set

$$\{\tau \in R : d(\pi(p, t), \pi(p, \tau + t)) < \varepsilon \text{ for all } t \text{ in } R\}$$

is relatively dense in R. It is known (see [18, pp. 384–385]) that every almost periodic motion (in a complete space X) is compact and recurrent. A motion $\pi(p, t)$ is said to be *periodic* (with period $T > 0$) if $\pi(p, t) = \pi(p, t + T)$ for all t in R.

A set $E \subset X$ is said to be *positively invariant* {or *invariant*} if $\gamma^+(E) \subset E$ {or, respectively, $\gamma(E) \subset E$}. A set $E \subset X$ is said to be *minimal* if E is nonempty, closed, invariant and contains no proper subset with these three properties. G. D. Birkhoff [5] proved that if X is complete, then a set E is a compact, minimal set if and only if $E=\mathrm{Cl}\,\gamma(p)$, where $\pi(p, t)$ is a compact, recurrent motion. We shall say that a set E is an *almost periodic-minimal set* (or, briefly an *a.p. minimal set*) if $E=\mathrm{Cl}\,\gamma(p)$, where $\pi(p, t)$ is an almost periodic motion. Also, E is *periodic minimal* if $E=\mathrm{Cl}\,\gamma(p)$, where $\pi(p, t)$ is a periodic motion.

We conclude this section by recalling two concepts of stability defined in [7] and [24]. A motion $\pi(p, t)$ is said to be *uniformly positively Lyapunov-stable* with respect to a set $D\subset X$ if for every $\varepsilon>0$, there is a $\delta>0$ such that

$$d(\pi(q, t), \pi(\hat{q}, t)) < \varepsilon \qquad (t \geqq 0),$$

whenever $q \in \gamma^+(p)$, $\hat{q} \in D$ and $d(q, \hat{q})<\delta$. A motion $\pi(p, t)$ is said to be *asymptotically stable* with respect to D if (i) it is uniformly positively Lyapunov-stable with respect to D and (ii) there is an $\eta_0>0$ with the property that for every q in D with $d(q, \gamma^+(p))<\eta_0$ there is a τ in R such that

$$d(\pi(q, t), \pi(p, \tau+t)) \to 0 \qquad (\text{as } t \to \infty).$$

3. **Construction of the dynamical system.** A. *The basic construction.* Let W be an open set in R^n, Euclidean n-space. The Euclidean norm on R^n will be denoted by $|x|$. We shall say that a function f: $W\times R \to R^n$ is *admissible* if

(i) f is continuous, and

(ii) the solutions of the differential equation $x'=f(x, t)$ are unique.

By the second condition we mean that given any point (x_0, t_0) in $W\times R$, there is precisely one solution ϕ of $x'=f(x, t)$ that satisfies $\phi(t_0)=x_0$.

In this section we shall be interested in only those admissible functions f that satisfy the global existence property. The purpose of this is to explain the basic construction while keeping the number of details at a minimum. The construction for the general case will be presented in §5.

DEFINITION. We shall say that an admissible function f in $\mathfrak{C}(W\times R, R^n)$ satisfies the *global existence property* if every solution of $x'=f(x, t)$ can be continued for all time t in R.

Let $\mathfrak{C}=\mathfrak{C}(W\times R, R^n)$ denote the set of all continuous functions f defined on $W\times R$ with values in R^n. It is evident that if $f\in\mathfrak{C}$ is an admissible function, then every translate f_τ of f (where $f_\tau(x, t)=f(x, \tau+t)$) is an admissible function. Also, if f satisfies the global existence property, then so does each f_τ. Now let $\mathfrak{F}=\{f_\tau : \tau \in R\}$ be the space of translates of f, then $\mathfrak{F}$ is a subset of $\mathfrak{C}$. If $\{\tau_n\}$ is any sequence in R with $\tau_n \to \tau$, then for each (x, t) in $W\times R$ one has

$$f_{\tau_n}(x, t) = f(x, \tau_n+t) \to f(x, \tau+t) = f_\tau(x, t), \tag{2}$$

since f is continuous. Furthermore, since f is continuous on $W\times R$, it is uniformly continuous on every compact set in $W\times R$. Consequently, the convergence in (2) is uniform on compact sets in $W\times R$. That is, $\{f_{\tau_n}\}$ converges to f in the compact-open topology on $\mathfrak{C}$. This topology is metrizable (see §4). Any metric ρ that generates this topology will be called a *basic metric* on $\mathfrak{C}$, or on $\mathfrak{F}$. We have now proved the following result.

LEMMA 1. *Let $f \in \mathfrak{C}(W \times R, R^n)$. Then the mapping $t \to f_t$ of R onto $\mathfrak{F}$ is continuous in the compact-open topology on $\mathfrak{F}$.*

As a direct consequence of this fact we can construct a dynamical system on $\mathfrak{C}$.

THEOREM 1. *The mapping $\pi^*: \mathfrak{C} \times R \to \mathfrak{C}$, defined by $\pi^*(f, \tau) = f_\tau$, defines a dynamical system on $\mathfrak{C} = \mathfrak{C}(W \times R, R^n)$, when $\mathfrak{C}$ has the compact-open topology. Each set $\mathfrak{F} = \{f_\tau : \tau \in R\}$ is a trajectory of π^*.*

Now let $f \in \mathfrak{C}(W \times R, R^n)$ be an admissible function that satisfies the global existence property and let X be the product space $W \times \mathfrak{F}$. A metric d on X is defined by

$$d((x, f), (\hat{x}, \hat{f})) = |x - \hat{x}| + \rho(f, \hat{f}),$$

where ρ is any basic metric on $\mathfrak{F}$. We define a mapping $\pi: X \times R \to X$ by

$$\pi(x, f; \tau) = (\phi(x, f, \tau), f_\tau), \tag{3}$$

where $\phi(x, f, t)$ denotes the solution of $x' = f(x, t)$ that satisfies $\phi(x, f, 0) = x$. Because of the assumption of global existence for all solutions of $x' = f(x, t)$, we see that π is defined on all of $X \times R$, with range in X. Furthermore,

$$\pi(x, f; 0) = (\phi(x, f, 0), f_0) = (x, f),$$

so π satisfies property (i) in the definition of a dynamical system.

Let us now show that π satisfies property (ii). Let $(x, f) \in X$, $\tau \in R$ and $\sigma \in R$. Let $\phi(t) = \phi(x, f, t)$ and $\psi(t) = \phi(\phi(\tau), f_\tau, t)$. That is $\phi(t)$ is the solution of $x' = f(x, t)$ that satisfies $\phi(0) = x$ and $\psi(t)$ is the solution of

$$x' = f_\tau(x, t) = f(x, \tau + t) \tag{4}$$

that satisfies $\psi(0) = \phi(\tau) = \phi(x, f, \tau)$. However, $\xi(t) = \phi(t + \tau)$ is also a solution of (4) and $\xi(0) = \phi(\tau)$. Hence, by the uniqueness of solutions of (4) we have $\psi(t) = \phi(t + \tau)$, for all t in R. Consequently,

$$\begin{aligned} \pi(\pi(x, f; \tau); \sigma) &= \pi(\phi(\tau), f_\tau; \sigma) = (\psi(\sigma), f_{\tau+\sigma}) \\ &= (\phi(\tau + \sigma), f_{\tau+\sigma}) = \pi(x, f; \tau + \sigma). \end{aligned}$$

In order to show that π is continuous, we will make use of a result of Kamke [11].

LEMMA 2 (KAMKE). (A) *Let $\{g_n\} \subset \mathfrak{C}(W \times R, R^n)$ for $n = 1, 2, \ldots$, and let $g = \lim g_n$, where the convergence is in the compact-open topology on $\mathfrak{C}$. For $n = 1, 2, \ldots$, let ϕ_n be a solution of $x' = g_n(x, t)$ with $\phi_n(0) \to x_0 \in W$. Then there is a subsequence of $\{\phi_n\}$ that converges to a solution ϕ of $x' = g(x, t)$ that satisfies $\phi(0) = x_0$, and the convergence is uniform on compact sets in the interval of definition of ϕ.*

(B) *If, in addition, the solutions of $x' = g(x, t)$ are unique, then $\phi = \lim \phi_n$, where the convergence is uniform on compact sets in the interval of definition of ϕ.*

To show that π is continuous, we let $\{(x_n, f_{\tau_n})\}$ be a sequence in X and $\{t_n\}$ a sequence in R with limits

$$(x_n, f_{\tau_n}) \to (x, f_\tau) \quad \text{and} \quad t_n \to t,$$

respectively. Let $\phi_n(t)=\phi(x_n, f_{\tau_n}, t)$ and $\phi(t)=\phi(x, f_\tau, t)$. Since $f_{\tau_n}\to f_\tau$ in the compact-open topology and since the solutions of $x'=f_\tau(x, t)$ are unique, and since $\phi_n(0)=x_n\to x=\phi(0)$, it follows that $\phi=\lim \phi_n$, where the convergence is uniform on compact sets in R. Furthermore, since $t_n\to t$, it follows that $\phi_n(t_n)\to\phi(t)$. Hence,

$$\pi(x_n, f_{\tau_n}; t_n) = (\phi_n(t_n), f_{\tau_n+t_n})\to(\phi(t), f_{\tau+t}) = \pi(x, f_\tau; t).$$

Therefore π is continuous. We have thus proved:

THEOREM 2. *Let $f\in\mathfrak{C}(W\times R, R^n)$ be an admissible function that satisfies the global existence property and let ρ be a basic metric on the space of translates $\mathfrak{F}$. Then the mapping π defined by* (3) *is a dynamical system on $X=W\times\mathfrak{F}$.*

B. *Stronger topologies.* We will also be interested in topologies on $\mathfrak{C}$, or more generally on subsets of $\mathfrak{C}$, that are stronger than the compact-open topology. A metric ρ on a subset $\mathfrak{G}$ of $\mathfrak{C}$ is said to be *stronger than* the basic metric provided $\rho(f_n, f)\to 0$ implies that the sequence $\{f_n\}$ converges to f in the compact-open topology, for every sequence $\{f_n\}$ and every function f in $\mathfrak{G}$.

Theorem 2 has a counterpart for metrics on $\mathfrak{F}$ that are stronger than the basic metric. However, in this case, one must assume that the mapping $(f, t)\to f_t$ of $\mathfrak{F}\times R$ onto $\mathfrak{F}$ is continuous in the stronger metric. Because of Theorem 1, this hypothesis is automatically satisfied when $\mathfrak{F}$ has the compact-open topology.

THEOREM 3. *Let $f\in\mathfrak{C}(W\times R, R^n)$ be an admissible function that satisfies the global existence property and let $\hat\rho$ be a metric on the space of translates $\mathfrak{F}$ that is stronger than the basic metric. If the mapping $(f, t)\to f_t$ of $\mathfrak{F}\times R$ onto $\mathfrak{F}$ is continuous with respect to $\hat\rho$, then the mapping π defined by* (3) *is a dynamical system on $X=W\times\mathfrak{F}$, where the metric $\hat d$ on X is given by*

$$\hat d((x, f), (\hat x, \hat f)) = |x-\hat x|+\hat\rho(f, \hat f).$$

The argument used for Theorem 2 also establishes the validity of Theorem 3.

In the sequel we shall say that a metric ρ on the space of translates $\mathfrak{F}$, of a function f in $\mathfrak{C}(W\times R, R^n)$, is *admissible* provided (i) ρ is stronger than the basic metric and (ii) the mapping $(f, t)\to f_t$ of $\mathfrak{F}\times R$ onto $\mathfrak{F}$ is continuous with respect to ρ.

C. *The hull of f.* Using the topological dynamical properties of π one expects to get information about the solutions of the given equation $x'=f(x, t)$. However, there is still one fault in the theory at this point. Many of the results of topological dynamics rely on the assumption that the base space X is complete, or rather, is homeomorphic to a complete metric space. Generally speaking this is not true for the function space $(\mathfrak{F}, \rho)$, and therefore it is not true for (X, d).

However, in the compact-open topology the space of translates $\mathfrak{F}$ is an invariant set of the dynamical system $\pi^*(f, t)=f_t$ on $\mathfrak{C}=\mathfrak{C}(W\times R, R^n)$. It is known (see [12, p. 186 and p. 231]) that the compact-open topology on $\mathfrak{C}$ is generated by a metric ρ where $(\mathfrak{C}, \rho)$ is a complete metric space. If we let $\mathfrak{F}^*_{co} = \mathrm{Cl}\,\mathfrak{F}$, in this topology, then $\mathfrak{F}^*_{co}$ is complete with respect to ρ. We shall call $\mathfrak{F}^*_{co}$ the *hull* of f.

It is natural to ask now whether the flow π on $W \times \mathfrak{F}$ can be extended to a flow on $W \times \mathfrak{F}^*_{co}$. In order to do this, we are forced to make a critical a priori assumption on the differential equations we wish to consider.

Before we do this though, we should examine the behavior of differential equations $x' = f(x, t)$, where f is admissible but does not satisfy the global existence property. To do this we introduce the concept of a "local dynamical system" and discuss some pertinent properties in the next section. The problem of extending π to $W \times \mathfrak{F}^*_{co}$ will be discussed in §5.

4. **Local dynamical systems.** A local dynamical system differs from a dynamical system in that the motions $\pi(p, t)$ may not be defined for all t in R. We now proceed with a formal definition.

Let (X, d) be a metric space and let $I = (\alpha, \beta)$ be an open interval in R. If $\phi: I \to X$ we define the phrase "$\phi(t) \to \omega$ as $t \to \mathrm{bdy}\, I$" as follows:

(i) $I \neq R$, that is, either $\alpha \neq -\infty$ or $\beta \neq +\infty$.

(ii) If $\alpha \neq -\infty$ {or, respectively, $\beta \neq +\infty$}, then for every compact $K \subset X$, there is a T, $\alpha < T < \beta$, such that $\phi(t) \in X - K$ for $\alpha < t \leqq T$ {or, respectively, $T \leqq t < \beta$}.

For each point $p \in X$ let $I_p = (\alpha_p, \beta_p)$ be an open interval in R containing 0. Let

$$F = \{(p, t) \in X \times R : t \in I_p\}.$$

A function $\pi: F \to X$ is said to be a *local dynamical system* on X if the following properties hold:

(i) $\pi(p, 0) = p$, for all p in X.

(ii) If $t \in I_p$ and $s \in I_{\pi(p,t)}$, then $t + s \in I_p$ and $\pi(\pi(p, t), s) = \pi(p, t + s)$.

(iii) π is continuous.

(iv) Each interval I_p is maximal in the sense that either $I_p = R$, or $\pi(p, t) \to \omega$ as $t \to \mathrm{bdy}\, I_p$.

(v) The intervals I_p are lower semicontinuous in p, that is, if $p_n \to p$, then $I_p \subset \liminf I_{p_n}$.

For p fixed, the mapping $\pi(p, t): I_p \to X$ is called the *motion through* p. The sets

$$\gamma(p) = \{\pi(p, t) : t \in I_p\},$$
$$\gamma^+(p) = \{\pi(p, t) : 0 \leqq t < \beta_p\},$$
$$\gamma^-(p) = \{\pi(p, t) : \alpha_p < t \leqq 0\}$$

are *trajectories through* p. The definitions of positive invariant and invariant sets are the same as in §2. Now define the sets

$$LB^+ = \{p \in X : \beta_p = +\infty\},$$
$$LB^- = \{p \in X : \alpha_p = -\infty\},$$
$$LB = LB^+ \cap LB^-.$$

We note that if LB is nonempty then $LB \times R \subset F$ and π, restricted to $LB \times R$, maps $LB \times R$ into LB. This means that $\pi: LB \times R \to LB$ is a dynamical system on LB.

If $p \in LB^+$, we define the ω-limit set Ω_p as in §2. Similarly, the α-limit set A_p is defined for all $p \in LB^-$. Also the definitions of positively compact, negatively compact and compact motions are the same as in §2. Because of the maximality of I_p we see that if $\pi(p, t)$ is positively compact, then $p \in LB^+$. Similarly, if $\pi(p, t)$ is negatively compact {compact}, then $p \in LB^-$ $\{p \in LB\}$. The following lemmas now show the relationship between local dynamical systems on X and dynamical systems on LB. We assume that the local dynamical system π is given on X together with the above definitions.

LEMMA 3. *Let $\pi(p, t)$ be a positively compact motion. Then Ω_p is nonempty, compact, and invariant. Moreover, for every $q \in \Omega_p$, $I_q = R$.*

Proof. The proof that Ω_p is nonempty, compact, and invariant is the same as in [18, pp. 338–340]. If $q \in \Omega_p$, then $\pi(q, t)$ lies in the compact set Ω_p for all t in I_q. Therefore, from the maximality of I_q we have $I_q = R$. Q.E.D.

THEOREM 4. *Let π be a local dynamical system on X. If there exists a positively compact motion, then LB is nonempty and the restriction of π to LB defines a dynamical system on LB.*

Proof. If $\pi(p, t)$ is a positively compact motion, then by Lemma 3, Ω_p is nonempty and lies in LB. Hence LB is nonempty and π defines a dynamical system on LB. Q.E.D.

There is one more question that should be settled and that is whether LB is complete or homeomorphic to a complete space.

THEOREM 5. *Let π be a local dynamical system on X and assume that LB is nonempty. If X is homeomorphic with a complete metric space, then LB is homeomorphic with a complete metric space.*

Proof. We can assume that X is complete with respect to the given metric d. In [23] we proved that LB is a G_δ-set in X. The conclusion now follows from Hausdorff's Theorem [9], which asserts that any G_δ-set in a complete metric space is homeomorphic with a complete metric space. Q.E.D.

We shall need the following formulation of continuity of π in the sequel.

LEMMA 4. *Let π be a local dynamical system on X. If $\{p_n\}$ is a sequence in X and $p_n \to p$, then the sequence of functions $\{\pi(p_n, t)\}$ converge to $\pi(p, t)$, and the convergence is uniform on compact sets in I_p.*

Proof. If J is a compact set in I_p, then by property (v), J is in I_{p_n} for n sufficiently large. By a standard argument (see [18, pp. 327–328]) one now can show that $\{\pi(p_n, t)\}$ converges to $\pi(p, t)$ uniformly on J. Q.E.D.

5. **Admissible differential equations.** A. *As local dynamical systems.* Let $f \in \mathfrak{C}(W \times R, R^n)$ be an admissible function and let $\mathfrak{F}$ be the space of translates of f. For each point $p=(x, g)$ in $X=W\times\mathfrak{F}$ let $I_p=I_{(x,g)}$ be the maximal interval of definition of the solution $\phi(x, g, t)$ of $x'=g(x, t)$ that satisfies $\phi(x, g, 0)=x$. It is known, by a result of Kamke [10], that either $I_p=R$, or $\phi(x, g, t) \to \omega$ as $t \to \text{bdy } I_p$. Let

$$F = \{(x, g; t) = (p; t) \in X \times R : t \in I_p\}$$

and define $\pi: F \to X$ by

$$\pi(x, g; \tau) = (\phi(x, g, \tau), g_\tau). \tag{5}$$

The argument used for Theorem 2 can be easily adapted to prove

THEOREM 6. *Let $f \in \mathfrak{C}(W \times R, R^n)$ be an admissible function and let ρ be a basic metric on the space of translates $\mathfrak{F}$. Then the mapping π, defined by* (5), *is a local dynamical system on $X=W\times\mathfrak{F}$.*

Similarly Theorem 3 has the following generalization.

THEOREM 7. *Let $f \in \mathfrak{C}(W \times R, R^n)$ be an admissible function and let ρ be an admissible metric on the space of translates $\mathfrak{F}$. Then the mapping π, defined by* (5), *is a local dynamical system on $X=W\times\mathfrak{F}$, where the metric d on X is given by*

$$d((x,f), (\hat{x},\hat{f})) = |x-\hat{x}|+\rho(f,\hat{f}).$$

B. *The hull, revisited.* In §3 we posed the problem of extending the dynamical system on $W\times\mathfrak{F}$ to one on $W\times\mathfrak{F}^*_{\text{co}}$. For reasons which we shall see, it is more natural to ask this question in the context of local dynamical systems. Let $f \in \mathfrak{C} = \mathfrak{C}(W \times R, R^n)$ be an admissible function and consider the space of translates $\mathfrak{F}$ in the compact-open topology on $\mathfrak{C}$. Let $\mathfrak{F}^*_{\text{co}}=\text{Cl}\,\mathfrak{F}$ (that is, the closure in the compact open topology) be the hull of f. We now seek conditions (on f) that the local dynamical system π, defined by (5), on $W\times\mathfrak{F}$ can be extended to a local dynamical system on $W\times\mathfrak{F}^*_{\text{co}}$.

The first step in solving this is to look at the projection of π onto $\mathfrak{F}$, as was done in Theorem 1. We can now prove the following result.

LEMMA 5. *Let $f \in \mathfrak{C}$ and let $\mathfrak{F}^*_{\text{co}}$ be the hull of f, in the compact-open topology. Then the mapping $\pi^*(f^*, t)=f^*_t$ defines a dynamical system on $\mathfrak{F}^*_{\text{co}}$ and is an extension of the dynamical system on $\mathfrak{F}$. Moreover, $\mathfrak{F}^*_{\text{co}}$ is homeomorphic with a complete metric space.*

Proof. The fact that the mapping $\pi^*(f^*, t)=f^*_t$ is a dynamical system on $\mathfrak{F}^*_{\text{co}}$ is a consequence of Theorem 1 and the fact that the closure of an invariant set is an invariant set. (See [18, p. 331].) To show that $\mathfrak{F}^*_{\text{co}}$ is homeomorphic with a complete metric space, we recall that $\mathfrak{C}$ is homeomorphic with a complete metric space $\hat{\mathfrak{C}}$. Since every closed subset of $\hat{\mathfrak{C}}$ is complete, $\mathfrak{F}^*_{\text{co}}$ is homeomorphic with a complete metric space. Q.E.D.

If ρ is an admissible metric on $\mathfrak{F}$ that is stronger than the basic metric, the situation is not as simple. Let us illustrate the difficulty with the following example.

EXAMPLE A. Consider the differential equation

$$x' = f(t) \tag{6}$$

where $f\colon R \to R$ is an even function defined (for $t \geqq 0$) by

$$\begin{aligned} f(t) &= 0, \qquad 4k \leqq t \leqq 4k+2, \qquad k = 0, 1, \ldots, \\ &= (k+1)^{-1} \sin^2 \pi(k+1)t, \qquad 4k+2 < t < 4k+4, \qquad k = 0, 1, \ldots. \end{aligned}$$

Then $f \in \mathfrak{C}^1$ and the space of translates $\mathfrak{F}$ lies in $\mathfrak{C}^1$. On $\mathfrak{C}^1$ we define a metric by

$$d(f, g) = \rho(f, g) + \sup \{|f'(t) - g'(t)| : 0 \leqq t \leqq 1\},$$

where ρ is a basic metric on $\mathfrak{C}(R, R)$. One can now show that the following statements are valid:

(i) The mapping $t \to f_t$ is continuous with respect to d.

(ii) If $g=0$, then $d(f_{4k}, g) \to 0$ as $k \to \infty$.

(iii) $d(f_{4k+3}, g_3) = \rho(f_{4k+3}, g_3) + 2\pi \to 2\pi$ as $k \to \infty$.

In other words, the mapping $\pi^*(f^*, t) = f_t^*$ on $\mathfrak{C}^1$ is not continuous in the given metric, and, in particular, it is not continuous on Cl $\mathfrak{F}$, where the closure is taken in $\mathfrak{C}^1$ in the topology generated by d.

Let $\mathfrak{G}$ be a subset of $\mathfrak{C}$ and assume that $\mathfrak{G}$ is translation-invariant, that is, if $g \in \mathfrak{G}$, then $g_\tau \in \mathfrak{G}$ for every $\tau \in R$. We say that a metric ρ is *admissible* on $\mathfrak{G}$ if (i) ρ is stronger than the basic metric and (ii) the mapping $\pi^*(g, t) = g_t$ defines a dynamical system on $\mathfrak{G}$. Now if $f \in \mathfrak{G}$, then the space of translates $\mathfrak{F}$ is in $\mathfrak{G}$ and ρ is admissible on $\mathfrak{F}$. In this case we define the *hull* of f (with respect to $(\mathfrak{G}, \rho)$) as closure of $\mathfrak{F}$ in $\mathfrak{G}$, with respect to the topology generated by ρ. We shall denote the hull by $\mathfrak{F}^*$, where $\mathfrak{G}$ and ρ will always be specified in the context. However, if ρ is a basic metric, then $\mathfrak{G}$ will always be the space $\mathfrak{C}$ itself, and in this case we denote the hull by the special symbol $\mathfrak{F}^*_{\mathrm{co}}$. One should note that since an admissible metric is stronger than a basic metric every hull $\mathfrak{F}^*$ is a subset of $\mathfrak{F}^*_{\mathrm{co}}$.

With this definition we now have the next result.

LEMMA 6. *Let $f \in \mathfrak{C}$ and let $\mathfrak{F}^*$ be the hull of f, with respect to $(\mathfrak{G}, \rho)$, then the mapping $\pi^*(f^*, t) = f_t^*$ is a dynamical system on $\mathfrak{F}^*$. Moreover, if $(\mathfrak{G}, \rho)$ is a complete metric space, then $(\mathfrak{F}^*, \rho)$ is complete.*

We shall only have occasion to use this lemma when $(\mathfrak{G}, \rho)$ is a translation-invariant topological vector space with an admissible metric ρ. Generally, in these cases, the space $(\mathfrak{G}, \rho)$ will be complete.

We are now prepared to investigate the question of extending the local dynamical system π on $W \times \mathfrak{F}$ to a local dynamical system on $W \times \mathfrak{F}^*$. We consider first the case of the compact-open topology on $\mathfrak{F}$.

THEOREM 8. *Let $f \in \mathfrak{C}(W \times R, R^n)$ be an admissible function, let $\mathfrak{F}$ be the space of translates of f and let ρ be a basic metric on $\mathfrak{F}$. Let $\mathfrak{F}^*_{co}$ be the hull of f, in the compact-open topology. Then the local dynamical system π on $W \times \mathfrak{F}$, defined by* (5), *can be extended to a local dynamical system on $W \times \mathfrak{F}^*_{co}$ if every function f^* in the hull is admissible. In this case, the extension is given by* (5).

Proof. This follows from Kamke's Lemma and Theorem 2. If every f^* in the hull is admissible, we can (formally) define the extension π by (5). It is then immediate that π satisfies properties (i), (ii), (iv), and (v) for a local dynamical system. To prove that π is continuous, we use Kamke's Lemma in the same manner as used in Theorem 2. Q.E.D.

The counterpart of Theorem 8 for more general metrics is the following:

THEOREM 9. *Let $\mathfrak{G}$ be a translation-invariant subset of $\mathfrak{C}(W \times R, R^n)$ and let ρ be an admissible metric on $\mathfrak{G}$. Let $\mathfrak{F}$ be the space of translates of f and $\mathfrak{F}^*$ the hull of f, with respect to $(\mathfrak{G}, \rho)$. Then the local dynamical system π on $W \times \mathfrak{F}$, defined by* (5), *can be extended to a local dynamical system on $W \times \mathfrak{F}^*$ if every function f^* in the hull is admissible. In this case, the extension is given by* (5).

The proof of this theorem is the same as that of Theorem 8.

These two theorems motivate the following definition. Let $f \in \mathfrak{C}(W \times R, R^n)$ be an admissible function and let $\mathfrak{F}^*_{co}$ be the hull of f in the compact-open topology. We shall say that f is *regular* (or *$\mathfrak{F}^*_{co}$-regular*) if every function f^* in the hull $\mathfrak{F}^*_{co}$ is admissible. Similarly, if $\mathfrak{F}^*$ is any other hull of f then we say that f is *$\mathfrak{F}^*$-regular* if every function in $\mathfrak{F}^*$ is admissible.

LEMMA 7. *Let $f \in \mathfrak{C}(W \times R, R^n)$ be a regular function (that is, f is $\mathfrak{F}^*_{co}$-regular). If $\mathfrak{F}^*$ is any hull of f, then f is $\mathfrak{F}^*$-regular.*

Proof. Since $\mathfrak{F}^* \subset \mathfrak{F}^*_{co}$, the statement is obvious. Q.E.D.

Theorems 8 and 9 can now be reformulated as follows:

COROLLARY. *Let $f \in \mathfrak{C}(W \times R, R^n)$ be $\mathfrak{F}^*$-regular, where $\mathfrak{F}^*$ is some hull of f. Then the local dynamical system π on $W \times \mathfrak{F}$, defined by* (5), *can be extended to a local dynamical system on $W \times \mathfrak{F}^*$. In this case the extension is given by* (5).

The assumption of regularity has been used elsewhere, although it is formulated differently. (See [17], [22].) If $f \in \mathfrak{C}(W \times R, R^n)$ and $\mathfrak{F}^*$ is any hull of f, then every function in $\mathfrak{F}^*$ is continuous. Therefore, f is $\mathfrak{F}^*$-regular if and only if the differential equation in the hull (that is, every differential equation of the form $x' = f^*(x, t)$, where $f^* \in \mathfrak{F}^*$) satisfies a uniqueness condition.

We now give a sufficient condition that an admissible function $f \in \mathfrak{C}(W \times R, R^n)$ be regular.

THEOREM 10. *Let $f \in \mathfrak{C}(W \times R, R^n)$ be a function that satisfies a local Lipschitz condition in x, where the Lipschitz constant is independent of t. Then f is regular.*

The Lipschitz condition stated above means that for every compact set $K \subset W$, there is a positive constant k such that

$$|f(x,t)-f(y,t)| \leqq k|x-y| \qquad (x \in K,\ y \in K,\ t \in R).$$

Proof. Let ρ be a basic metric on $\mathfrak{C} = \mathfrak{C}(W \times R, R^n)$, $\mathfrak{F}$ the space of translates of f and $\mathfrak{F}^*_{co}$ the hull of f, in the compact-open topology on $\mathfrak{C}$. We shall now show that every function f^* in $\mathfrak{F}^*_{co}$ satisfies the same Lipschitz condition as f. This will imply that the solutions of $x' = f^*(x, t)$ are unique, and therefore f is regular.

Let K be a compact set in W and let f_τ be any translate f. Then there is a positive constant k such that

$$|f_\tau(x,t)-f_\tau(y,t)| = |f(x,\tau+t)-f(y,\tau+t)| \leqq k|x-y|,$$

for all x and y in K and all t in R. In other words, every translate of f satisfies the same Lipschitz condition. Now let f^* be any element of $\mathfrak{F}^*_{co}$. Then for every $\varepsilon > 0$ there is a translate f_τ such that $\rho(f^*, f_\tau) < \varepsilon$.

Now let I be any compact set in R. Then $M = K \times I$ is compact in $W \times R$. Since the metric ρ generates the topology of uniform convergence on compact sets (this topology is the same as the compact-open topology on $\mathfrak{C}(W \times R, R^n)$ [12, pp. 186 and 230]), there is a nonnegative function $l(M; \varepsilon)$ such that

$$l(M;\varepsilon) \to 0 \text{ as } \varepsilon \to 0,$$

and

$$\sup\{|f^*(x,s)-g^*(x,s)| : (x,s) \in M\} \leqq l(M;\varepsilon),$$

whenever $g^* \in \mathfrak{C}(W \times R, R^n)$ and $\rho(f^*, g^*) < \varepsilon$. (The function l depends on f^*, which is fixed for our argument.) Thus if $\rho(f^*, f_\tau) < \varepsilon$, then

$$\begin{aligned} |f^*(x,t)-f^*(y,t)| &\leqq |f^*(x,t)-f_\tau(x,t)| + |f_\tau(x,t)-f_\tau(y,t)| + |f_\tau(y,t)-f^*(y,t)| \\ &\leqq 2\cdot l(M;\varepsilon) + k|x-y|, \end{aligned}$$

whenever (x, t) and (y, t) are in M. If we let $\varepsilon \to 0$ we get

$$|f^*(x,t)-f^*(y,t)| \leqq k|x-y| \qquad (x \in K,\ y \in K,\ t \in I).$$

However, since I is arbitrary we get

$$|f^*(x,t)-f^*(y,t)| \leqq k|x-y| \qquad (x \in K,\ y \in K,\ t \in R),$$

which completes the proof.

As an immediate consequence of Theorem 10 and Lemma 7 we get the next result.

Corollary. *Let $f \in \mathfrak{C}(W \times R, R^n)$ be a function that satisfies a local Lipschitz condition, where the Lipschitz constant is independent of t. If $\mathfrak{F}^*$ is any hull of f, then f is $\mathfrak{F}^*$-regular.*

The assumption that f satisfy a Lipschitz condition, where the Lipschitz constant is independent of t, cannot be removed entirely, as is shown in the following example.

EXAMPLE B. Consider the differential equation

$$x' = f(x, t) \tag{7}$$

on $R \times R$, where

$$\begin{aligned} f(x, t) &= |x|^{1/2}, \quad |x| \geqq e^{-2t}, \qquad t \in R, \\ &= e^{-t}, \qquad |x| < e^{-2t}, \qquad t \in R. \end{aligned}$$

The solutions of (7) are unique, but $f^*(x, t) = (|x|)^{1/2}$ is in $\mathfrak{F}^*_{co}$ and is not admissible.

The following fact may prove useful.

LEMMA 8. *Let $f \in \mathfrak{C}(W \times R, R^n)$ where f is either autonomous, or periodic in t. Then f is regular if and only if f is admissible.*

Proof. If f is autonomous, or periodic in t, and $\mathfrak{F}^*$ is any hull of f, then $\mathfrak{F}^* = \mathfrak{F}^*_{co} = \mathfrak{F}$. The lemma now follows from the fact that f is admissible if and only if every translate f_τ is admissible. Q.E.D.

Lemma 8 is not true if we assume that f is almost periodic in t as is seen in the following example due to Y. Sibuya.

EXAMPLE C. Consider the differential equation

$$x' = f(x, t) = (|x| + a(t))^{1/2} \tag{8}$$

on $R \times R$, where $a(t)$ is continuous and (Bohr) almost periodic in t.

LEMMA 9. *There exists a continuous, Bohr almost periodic function $a(t)$ satisfying*

(i) $a(t) > 0$, *for all t, and*

(ii) $a_{\tau_n}(t) = a(\tau_n + t) \to 0$ *(for $|t| \leqq 1/2$) as $n \to \infty$, for some sequence $\{\tau_n\}$ with $\tau_n \to \infty$.*

Proof. We shall illustrate the proof by constructing a discontinuous, almost periodic function with the required properties. This function can be made continuous by one of the standard smoothing processes.

For $n = 1, 2, \ldots$, let $b_n(t)$ be the periodic function, of period 2^{n+1}, defined by

$$\begin{aligned} b_n(t) &= 0, \qquad 0 \leqq t \leqq 2^n, \\ &= -2^{-n}, \qquad 2^n < t < 2^{n+1}. \end{aligned}$$

Let $b_0(t) = 1$ for all t. Since the infinite series

$$a(t) = \sum_{n=0}^{\infty} b_n(t)$$

reduces to a finite series for each t, it converges everywhere and $a(t)$ is almost periodic. Furthermore, $a(t) > 0$ for all t. If we let $\sigma_n = 2^n - 1$, for $n = 1, 2, \ldots$, then one establishes by induction that

$$a_{\sigma_n}(t) = 2^{-n+1} \qquad (0 \leqq t \leqq 1).$$

If $\tau_n = 2^n - 1/2$, then

$$a_{\tau_n}(t) \to 0 \text{ as } n \to \infty \qquad (|t| \leqq 1/2).$$

LEMMA 10. *If $a(t)$ is a continuous, Bohr almost periodic function that satisfies properties* (i) *and* (ii) *of Lemma* 9, *then the function f given by* (8) *is admissible but not regular.*

Proof. Since f is continuous and locally Lipschitzian in x, f is admissible. Now let $\{\tau_n\}$ be a sequence that satisfies

$$a_{\tau_n}(t) \to 0 \text{ as } n \to \infty \qquad (|t| \leqq 1/2).$$

It is known that the sequence $\{a_{\tau_n}\}$ has a uniformly convergent subsequence, say that $a_{\tau_n} \to \hat{a}$, and $\hat{a}$ is Bohr almost periodic. The function

$$f^*(x, t) = (|x| + \hat{a}(t))^{1/2} \tag{9}$$

is in $\mathfrak{F}^*_{co}$. Since $\hat{a}(t) = 0$, $|t| \leqq 1/2$, equation (9) does not satisfy the uniqueness property. So f^* is not admissible. Q.E.D.

We conclude this section with an example illustrating that if $f \in \mathfrak{C}(W \times R, R^n)$ is regular and satisfies the global existence property (that is, the mapping π on $W \times \mathfrak{F}$, defined by (5), is a dynamical system and not merely a local dynamical system), this does not imply that every function f^* in the hull $\mathfrak{F}^*_{co}$ satisfies the global existence property. In other words, if $f \in \mathfrak{C}(W \times R, R^n)$ is regular and π is a dynamical system on $W \times \mathfrak{F}$, then the extension of π to $W \times \mathfrak{F}^*_{co}$ may be only a local dynamical system.

EXAMPLE D. Consider the differential equation

$$x' = f(x, t) \tag{10}$$

on $R \times R$, where

$$\begin{aligned} f(x, t) &= x^2, & |x| &\leqq e^t, & t &\in R, \\ &= e^{2t}, & |x| &> e^t, & t &\in R. \end{aligned}$$

The solutions of (10) are unique and, since

$$|f(x, t)| \leqq e^t|x| + e^{2t},$$

f satisfies the global existence property. However, $f^*(x, t) = x^2$ is in the hull $\mathfrak{F}^*_{co}$ and this function does not satisfy the global existence property.

6. **Topologies on $\mathfrak{F}$.** In this section we shall discuss several topologies on $\mathfrak{C} = \mathfrak{C}(W \times R, R^n)$, or rather on subsets of $\mathfrak{C}$. The first topology, the compact-open topology, has already been discussed at length. Here we shall only illustrate how one can construct a metric for this topology. The second topology, the "Bohr topology," which is generated by the "Bohr metric," is of great importance in the theory of ordinary differential equations. We shall discuss this at some length while treating the concepts of admissibility and regularity in this topology. Some additional topologies, the "uniform topology" and the "C^1-topology," will be treated briefly at the end.

A. *The compact-open topology.* A metric for the compact-open topology on $\mathfrak{C}(W\times R, R^n)$ can be constructed as follows: Let $\{K_n\}$ be a sequence of compact sets in $W\times R$ such that $W\times R=\bigcup_{n=1}^{\infty} K_n$. For each n we construct a pseudo-metric on $\mathfrak{C}(W\times R, R^n)$ as follows: Let

$$\|f-g\|_n = \sup\{|f(x,t)-g(x,t)| : (x,t)\in K_n\}$$

$$\rho_n(f,g) = \frac{\|f-g\|_n}{1+\|f-g\|_n}.$$

The required metric is given by

$$\rho(f,g) = \sum_{n=1}^{\infty} 2^{-n}\rho_n(f,g). \tag{11}$$

The metric ρ depends on the choice of the sequence $\{K_n\}$, however any other sequence of compact sets would generate an equivalent metric. If $K_n\subset K_{n+1}$, $n=1,2,\ldots$, then the space $\mathfrak{C}(W\times R, R^n)$ is complete with respect to the metric given by (11).

If $W=R^n$, that is on $\mathfrak{C}(R^n\times R, R^n)$, another basic metric is defined by

$$\rho(f,g) = \sup_{0<T}[\min(\sup\{|f(x,t)-g(x,t)| : |x|+|t| \leqq T\}, 1/T)].$$

This particular metric appears to be simpler for computational purposes.

B. *The Bohr topology.* This topology is defined on the subspace $\mathfrak{B}(W\times R, R^n)$ of $\mathfrak{C}(W\times R, R^n)$ consisting of all continuous functions $f(x,t)$ that are bounded (in t) for x in compact sets in W. That is, $f\in\mathfrak{B}(W\times R, R^n)$ if and only if $f\in\mathfrak{C}(W\times R, R^n)$ and for every compact $M\subset W$, there is a $k\geqq 0$ such that

$$|f(x,t)| \leqq k \qquad (x\in M,\ t\in R).$$

We shall now define a Bohr metric on $\mathfrak{B}(W\times R, R^n)$. Let $\{M_n\}$ be a sequence of compact sets in W such that $W=\bigcup_{n=1}^{\infty} M_n$. For each n we construct a pseudo-metric on $\mathfrak{B}(W\times R, R^n)$ as follows; let

$$\|f-g\|_{n,\infty} = \sup\{|f(x,t)-g(x,t)| : x\in M_n,\ t\in R\},$$

$$b_n(f,g) = \frac{\|f-g\|_{n,\infty}}{1+\|f-g\|_{n,\infty}}.$$

A *Bohr metric* is defined by

$$b(f,g) = \sum_{n=1}^{\infty} 2^{-n} b_n(f,g). \tag{12}$$

Any metric that is equivalent to $b(f,g)$, as given by (12), is also said to be a *Bohr metric.* If $M_n\subset M_{n+1}$, $n=1,2,\ldots$, then the space $\mathfrak{B}(W\times R, R^n)$ is complete with respect to the metric defined by (12). The topology generated by a Bohr metric is called the *Bohr topology.*

Let us now consider the following question. Let $f \in \mathfrak{B}(W \times R, R^n)$ so that the space of translates $\mathfrak{F}$ also lies in $\mathfrak{B}(W \times R, R^n)$. When is the Bohr metric b an admissible metric on $\mathfrak{F}$? Or equivalently, when is the mapping $(f, t) \to f_t$ continuous with respect to the Bohr metric b? The answer is that $f(x, t)$ must satisfy a uniform-continuity condition.

LEMMA 11. *Let $f \in \mathfrak{B}(W \times R, R^n)$. Then the Bohr metric b is admissible on the space of translates $\mathfrak{F}$ if and only if f satisfies the following condition*:

(U) *For every compact set $M \subset W$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$|f(x, t) - f(x, \tau + t)| \leqq \varepsilon \qquad ((x, t) \in M \times R),$$

whenever $|\tau| \leqq \delta$.

Proof. It is clear that a sequence of functions $\{f^{(n)}\}$ in $\mathfrak{B}(W \times R, R^n)$ converge (in the Bohr topology) to a function f in $\mathfrak{B}(W \times R, R^n)$ if and only if $f^{(n)}(x, t)$ converges to $f(x, t)$ uniformly on every set $M \times R$, where M is compact in W. The Bohr metric given by (12) is translation-invariant in the sense that $b(f_t, g_t) = b(f, g)$ for all t in R. Therefore, b is admissible on $\mathfrak{F}$ if and only if the mapping $t \to f_t$ is continuous (with respect to b) at $t = 0$. The proof of the lemma is complete once we observe that condition (U) is equivalent to saying that for every sequence $\{\tau_n\}$, with $\tau_n \to 0$, the sequence $f_{\tau_n}(x, t)$ converges to $f(x, t)$ uniformly on every set $M \times R$, where M is compact in W. Q.E.D.

Property (U) implies that $f(x, t)$ is uniformly continuous in t, for each x. It is easy to construct an example showing that the converse is not true. However, uniform continuity in both variables, implies condition (U).

LEMMA 12. *Let $f \in \mathfrak{B}(W \times R, R^n)$ and assume that $f(x, t)$ is uniformly continuous on every set $M \times R$, where M is compact in W. Then the Bohr metric b is admissible on the space of translates.*

This follows from a standard argument, and we omit the details.

We now define the class $\mathfrak{B}_U(W \times R, R^n)$ as the class of all functions $f \in \mathfrak{B}(W \times R, R^n)$ that satisfy condition (U). This class is translation-invariant and we can prove the following result:

THEOREM 11. *The mapping $\pi^*(f, t) = f_t$ defines a dynamical system on $\mathfrak{B}_U(W \times R, R^n)$, with respect to the Bohr topology.*

Proof. Since $\mathfrak{B}_U = \mathfrak{B}_U(W \times R, R^n)$ is translation-invariant, we need only check that π^* is continuous on $\mathfrak{B}_U \times R$. Let $\{f^{(n)}\}$ and $\{\tau_n\}$ be sequences in $\mathfrak{B}_U$ and R, respectively, with limits $f^{(n)} \to f$ and $\tau_n \to \tau$. Let b be the Bohr metric given by (12), so b is translation-invariant. Then

$$b(f^{(n)}_{\tau_n}, f_\tau) = b(f^{(n)}, f_{\tau - \tau_n}) \leqq b(f^{(n)}, f) + b(f, f_{\tau - \tau_n}),$$

which can be made arbitrarily small. Q.E.D.

Let us now look at regularity for admissible functions in $\mathfrak{B}_U$. If $f \in \mathfrak{B}_U$, then define the *Bohr-hull* of f as the closure of $\mathfrak{F}$ in the space $\mathfrak{B}_U$ with the Bohr topology. Denote this by $\mathfrak{F}_b^*$. As already observed, $\mathfrak{F}_b^* \subset \mathfrak{F}_{co}^*$. We shall say that f is *Bohr-regular* if f is $\mathfrak{F}_b^*$-regular, that is, every function in the hull $\mathfrak{F}_b^*$ is admissible. Using techniques which will be developed later, one can show that if

$$f(x, t) = (|x|)^{1/2} + 1 + e^{-t} + \sin t^{1/2},$$

then f is Bohr-regular but not regular.

C. *The uniform topology*. On the space $\mathfrak{B}\mathfrak{C}(W \times R, R^n)$, of bounded, continuous functions defined on $W \times R$ with values in R^n, one defines a metric by

$$\rho(f, g) = \|f-g\|_\infty = \sup\{|f(x, t)-g(x, t)| : (x, t) \in W \times R\}.$$

The mapping $\pi^*(f, t) = f_t$ will be a dynamical system on the space $\mathfrak{B}_U \cap \mathfrak{B}\mathfrak{C}$, that is, on the space of bounded, continuous functions that satisfy condition (U).

If $W = R^n$, then every differential equation $x' = f(x, t)$, where $f \in \mathfrak{B}\mathfrak{C}(R^n \times R, R^n)$, satisfies the global existence property. Therefore, if $f \in \mathfrak{B}\mathfrak{C}(R^n \times R, R^n)$ is admissible, then the local dynamical system, given by (5), is a dynamical system.

D. *The C^1-topology*. Let $\mathfrak{C}^1 = \mathfrak{C}^1(W \times R, R^n)$ be the space of functions f from $\mathfrak{C} = \mathfrak{C}(W \times R, R^n)$ where $D_{x_i} f = \partial f / \partial x_i$ is also in $\mathfrak{C}$. Define a metric on $\mathfrak{C}^1$ by

$$\hat{\rho}(f, g) = \rho(f, g) + \rho(D_{x_1} f, D_{x_1} g) + \cdots + \rho(D_{x_n} f, D_{x_n} g), \tag{13}$$

where ρ is a basic metric on $\mathfrak{C}$. One can easily prove the following result.

THEOREM 12. *The mapping $\pi^*(f, t) = f_t$ is a dynamical system on $\mathfrak{C}^1$, where the topology on $\mathfrak{C}^1$ is generated by the metric $\hat{\rho}$ given by* (13). *If* $(\mathfrak{C}, \rho)$ *is complete, then* $(\mathfrak{C}^1, \hat{\rho})$ *is complete. Furthermore, every function $f \in \mathfrak{C}^1$ is admissible. Therefore, if $\mathfrak{F}_{c-1}^*$ denotes the hull of f, with respect to* $(\mathfrak{C}^1, \hat{\rho})$, *then f is $\mathfrak{F}_{c-1}^*$-regular.*

7. **Topological dynamics on $\mathfrak{F}$ and $\mathfrak{F}^*$.** Before we examine the topological-dynamical behavior of solutions of $x' = f(x, t)$, we shall first consider the behavior on the space of translates $\mathfrak{F}$, or on the hull $\mathfrak{F}^*$. Although we shall be primarily interested in the compact-open topology on $\mathfrak{F}$, we shall also discuss the relevant facts for the Bohr topology and the C^1-topology. The material in this section generalizes some results of V. V. Nemyckiĭ [19], J. Auslander and F. Hahn [1], and L. Auslander and F. Hahn [2], who considered the case where f is independent of x.

Let $\mathfrak{G}$ be a nonempty translation-invariant subset of $\mathfrak{C}(W \times R, R^n)$, with an admissible metric ρ. That is, $\pi^*(f, t) = f_t$ is a dynamical system on $(\mathfrak{G}, \rho)$. The first result is obvious.

THEOREM 13. (A) *A point f in $\mathfrak{G}$ is a fixed point for π^* if and only if f is an autonomous function, $f(x)$.*

(B) *A motion $\pi^*(f, t)$ is periodic if and only if $f(x, t)$ is periodic in t.*

Let us now consider the case where $\mathfrak{G}=\mathfrak{C}=\mathfrak{C}(W\times R, R^n)$ and the metric ρ is a basic metric. We ask then, when is a motion $\pi^*(f, t)$ positively compact, or compact. Recall that this means: when is $\gamma^+(f)$, or $\gamma(f)$, relatively compact in $\mathfrak{C}$. The answer to this is given by Ascoli's Theorem [21, p. 155].

THEOREM 14. (A) *A motion $\pi^*(f, t)$ is compact (in the compact-open topology on $\mathfrak{C}$) if and only if f is bounded and uniformly continuous on every set $M\times R$, where M is a compact set in W.*

(B) *A motion $\pi^*(f, t)$ is positively-compact (in the compact-open topology on $\mathfrak{C}$) if and only if f is bounded and uniformly continuous on every set $M\times R^+$, where M is a compact set in W and $R^+=\{t : 0\leqq t<\infty\}$.*

Proof. Let X and Y be two separable metric spaces. Ascoli's Theorem states that a subset $\mathfrak{A}$ of the space $\mathfrak{C}(X, Y)$ of continuous functions from X into Y is relatively compact in the compact-open topology if and only if

(i) the set $\{f(x) : f\in\mathfrak{A}\}$ is relatively compact in Y, for each x in X, and

(ii) $\mathfrak{A}$ is equicontinuous.

In proving statement (A) we first note that the motion $\pi^*(f, t)$ is compact if and only if the trajectory $\gamma(f)=\mathfrak{F}=\{f_\tau : \tau\in R\}$ is relatively compact. Now the set $\{f_\tau(x, t) : \tau\in R\}$ is relatively compact in R^n if and only if it is bounded, in other words, for each x, the function $f(x, t)$ is bounded in t. Also, the set $\{f_\tau : \tau\in R\}$ is equicontinuous at the point (x_0, t_0) if for every $\varepsilon>0$ there is a $\delta=\delta(\varepsilon, x_0, t_0)>0$ such that

$$|f_\tau(x, t)-f_\tau(x_0, t_0)| = |f(x, \tau+t)-f(x_0, \tau+t_0)|<\varepsilon \qquad (\tau\in R) \tag{14}$$

whenever $|x-x_0|<\delta$ and $|t-t_0|<\delta$. This shows that the δ can be chosen independent of t_0. Then (14) is equivalent to saying that f is uniformly continuous on sets of the form $M\times R$, where M is a compact set in W. We have thus shown that the motion $\pi^*(f, t)$ is compact if and only if

(a) for each x, $f(x, t)$ is bounded in t, and

(b) f is uniformly continuous on every set of the form $M\times R$, where M is compact in W.

Finally we note that this is equivalent to saying that f is bounded and uniformly continuous on every set of the form $M\times R$, where M is compact in W. This completes the proof of (A). The proof of statement (B) is similar. Q.E.D.

COROLLARY. *If a motion $\pi^*(f, t)$ is compact (in the compact-open topology), then f is in $\mathfrak{B}_U$. In other words, the mapping $t\to f_t$ is continuous in the Bohr topology.*

Proof. This follows from Lemmas 11 and 12 and the last theorem. Q.E.D.

Let us now look at compact motions in $\mathfrak{B}_U(W\times R, R^n)$ in the Bohr topology. We need the following

DEFINITION. A function $f\in\mathfrak{C}(W\times R, R^n)$ is said to be *Bohr almost periodic* (in t) if for every compact set $M\subset W$ and every $\varepsilon>0$ the set

$$\{\tau : |f(x, t)-f(x, \tau+t)| \leqq \varepsilon, \text{ for all } x \text{ in } M \text{ and all } t \text{ in } R\}$$

is relatively dense in R. (A similar definition holds for functions on $W\times R$ with values in R^m instead of R^n.)

The following results are standard generalizations of the theory of real-valued almost periodic functions. (See [4].)

LEMMA 13. *Assume that* $f\in\mathfrak{C}(W\times R, R^n)$ *is Bohr almost periodic. Then* $f(x, t)$ *is bounded and uniformly continuous on every set* $M\times R$, *where* M *is compact in* W. *In particular*, $f\in\mathfrak{B}_U$.

THEOREM 15. *Let* $f\in\mathfrak{B}_U$. *Then the motion* $\pi^*(f, t)=f_t$ *is compact in the Bohr topology if and only if* f *is Bohr almost periodic.*

We shall see later that f is Bohr almost periodic if and only if the motion f_t is almost periodic in the Bohr topology on $\mathfrak{F}$.

The concept of Bohr almost periodic functions f defined on $W\times R$ can be reformulated in another manner. If we define $f^t(x)=f(x, t)$, then for each t, $f^t\in\mathfrak{C}(W, R^n)$; that is, f^t is a continuous function from W into R^n. By restricting the domain of f^t to a subset $M\subset W$, we then have $f^t\in\mathfrak{C}(M, R^n)$ for every set $M\subset W$. Thus f^t can be viewed as a motion in $\mathfrak{C}(M, R^n)$. If M is compact, then the space $\mathfrak{C}(M, R^n)$ is a Banach space with the uniform norm $\|\cdot\|_\infty$ and the motion f^t is continuous with respect to this norm. The following result is now an immediate consequence of the definitions.

LEMMA 14. *The function* $f\in\mathfrak{C}(W\times R, R^n)$ *is Bohr almost periodic if and only if the mapping* $t\to f^t$ *of* R *into* $(\mathfrak{C}(M, R^n), \|\cdot\|_\infty)$ *is almost periodic, for every compact set* $M\subset W$.

It is also possible to characterize compact motions in $\mathfrak{C}^1(W\times R, R^n)$, in the C^1-topology (13). This is given by

THEOREM 16. *Let* $f\in\mathfrak{C}^1(W\times R, R^n)$. *Then the motion* $\pi^*(f, t)=f_t$ *is compact in the* C^1-*topology* (*that is, the hull* $\mathfrak{F}^*_{c-1}$ *is compact*) *if and only if the functions* f, $D_{x_1}f,\dots, D_{x_n}f$ *are in* $\mathfrak{B}_U$. *In particular, if the functions* f, $D_{x_1}f,\dots, D_{x_n}f$ *are uniformly continuous on every set* $M\times R$, *where* M *is compact in* W, *then the motion* $\pi^*(f, t)=f_t$ *is compact.*

A result, similar to Theorem 14(B), is valid for positive-compactness in the C^1-topology. We shall not give a formal statement of this.

The concepts of Poisson-stability and recurrence can easily be formulated in terms of the flow f_t. We shall not do this. However, the concept of recurrence, under the condition of compactness, is somewhat interesting.

Let $f\in\mathfrak{C}(W\times R, R^n)$ and assume that $\mathfrak{F}^*_{co}$ is compact in the compact-open topology, see Theorem 14. Then the motion f_t is recurrent (in the compact-open topology) if for every neighborhood $\mathfrak{U}$ of f, the set $\{\tau : f_\tau\in\mathfrak{U}\}$ is relatively dense in R (see §2). (We formulated recurrence this way to show that, in this case, it does not depend on the metric that generates the topology.) By Birkhoff's Theorem (see [18, pp. 375–377] and §2), we see that a compact motion f_t is recurrent in the compact-open topology if and only if the hull $\mathfrak{F}^*_{co}$ is a (compact) minimal set. A similar result is, of course, true in the other topologies we have considered. However, as we shall see, in the Bohr topology a compact motion is almost periodic.

Let us now turn to the question of almost periodicity. First consider the compact-open topology. Let $f \in \mathfrak{C}(W \times R, R^n)$ and let ρ be a basic metric. The motion f_t is almost periodic (in the compact-open topology) if for every $\varepsilon > 0$ the set

$$\{\tau : \rho(f_t, f_{t+\tau}) < \varepsilon \text{ for all } t \text{ in } R\}$$

is relatively dense in R. Since the hull $\mathfrak{F}^*_{co}$ is homeomorphic with a complete metric space it follows (see §2) that the motion f_t is compact and recurrent.

Now consider the Bohr topology. Let $f \in \mathfrak{B}_U(W \times R, R^n)$ and let b be a Bohr metric. The motion f_t is (Bohr) almost periodic if for every $\varepsilon > 0$ the set

$$\{\tau : b(f_t, f_{t+\tau}) < \varepsilon \text{ for all } t \text{ in } R\}$$

is relatively dense in R. If b is given by (12), then $b(f_t, f_{t+\tau}) = b(f, f_\tau)$, for all t. Therefore, recurrence and almost periodicity are equivalent for this metric.

It appears that the concept of almost periodicity, either in the compact-open topology or in the Bohr topology, depends on the choice of the metric. However, as we shall see (Theorem 18), this is not the case. Also, it may appear that the concepts of almost periodicity for the compact-open topology and for the Bohr topology are different. We shall also show (Theorem 18) that this is not the case.

We shall need the following extension of a theorem of Kakutani and Baum [3].

THEOREM 17. *Let X be a compact metric space and let $\pi: X \times R \to X$ be a dynamical system on X, where X is a minimal set. Then X is a.p. minimal if and only if there is a point x in X such that if $\psi: X \to \mathfrak{C}(M, R^n)$ is any continuous function* (*where M is any compact metric space*), *then $\psi_x: R \to \mathfrak{C}(M, R^n)$* (*where $\psi_x(t) = \psi(\pi(x, t))$*) *is almost periodic with respect to the uniform metric on $\mathfrak{C}(M, R^n)$.*

Using this, we can now prove the following result. (See the remark following the statement of the theorem.)

THEOREM 18. *Let $f \in \mathfrak{C}(W \times R, R^n)$. Then the following statements are equivalent*:

(A) *f is Bohr almost periodic.*

(B) *The motion f_t is almost periodic with respect to a basic metric ρ.*

(C) *The motion f_t is almost periodic with respect to a Bohr metric b.*

(D) *The motion f_t is recurrent in the Bohr topology.*

REMARK. The statement of this theorem is somewhat simplified. We shall prove that (B) $\Rightarrow$ (A) for *any* basic metric ρ, and that (C) $\Rightarrow$ (A) for *any* Bohr metric b. Conversely, we shall prove that (A) $\Rightarrow$ (B) {or (A) $\Rightarrow$ (C)} whenever the metric ρ {or b} is generated as indicated in (11) {or (12)}. We have already observed that (C) $\Leftrightarrow$ (D) whenever the Bohr metric b is given by (12).

Proof. (A) $\Rightarrow$ (D). Let $\{M_n\}$ be a sequence of compact sets in W with $W=\bigcup_{n=1}^{\infty} M_n$. Let $\|f-g\|_{n,\infty}$, $b_n(f,g)$ and $b(f,g)$ be given by (12). Then $b_n(f,g)\leqq 1$ for $n=1,2,\ldots$. If $\varepsilon>0$ is given, choose N so that $1/2^N<\varepsilon$. Let $T(\varepsilon)$ be defined by

$$T(\varepsilon) = \{\tau : |f(x,t)-f(x,\tau+t)| \leqq \varepsilon, x\in M_n, t\in R, n = 1,\ldots,N\}.$$

Then $T(\varepsilon)$ is relatively dense in R since f is Bohr almost periodic. We now have for $\tau\in T(\varepsilon)$

$$b_n(f,f_\tau) \leqq \|f-f_\tau\|_{n,\infty} \leqq \varepsilon, \qquad n = 1,2,\ldots,N,$$
$$b_n(f,f_\tau) \leqq 1, \qquad n = N+1,\ldots.$$

Hence $b(f,f_\tau)<2\varepsilon$ for $\tau\in T(\varepsilon)$, so the motion f_t is recurrent in the Bohr topology.

(C) $\Rightarrow$ (B). If f_t is almost periodic with respect to a Bohr metric given by (12), then we construct a basic metric ρ by using the sequence of compact sets $\{K_n\}$ where $K_n=M_n\times I_n$, where $\{I_n\}$ is any sequence of compact sets in R with $R=\bigcup_{n=1}^{\infty} I_n$. Since $\rho(f,g)\leqq b(f,g)$ on $\mathfrak{B}(W\times R, R^n)$, the motion f_t is almost periodic with respect to ρ.

(B) $\Rightarrow$ (A). If f_t is almost periodic with respect to a basic metric ρ, then $\mathfrak{F}^*_{\text{co}}$ is an a.p. minimal set. Let M be any compact set in W and let ψ: $\mathfrak{F}^*_{\text{co}}\to\mathfrak{C}(M, R^n)$ be defined by $(\psi f)(x)=f(x,0)$. ψ is continuous and, by Theorem 17,

$$(\psi_f(t))(x) = (\psi(f_t))(x) = f(x,t) = f^t(x)$$

is almost periodic with respect to the uniform norm. Since M is arbitrary, f is a Bohr almost periodic function, by Lemma 14. Q.E.D.

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