GENERALIZATION OF SCHWARZ-PICK LEMMA TO INVARIANT VOLUME IN A KÄHLER MANIFOLD

BY

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1. **Introduction.** Let $\mathscr{D}$ be the class of bounded homogeneous star-shaped domains $D \subset$ space C^n of n complex variables $z=(z^1, \dots, z^n)$. A domain D is homogeneous if any point of D can be transformed into any other by a holomorphic automorphism; D is star-shaped with respect to a point $z_0 \in D$ if $z \in D$ implies that $r(z-z_0) \in D$ for $0<r\leqq 1$. A bounded domain D possesses the Bergman metric, which is invariant under biholomorphic mappings. Let $\mathscr{K}$ be a class of Kähler manifolds Δ such that the components of its Ricci curvature tensor satisfy certain boundedness conditions (see formulas (2.9)). We consider biholomorphic mappings $w=w(z)$ of $D \in \mathscr{D}$ into $\Delta \in \mathscr{K}$, that is, $w=(w^1, \dots, w^n)$, where w is local coordinate on Δ, and $w^j=w^j(z^1, \dots, z^n)$, are holomorphic functions on D with Jacobian determinant

$$J_w(z) \equiv \partial(w)/\partial(z) \neq 0.$$

In §2 we generalize the Ahlfors version of the Schwarz-Pick lemma in C^1 to invariant volume in bounded homogeneous domains $D \in \mathscr{D}$ in C^n. This theorem states that if $w=w(z)$ is a holomorphic mapping of the disk $|z|<1$ into a Riemann surface W and if the metric $d\sigma=\lambda\, |dw|$, $\lambda>0$, of W has a negative curvature $\leqq -4$ everywhere on W, then

$$\lambda\, |dw/dz| \leqq 1/(1-|z|^2)$$

for $|z|<1$ [1]. Theorem 1 gives the invariant form of this generalization (with respect to biholomorphic mappings) and Theorem 2 an inequality which generalizes an inequality obtained by Dinghas when D is the unit hypersphere [4]. The proof of these theorems uses the method of Ahlfors in [1] and depends on properties of certain relative invariants of D, in particular, the fact that the Bergman kernel function of a bounded homogeneous domain is infinite everywhere on the boundary (Lemma 1). In §3 various applications and extensions of the ideas in §2 are given in Theorems 3-6 and corollary. In §4 the results of §2 are applied to a study of the relative invariants of the classical Cartan domains R_j ($j=\text{I}, \dots, \text{IV}$), in particular, the invariant I_j (see (2.4)) is calculated. This procedure leads to a solution of a certain nonhomogeneous partial differential equation formed from the Hessian determinant of a holomorphic function on R_j.

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2. **Generalization of Schwarz-Pick lemma.** 1. Let D be a bounded homogeneous star-shaped domain of class $\mathscr{D}$. Its Bergman metric is given by

$$ds_D^2 = T_{\alpha\beta}\, dz^\alpha\, d\bar{z}^\beta \tag{1}$$

(the summation convention is used), where

$$T_{\alpha\beta} = T_{\alpha\beta}(z, \bar{z}) = \frac{\partial^2 \log K_D(z, \bar{z})}{\partial z^\alpha\, \partial \bar{z}^\beta},$$

(2)

$$T_D = T_D(z, \bar{z}) = \det(T_{\alpha\beta}),$$

$K_D\,(z, \bar{z})$ the Bergman kernel function of D [2]. The Bergman metric is a Kähler metric on D which is invariant under biholomorphic mappings of D. The domains D in $\mathscr{D}$ have the additional properties:

(i) The image domains D_r of D under the similarity transformation

$$w = r(z - z_0), \qquad 0 < r \leqq 1, \tag{3}$$

are such that $D_{r_1} \subset D_{r_2}$ if $r_1 \leqq r_2$. Also $D = \bigcup_{j=1}^\infty D_{r_j}$, where r_j, $0 < r_j < 1$, is an increasing sequence with limit 1. These facts follow since D is star-shaped. (Without loss of generality we may take $z_0 = 0$ in (3).)

(ii) The kernel function $K_D(z, \bar{z})$ becomes infinite on the boundary ∂D of D (This means that the set $[z\colon z \in D$ and $K_D(z, \bar{z}) < M]$ is relatively compact on D.) This result is included in Lemma 1.

Definition. A real-valued function $R_D(z, \bar{z})$ on D is a relative invariant of D if under any biholomorphic mapping $w\colon D \to D^*$

$$R_{D^*}(w, \bar{w})\, |J_w(z)|^2 = R_D(z, \bar{z}).$$

The functions $K_D(z, \bar{z})$ and $T_D(z, \bar{z})$ are relative invariants of D [2] and consequently the function

$$I_D(z, \bar{z}) = K_D(z, \bar{z})/T_D(z, \bar{z}) \tag{4}$$

is invariant under biholomorphic mappings:

$$I_D(z, \bar{z}) = I_{D^*}(w, \bar{w}). \tag{5}$$

It is clear that if J_D is another invariant of D, then $J_D = kI_D$ for some constant k. Therefore, an invariant on a homogeneous domain is uniquely determined up to a constant multiple.

Lemma 1. *Any relative invariant $R_D(z, \bar{z})$ of a bounded homogeneous domain D becomes infinite on ∂D.*

Proof. Let Γ be the group of holomorphic automorphisms of D. Since the set of elements of Γ is uniformly bounded it forms a normal family. Let a be an arbitrary point and a_0 a fixed point of D. Since D is homogeneous there is an automorphism $t=t_a(z)$ which maps a_0 into a. Since $R_D(a, \bar{a})$ is a relative invariant of D

$$R_D(a, \bar{a}) = R_D(a_0, \bar{a}_0)\,|J_{t_a}(a_0)|^{-2}. \tag{6}$$

Let $b \in \partial D$. It follows from a well-known theorem of H. Cartan [3] that

$$\lim_{a_i \to b} J_{t_{a_i}}(z) = 0. \tag{7}$$

The lemma follows from (6) and (7).

Let $\mathscr{K}$ be the class of Kähler manifolds Δ with metric given by

$$\begin{aligned} d\sigma_\Delta^2 &= g_{\alpha\beta}(w, \bar{w})\, dw^\alpha\, d\bar{w}^\beta, \\ g_\Delta &= g_\Delta(w, \bar{w}) = \det(g_{\alpha\beta}), \end{aligned} \tag{8}$$

where w is a local coordinate of a point on Δ. We also assume

$$-r_{\alpha\beta}u^\alpha\bar{u}^\beta \geqq 0, \tag{9a}$$

$$\det(-r_{\alpha\beta}) \geqq g_\Delta, \tag{9b}$$

where

$$r_{\alpha\beta} = -\frac{\partial^2 \log g_\Delta}{\partial w^\alpha\, \partial\bar{w}^\beta} \tag{9c}$$

are the components of the Ricci curvature tensor of the metric (8) [8, 126]. Since $I_D = K_D/T_D$ is constant for a homogeneous domain, it follows that the components of the Ricci curvature tensor of the metric (1) have the form

$$-\frac{\partial^2 \log T_D}{\partial z^\alpha\, \partial\bar{z}^\beta} = -T_{\alpha\beta}$$

so that (9) is satisfied. Hence $\mathscr{D}$ is a subclass of $\mathscr{K}$.

2. THEOREM 1. *If a bounded homogeneous domain D of class $\mathscr{D}$ can be mapped biholomorphically by $w = w(z)$ into a Kähler manifold $\Delta \in \mathscr{K}$, then*

$$g_\Delta(w, \bar{w})\,|J_w(z)|^2 \leqq T_D(z, \bar{z}) \tag{10}$$

on D. Equality holds if the mapping is onto and the Kähler metric of Δ equals the Bergman metric of Δ.

Proof. Let $z \in D$. By (i) there exists an $r<1$, such that $z \in D_r$. Now D_r is a homogeneous domain: $z_1, z_2 \in D_r$ implies $z_1/r, z_2/r \in D$ by the similarity transformation s given by (3) with $z_0 = 0$; thus there is an automorphism t of D which takes z_1/r into z_2/r and $s^{-1}ts$ is an automorphism of D_r and takes z_1 into z_2. Let $\zeta = \zeta(z)$ be an automorphism of D_r which takes an arbitrary point z of D_r into 0. From the definition of relative invariant

$$I_{D_r} = \frac{K_{D_r}(0, 0)}{T_{D_r}(0, 0)} = \frac{K_{D_r}(z, \bar{z})}{T_{D_r}(z, \bar{z})} = I_{D_r}(z, \bar{z}) \tag{11}$$

so that the invariant $I_{D_r}(z, \bar{z})$ is a constant on D_r, and by (5)

$$I_{D_r} = I_D. \tag{12}$$

Let

$$G_{\alpha\beta}\, dz^\alpha\, d\bar{z}^\beta$$

be the hermitian form on D corresponding to the metric (8) on Δ under the inverse mapping $z = z(w)$ of $w\colon D \to \Delta$ [5, 79]. Then

$$G_{\mu\bar{\nu}}(z, \bar{z}) = g_{\alpha\beta}(w, \bar{w}) \frac{\partial w^\alpha}{\partial z^\mu} \frac{\partial \bar{w}^\beta}{\partial \bar{z}^\nu}$$

and

$$G_D(z, \bar{z}) = g_\Delta(w, \bar{w})\, |J_w(z)|^2 > 0. \tag{13}$$

Let

$$R_{\alpha\beta}(z, \bar{z}) = \frac{-\partial^2}{\partial z^\alpha\, \partial \bar{z}^\beta} \log G_D(z, \bar{z}).$$

Then

$$r_{\alpha\beta}(w, \bar{w}) \frac{\partial w^\alpha}{\partial z^\mu} \frac{\partial \bar{w}^\beta}{\partial \bar{z}^\nu} = R_{\mu\bar{\nu}}(z, \bar{z})$$

and

$$\det(-r_{\alpha\beta}(w, \bar{w})) = \det(-R_{\alpha\beta}(z, \bar{z}))\, |J_w(z)|^{-2}.$$

Thus from hypothesis (9b)

$$\det(-R_{\alpha\beta}(z, \bar{z})) \geqq g_\Delta(w, \bar{w})\, |J_w(z)|^2 = G_D(z, \bar{z}). \tag{14}$$

Following the proof in Dinghas and Ahlfors [4, 11] let

$$\begin{aligned} U &= \log \frac{G_D(z, \bar{z})}{T_D(0, 0)}, \quad V = \log \frac{K_{D_r}(z, \bar{z})}{K_D(0, 0)}, \\ \Psi &= U - V, \end{aligned} \tag{15}$$

and set $E = [z \in D\colon U > V]$ (E open). Since under the transformation (3) $\partial D \to \partial D_r$, from Lemma 1 $K_{D_r}(z, \bar{z})$ becomes infinite on ∂D_r. Thus since U is continuous on $\bar{D}_r \subset D$ and V on D_r, $\bar{E} \subset D_r$. Let O be any component of E. Then $\bar{O} \subset \bar{E} \subset D_r$ so that $\bar{O}$ is compact. Thus the continuous function Ψ takes its maximum at a point $z_0 \in \bar{O}$ but $\Psi(z_0) = \max \Psi > 0$ so that $z_0 \in E$. Since Ψ has a maximum on E,

$$\frac{\partial^2 \Psi}{\partial z^\alpha\, \partial \bar{z}^\beta} u^\alpha \bar{u}^\beta \leqq 0$$

at z_0 for any vector (u^α) or by (15)

$$\frac{\partial^2 U}{\partial z^\alpha\, \partial \bar{z}^\beta} u^\alpha \bar{u}^\beta \leqq \frac{\partial^2 V}{\partial z^\alpha\, \partial \bar{z}^\beta} u^\alpha \bar{u}^\beta. \tag{16}$$

From (15), (13) and (9a) and (c)

$$\frac{\partial^2 U}{\partial z^\alpha \, \partial \bar{z}^\beta} u^\alpha \bar{u}^\beta = -r_{\mu\bar{\nu}} \tilde{u}^\mu \bar{\tilde{u}}^\nu \geqq 0, \tag{17}$$

where $(\tilde{u}^\mu) = ((\partial w^\mu/\partial z^\alpha) u^\alpha)$. From the definition of V the matrix $A = (\partial^2 V/\partial z^\alpha \, \partial \bar{z}^\beta)$ is positive definite. Hence by a classical theorem on the simultaneous reduction of a pair of hermitian quadratic forms there exists a nonsingular matrix T such that

$$A = T\bar{T}', \; B = T\Lambda\bar{T}', \; \Lambda = [\lambda_1, \ldots, \lambda_n]$$

[7, 191], $B = (\partial^2 U/\partial z^\alpha \, \partial \bar{z}^\beta)$, and from (16) and (17)

$$0 \leqq \lambda_i \zeta^i \bar{\zeta}^i \leqq \sum_{i=1}^{n} \zeta^i \bar{\zeta}^i$$

where $\zeta = uT$. By taking ζ successively equal to $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0), \ldots,$ $(0, \ldots, 0, 1)$, we get $0 \leqq \lambda_i \leqq 1$ $(i = 1, \ldots, n)$. Hence at z_0 since $\det T\bar{T}' > 0$

$$0 \leqq \det\left(\frac{\partial^2 U}{\partial z^\alpha \, \partial \bar{z}^\beta}\right) \leqq \det\left(\frac{\partial^2 V}{\partial z^\alpha \, \partial \bar{z}^\beta}\right).$$

But from (15) and (14)

$$\det\left(\frac{\partial^2 U}{\partial z^\alpha \, \partial \bar{z}^\beta}\right) \geqq G_D(z, \bar{z})$$

while from (15), (2), (11) and (12)

$$\det\left(\frac{\partial^2 V}{\partial z^\alpha \, \partial \bar{z}^\beta}\right) = T_{D_r}(z, \bar{z}) = K_{D_r}(z, \bar{z}) \frac{T_D(0, 0)}{K_D(0, 0)}$$

so that at z_0

$$\frac{G_D(z, \bar{z})}{T_D(0, 0)} \leqq \frac{K_{D_r}(z, \bar{z})}{K_D(0, 0)}, \tag{18}$$

that is, $U \leqq V$, which is a contradiction. Thus O and E are empty so that $U \leqq V$ on D_r, and relation (18) holds on D_r. Since K_{D_r} is a relative invariant,

$$\frac{G_D(z, \bar{z})}{T_D(0, 0)} \leqq \frac{r^{-2n} K_D\left(\dfrac{z}{r}, \dfrac{\bar{z}}{r}\right)}{K_D(0, 0)}$$

and letting $r \to 1$ from the continuity of $K_D(z, \bar{z})$ on D follows

$$\frac{G_D(z, \bar{z})}{T_D(0, 0)} \leqq \frac{K_D(z, \bar{z})}{K_D(0, 0)}$$

which gives (10) by (13), (4) and the fact that $I_D(0, 0) = I_D(z, \bar{z})$.

To prove the last part of the theorem we note that $d\sigma_\Delta = ds_\Delta$ and $g_\Delta = T_\Delta$ and since T_Δ is a relative invariant equality follows in (10).

REMARK. The inequality of Theorem 1 is invariant under biholomorphic mappings in the following sense. Let $D \in \mathscr{D} \to D^*$ and $\Delta \in \mathscr{K} \to \Delta^*$ under the biholomorphic maps $z^* = z^*(z)$ and $w^* = w^*(w)$ respectively. Suppose there exists a biholomorphic mapping $w^* = f(z^*)$ of D^* into Δ^*. Then

$$g_{\Delta^*}(w^*, \bar{w}^*)\,|J_f(z^*)|^2 \leqq T_{D^*}(z^*, \bar{z}^*) \tag{19}$$

for $z^* \in D^*$.

Proof. Set $h = w^{*-1} f z^*$. Then $w = h(z)$ is a biholomorphic mapping of D into Δ and by Theorem 1

$$g_\Delta(w, \bar{w})\,|J_h(z)|^2 \leqq T_D(z, \bar{z}).$$

Since

$$T_D(z, \bar{z}) = T_{D^*}(z^*, \bar{z}^*)\,|J_{z^*}(z)|^2 \text{ and } g_\Delta(w, \bar{w}) = g_{\Delta^*}(w^*, \bar{w}^*)\,|J_{w^*}(w)|^2,$$

and from the definition of h

$$|J_h(z)|^2 = |J_f(z^*)|^2\,|J_{z^*}(z)|^2\,|J_{w^*}(w)|^{-2},$$

from which (19) follows.

The following form of Theorem 1 is not invariant under biholomorphic mappings. It may be proved by replacing U in (15) by $U = \log G_D(z, \bar{z})$.

THEOREM 2. *If there exists a biholomorphic mapping $w = w(z)$ of a domain $D \in \mathscr{D}$ into a Kähler manifold $\Delta \in \mathscr{K}$ with condition* (9b) *replaced by*

$$\det(-r_{\alpha\beta}) \geqq T_D(0, 0) g_\Delta(w, \bar{w}),$$

then on D

$$g_\Delta(w, \bar{w})\,|J_w(z)|^2 \leqq T_D(z, \bar{z})/T_D(0, 0).$$

REMARKS.

1. The result obtained by Dinghas in [4] is a special case of Theorem 2 when D is the unit hypersphere.

2. A disadvantage of Theorem 2 is the fact that the original domain D furnished with the Bergman metric cannot belong to our admissible class of Kähler manifolds Δ.

3. Further extensions and applications of Theorem 1.

1. *Area theorem for manifolds of n real dimensions*. We also may derive an inequality for manifolds of n (real) dimensions. Let $M \subset D$ be a continuously differentiable manifold of dimension n parametrized by $z^i = z^i(u)$, u a point of the n-cube I^n. The n-dimensional noneuclidean analytic volume of M is

$$dV_M(z) = T_D^{1/2}(z, \bar{z})\,|\partial(z^1, \ldots, z^n)/\partial(u^1, \ldots, u^n)|\,d\omega_u,$$

and is invariant under biholomorphic mappings of D [5, 330]. Then

THEOREM 3. *Let M be a continuously differentiable manifold of real dimension n in D and $w(M)$ its image under a biholomorphic mapping of $D \in \mathscr{D}$ into $\Delta \in \mathscr{K}$. Then $V_\Delta(w(M)) \leqq V_D(M)$ and equality holds if the mapping is onto and $d\sigma_\Delta = ds_D$.*

We note that if we take $\Delta = D$ and $d\sigma_\Delta = ds_D$, we obtain a generalization of the Schwarz-Pick lemma to this real n-dimensional noneuclidean volume (see also Theorem 25.1 in [5]).

Theorem 3 is invariant under biholomorphic mappings of D and Δ in the sense described in §2.2 since all quantities involved are invariant under such biholomorphic mappings.

2. *Properties of certain domain functions.* The following theorems give some useful inequalities connecting the relative invariants of a domain D and $I_D(z, \bar{z})$. The first theorem follows easily from Theorem 1 for homogeneous star-shaped domains but also holds for any bounded domain.

THEOREM 4. *Let D be any bounded domain and $w = w(z)$ a biholomorphic mapping of D into D. The invariant $I_D(z, \bar{z}) \leqq 1$ on D if and only if $K_D(w, \bar{w})\,|J_w(z)|^2 \leqq T_D(z, \bar{z})$.*

Proof. Suppose that $I_D(z, \bar{z}) \leqq 1$ on D and assume that the conclusion does not hold, that is, there is a point $z_0 \in D$ such that

$$K_D(w_0, \bar{w}_0)\,|J_w(z_0)|^2 > T_D(z_0, \bar{z}_0)\ (w_0 = w(z_0)).$$

Now $D^* = w(D) \subset D$ and hence $K_{D^*}(w_0, \bar{w}_0) \geqq K_D(w_0, \bar{w}_0)$ [2, 45]. Therefore

$$K_{D^*}(w_0, \bar{w}_0)\,|J_w(z_0)|^2 = K_D(z_0, \bar{z}_0) > T_D(z_0, \bar{z}_0) \text{ or } I_D(z_0, \bar{z}_0) > 1$$

which is a contradiction. Since the Jacobian of the identity mapping is 1, the converse of the theorem is trivial.

A useful application of Theorem 4 is

THEOREM 5. *Let D be a bounded complete circular domain in C^n with center at the origin and $w = w(z)$ a biholomorphic mapping of D into D. If $I_D \leqq 1$ on D, then $|J_w(z)|^2 \leqq \omega(D) T_D(z, \bar{z})$. Also $\omega(w(G)) \leqq \omega(D) V_D(G)$* ($\omega$ euclidean volume) *for any measurable $G \subset D$, where $V_D(G) = \int_G T_D(z, \bar{z})\, d\omega_z$.*

Proof. Since D is a bounded complete circular domain, $K_D(z, \bar{z})$ attains its minimum at $z = 0$ [6, 79] and the minimum value is $1/\omega(D)$. Then the conclusions of the theorem follow from Theorem 4.

Since $V_D(G) = V_D(w(G))$ we have

COROLLARY. *Under the hypotheses of Theorem 5*

$$\omega(w(G))/V_D\,(w(G)) \leqq \omega(D)$$

for any measurable set $G \subset D$ with nonzero measure. In particular if $w = w(z)$ is the identity mapping, then

$$\omega(G)/V_D(G) \leqq \omega(D).$$

We remark that $I_D < 1$ for the classical Cartan domains. (See §4.) In fact we do not know examples of domains for which $I_D(z_0, \bar{z}_0) \geqq 1$ at some point $z_0 \in D$.

Finally if we apply Theorem 1 under the identity mapping $w = z$, we get

THEOREM 6. *Let Δ be a homogeneous bounded domain in C^n and D a subdomain which is equivalent to a domain in $\mathscr{D}$. Then for $z \in D$*

$$T_\Delta(z, \bar{z}) \leqq T_D(z, \bar{z}).$$

4. **Relative invariants on the classical Cartan domains.**

1. The theorems in §2 give interesting results for the classical Cartan domains. These domains along with two special domains are the 4 types of bounded irreducible symmetric domains in $C^{n'}$, into which all bounded symmetric domains in $C^{n'}$ can be mapped biholomorphically. Let z be a matrix of complex elements, z' its transpose, z^* its conjugate transpose, and I the identity matrix. The first 3 types are represented by

$$R_j = [z\colon I - zz^* > 0]$$

($j = \mathrm{I}, \mathrm{II}, \mathrm{III}$) where z is a matrix of type (n, m) on R_{I}, z is a symmetric matrix of order n on R_{II} and a skew-symmetric matrix of order n on R_{III}, and ">0" means that the quadratic form is positive definite. The fourth type R_{IV} is the set of n dimensional vectors such that

$$|zz'| < 1,\ 1 - 2\bar{z}z' + |zz'|^2 > 0.$$

These domains belong to class $\mathscr{D}$. The Bergman kernel function of these domains is known [6] so that to get inequality (2.10) it is sufficient to find $T_j = T_{R_j}(0, 0)$ and use formula (2.11). For the first 3 types

$$K_j(z, z^*) = \frac{1}{\omega_j \det^p(I - zz^*)},$$

$p = m + n$ for R_{I}, $n + 1$ for R_{II} and $n - 1$ for R_{III} and ω_j is the euclidean volume of R_j. For R_{IV}

$$K_{\mathrm{IV}}(z, \bar{z}) = \frac{1}{\omega_{\mathrm{IV}}(1 + |zz'|^2 - 2\bar{z}z')^n}.$$

In case I since $\log K_{\mathrm{I}}(z, z^*) = -(m+n)\log Q_{\mathrm{I}} - \log \omega_{\mathrm{I}}$, where $Q_{\mathrm{I}} = \det(I - zz^*)$ we need the value of $\partial^2 Q_{\mathrm{I}}/\partial z^{\alpha\beta}\partial \bar{z}^{\gamma\delta}$ at $z = 0$. To evaluate this use the expansion for the characteristic equation of zz^* with $\lambda = 1$:

$$\det(\lambda I - zz^*) = \lambda^n - \sigma\lambda^{n-1} + t_{n-2}\lambda^{n-2} - \cdots \pm \det zz^*,$$

where t_i is the sum of the principal i-rowed minors of zz^*, σ being the trace. Since t_i is a homogeneous polynomial of degree $2i$, only the second derivatives of σ

contribute nonzero terms at $z=0$. Also all the first derivatives of t_i and σ are 0 at $z=0$. Now

$$\frac{\partial}{\partial z^{\alpha\beta}} \log K_{\mathrm{I}}(z, z^*) = -(m+n)Q_{\mathrm{I}}^{-1} \frac{\partial}{\partial z^{\alpha\beta}} Q_{\mathrm{I}},$$

$$\frac{\partial^2}{\partial z^{\alpha\beta}\,\partial \bar{z}^{\gamma\delta}} \log K_{\mathrm{I}}(z, z^*) = (m+n)Q_{\mathrm{I}}^{-2} \frac{\partial}{\partial z^{\alpha\beta}} Q_{\mathrm{I}} \frac{\partial}{\partial \bar{z}^{\gamma\delta}} Q_{\mathrm{I}} - (m+n)Q_{\mathrm{I}}^{-1} \frac{\partial^2 Q_{\mathrm{I}}}{\partial z^{\alpha\beta}\,\partial \bar{z}^{\gamma\delta}}.$$

These remarks and formulas apply also to cases II and III.

Since $\sigma = \sum_{j,k} z^{jk}\bar{z}^{jk}$, at $z=0$

$$\frac{\partial^2 \sigma}{\partial z^{\alpha\beta}\,\partial \bar{z}^{\gamma\delta}} = \delta_{j\alpha}\delta_{k\beta}\delta_{j\gamma}\delta_{k\delta},$$

$$\frac{\partial^2 \log K_{\mathrm{I}}(z, z^*)}{\partial z^{\alpha\beta}\,\partial \bar{z}^{\alpha\beta}} = m+n,$$

and all other derivatives are zero so that

$$T_{\mathrm{I}}(0, 0) = \det[(m+n)I] = (m+n)^{mn},$$

$$I_{\mathrm{I}} = \frac{1}{\omega_{\mathrm{I}}(m+n)^{mn}},$$

and from (2.4)

$$T_{\mathrm{I}}(z, z^*) = \frac{(m+n)^{mn}}{\det^{m+n}(I - zz^*)}. \tag{1}$$

The trace of zz^* for a symmetric matrix is

$$\sigma = \sum_{j} z^{jj}\bar{z}^{jj} + 2 \sum_{j \neq k} z^{jk}\bar{z}^{jk}$$

so that at $z=0$

$$\frac{\partial^2 \sigma}{\partial z^{jj}\,\partial \bar{z}^{jj}} = 1, \quad \frac{\partial^2 \sigma}{\partial z^{jk}\,\partial \bar{z}^{jk}} = 2 \qquad (j \neq k)$$

and all other derivatives of σ are zero. Thus

$$T_{\mathrm{II}}(0, 0) = (n+1)^{n((n+1)/2)} 2^{n(n-1)/2},$$

$$I_{\mathrm{II}} = \frac{1}{\omega_{\mathrm{II}} 2^{n(n-1)/2} (n+1)^{n(n+1)/2}}$$

and

$$T_{\mathrm{II}}(z, z^*) = \frac{2^{n(n-1)/2}(n+1)^{n(n+1)/2}}{\det^{n+1}(I - z\bar{z})}. \tag{2}$$

For a skew symmetric matrix $\sigma=2\sum_{j<k} z^{jk}\bar{z}^{jk}$, where the matrix z has only $n(n-1)/2$ distinct nonzero elements, so that we get

$$T_{\mathrm{III}}(0,0) = [2(n-1)]^{n(n-1)/2}, \qquad I_{\mathrm{III}} = \frac{1}{\omega_{\mathrm{III}}[2(n-1)]^{n(n-1)/2}},$$

(3)

$$T_{\mathrm{III}}(z,\bar{z}) = \frac{[2(n-1)]^{n(n-1)/2}}{\det^{n-1}(I+z\bar{z})}.$$

For case IV setting $\Delta = 1+|zz'|^2-2\bar{z}z'$

$$\frac{\partial}{\partial z^\alpha}\Delta = 2z^\alpha[(\bar{z}^1)^2+\cdots+(\bar{z}^n)^2]-2z^\alpha,$$

which is 0 at $z=0$ and similarly for $\partial\Delta/\partial\bar{z}^\alpha$ and

$$\frac{\partial^2\Delta}{\partial z^\alpha\,\partial\bar{z}^\beta} = 4z^\alpha\bar{z}^\beta-2\delta_{\alpha\beta}$$

so that at $z=0$

$$\frac{\partial^2 \log K_{\mathrm{IV}}(z,\bar{z})}{\partial z^\alpha\,\partial\bar{z}^\beta} = -n(-2\delta_{\alpha\beta})$$

and

$$T_{\mathrm{IV}}(0,0) = (2n)^n, \qquad I_{\mathrm{IV}} = \frac{1}{\omega_{\mathrm{IV}}(2n)^n},$$

(4)

$$T_{\mathrm{IV}}(z,\bar{z}) = \frac{(2n)^n}{(1+|zz'|^2-2\bar{z}z')^n}.$$

Formulas (1), (2) and (3) give the interesting result

THEOREM 7. *The function* $\log\det^{-1}(I-zz^*)$ *satisfies the partial differential equation*

$$\det\left(\frac{\partial^2 V}{\partial z^{\alpha\beta}\,\partial\bar{z}^{\gamma\delta}}\right) = ae^{bV} \tag{5}$$

on R_j ($j=$I, II, III) *where* $a=1$, $b=m+n$ *for an* (n,m) *matrix*, $a=2^{n(n-1)/2}$ *for a symmetric or skew-symmetric matrix and* $b=n+1$ *for a symmetric and* $n-1$ *for a skew-symmetric matrix. For case* IV $-\log(1+|zz'|^2-2\bar{z}z')$ *satisfies the partial differential equation*

$$\det\left(\frac{\partial^2 V}{\partial z^{\alpha}\,\partial\bar{z}^{\beta}}\right) = 2^n e^{nV} \tag{6}$$

on R_{IV}.

This theorem corresponds to Lemma 4 of Dinghas [4] for the function

$$-\log(1-z^j\bar{z}^j).$$

Using the values for the euclidean volume given in [6] and induction on n we find that the invariant $I_j<1$ ($j=$I, II, III, IV). Thus Theorem 5 holds for the

Cartan domains and from this result, Theorem 4.2.1 of [6] and the expressions (1)–(4) for $T_j(z, z^*)$ we obtain an interesting distortion theorem on the Jacobian of an interior mapping of a Cartan domain:

THEOREM 8. *Let $w = w_j(z)$ be a biholomorphic mapping of the Cartan domain R_j into itself. Then*

$$|J_{w_j}(z)|^2 \leqq \frac{|J_j(z, \bar{z})|^2}{I_j}$$

for $z \in R_j$, where I_j is the invariant of R_j and $J_j(z, \bar{z})$ the Jacobian of the holomorphic automorphism of R_j which maps z into the origin (j=I, II, III, IV).

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